FOLIA 84

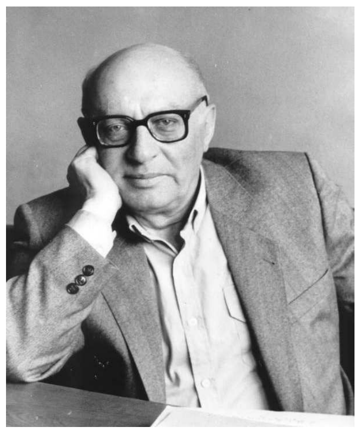
Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica IX (2010)

G. Akhalaia, N. Manjavidze

Boundary value problems with shift for generalized analytic vectors

(In honor of Professor Giorgi Manjavidze)



Professor Giorgi Manjavidze
(6.05.1924 – 13.04.1999)

The present work is devoted to boundary value problems of linear conjugation with shift for generalized analytic vectors.

A great place in the works of Giorgi Manjavidze takes the investigation of boundary value problems of the theory of functions with shifts. In such problems the boundary values of the desired functions are conjugating in the points which are displaced to each other. The model problem is to find a function $\Phi(z)$ holomorphic on the complex plane z, cut along some simple closed curves, the boundary values of which $\Phi^+(t)$ and $\Phi^-(t)$ are satisfying the condition

$$\Phi^+[\alpha(t)] = G(t)\Phi^-(t) + g(t), \qquad t \in \Gamma$$

from both sides of Γ, where $G(t)$, $g(t)$ are given continuous functions on Γ, $\alpha(t)$ is continuous function mapping Γ onto γ in one-to-one manner.

AMS (2000) Subject Classification: 31A25, 35J25.

Volumes I-VII appeared as *Annales Academiae Paedagogicae Cracoviensis Studia Mathematica*.

Later on result obtained by him in this direction became known as the theory of conformal sewing. He also studied the problem with shift depended on parameter and proved the invariance of partial indices for conformal mappings. A special cycle of G. Manjavidze's works is devoted to the investigation of boundary value problems of the function theory by method of successive approximations by means of which solution of the problems with shifts in case of several unknown functions was considerably simplified.

G. Manjavidze obtained marked results for discontinuous boundary value problems with shift for generalized analytic vectors. He studied the Riemann–Hilbert problem in domains with non-smooth boundaries, the Riemann–Hilbert–Poincare problem for generalized analytic functions, and the differential boundary value problem of linear conjugation. In particular, he established both the solvability conditions of these problems and the index formulas and discovered the connection between the problems with shift for analytic and generalized analytic functions.

1. Generalized analytic vectors (definitions and notations)

A vector $w(z) = (w_1, \dots, w_m)$ is called generalized analytic in domain D if it is a solution of the elliptic system

$$\partial_{\bar z} w - Q \partial_z w + Aw + B\overline{w} = 0, \tag{1.1}$$

where $A(z)$, $B(z)$ are given square matrices of order n of the class $L_{p_0}(D)$, $p_0 > 2$ and where $Q(z)$ is a matrix of the special form: it is quasi-diagonal and every block $Q^r = (q^r_{ik})$ is lower (upper) triangular matrix satisfying the conditions:

$$\begin{aligned} &q^r_{11} = \ldots = q^r_{m_r,m_r} = q^r, \qquad && |q^r| \le q_0 < 1, \\ &q^r_{ik} = q^r_{i+s,k+s} && (i+s \le n,\ k+s \le n). \end{aligned}$$

Moreover, $Q(z) \in W^1_p(\mathbb{C})$, $p > 2$ and $Q(z) = 0$ outside of some circle. Under the solution of (1.1) we mean so-called regular solution, i.e., $w(z) \in L_2(\overline{D})$, the generalized derivatives of which $w_{\bar z}, w_z \in L_\omega(D')$, $\omega > 2$, where D' is arbitrary closed subset of D.

If $A(z) \equiv B(z) \equiv 0$, then

$$\partial_{\bar z} w - Q \partial_z w = 0 \tag{1.2}$$

and the solutions of (1.2) are called Q-holomorphic vectors.

The equation (1.2) has a solution of the form

$$\zeta(z) = zI + T\omega, \tag{1.3}$$

where I is unit matrix and $\omega(z)$ is a solution of the equation

$$\omega(z) - Q(z)\Pi\omega = Q(z)$$

belonging to $L_p(\mathbb{C})$, $p > 2$. T, Π are well-known integral operators. The solution (1.3) is analogous of the fundamental homeomorphism of Beltrami equation. The matrix

$$V(t,z) = \partial_t \zeta(t)[\zeta(t) - \zeta(z)]^{-1}$$

is called generalized Cauchy kernel for the equation (1.2), and consider Cauchy type generalized integral

$$\Phi(z)=\frac{1}{2\pi i}\int_{\Gamma} V(t,z)\,d_Q t\mu(t), \tag{1.4}$$

where Γ is closed simple smooth curve, $\mu(t)\in l_1(\Gamma)$ and

$$d_Q t = I\,dt + Q(t)\,d\bar{t}.$$

If the density $\mu(t)$ in (1.4) is Holder-continuous on Γ, then (1.4) is Holder-continuous in $\overline{D^+}$ and in $\overline{D^-}$; the boundary values of Φ on Γ are given by

$$\Phi^{\pm}(t)=\pm\frac{1}{2}\mu(t)+\frac{1}{2\pi i}\int_{\Gamma} V(\tau,z)\,d_Q\tau\mu(\tau). \tag{1.5}$$

If $\mu(t)\in L_p(\Gamma)$, $p>1$, then (1.5) are fulfilled almost everywhere on Γ, provided that $\Phi^{\pm}(t)$ are angular boundary values of the vector $\Phi(z)$. Here very important role play the analogous integral operators

$$(\widetilde{T}f)[z]=-\frac{1}{\pi}\int_{D} V(t,z)f(t)\,d\sigma_t,$$

$$(\widetilde{\Pi}f)[z]=-\frac{1}{\pi}\int_{D} \partial_z V(t,z)f(t)\,d\sigma_t.$$

If $Q\in H^{\alpha_0}(\mathbb{C})$, then $(\widetilde{T}f)$ is completely continuous operator from $L_p(\overline{D})$, $p>2$, onto $H^{\alpha}(D)$, $\alpha=\min\{\alpha_0,\frac{p-2}{p}\}$, moreover the operator $\widetilde{\Pi}$ is linear bounded operator from $L_p(\overline{D})$ to $L_p(\overline{D})$, and

$$(\partial_{\bar{z}}-Q\partial_z)\widetilde{T}f=f,\qquad \partial_z\widetilde{T}f=\widetilde{\Pi}f.$$

Using Q-holomorphic vectors the generalized analytic vectors can be represented as follows (Bojarski B. Theory of Generalized Analytic Vector.)

$$\begin{aligned} w(z)&=\Phi(z)\\ &=\iint_{D}\Gamma_1(z,t)\Phi(t)\,d\sigma_t+\iint_{D}\Gamma_2(z,t)\overline{\Phi(t)}\,d\sigma_t+\sum_{k=1}^{N}c_k w_k(z), \end{aligned} \tag{1.6}$$

where $\Phi(z)$ is Q-holomorphic vector, $w_k(z)$, $k=1,\dots,N$, is a complete system of linearly independent solutions of Fredholm equation

$$Kw\equiv w(t)-\frac{1}{\pi}\iint_{D} V(t,z)\big[A(t)w(t)+B(t)\overline{w(t)}\big]\,d\sigma_t=0.$$

$w_k(z)$ turn to be continuous vectors in whole plane vanishing at infinity, c_k are arbitrary real constants, the kernels $\Gamma_1(z,t)$, $\Gamma_2(z,t)$ satisfy integral equations in turn.

Finally the vector $\Phi(z)$ has to satisfy the following conditions

$$\operatorname{Re}\iint\limits_{D}\Phi(z)v_k(z)\,d\sigma_t=0,\qquad k=1,\dots,N,\tag{1.7}$$

where $v_k(z)\in L_p(\overline{D})$ $(k=1,\dots,N)$ form system of linearly independent solutions of Fredholm integral equation

$$v(z)+\frac{\overline{A'(z)}}{\pi}\iint\limits_{D}\overline{V(z,t)}v(t)\,d\sigma_t+\frac{\overline{B'(z)}}{\pi}\iint\limits_{D}V(z,t)\overline{v(t)}\,d\sigma_t=0.$$

It should be mentioned that, generally speaking, the Liouville theorem is not true for the solutions of (1.1). This explains the appearance of the constants c_k in (1.6) and that the condition (1.7) has to be satisfied.

2. Relation between BVP of linear conjugation and generalized analytic functions

Now we show the connection of linear conjugation problem with shift and the theory of generalized analytic functions. This gives the possibility to consider the problem of linear conjugation in somewhat different formulation.

Let Γ_1 and Γ_2 be Liapunov curves, $\alpha(t)$ is function mapping Γ_1 onto Γ_2 in one-to-one manner preserving orientation, $\alpha(t(s))$ is absolutely continuous function, $M\geq|a'(t)|\geq m>0$ (M and m are constants), $a(t)$ and $b(t)$ are given matrices of the class $H^{\mu}(\Gamma_1)\,(\mu>\frac{1}{2})$, $a(t)$ is nonsingular square matrix of order n, $b(t)$ is $(n\times l)$-matrix.

Find piecewise-holomorphic matrix $\varphi(z)$ having finite order at infinity, $\varphi^+(t),\varphi^-(t)\in H(\Gamma)$ and satisfying the boundary condition

$$\varphi^+[\alpha(t)]=a(t)\varphi^-(t)+b(t),\qquad t\in\Gamma_1.\tag{2.1}$$

We call the piecewise-holomorphic matrix $\chi(z)$ with finite order at infinity the canonical matrix of the problem (2.1) if $\det\chi(z)\neq 0$ everywhere except perhaps at the point $z=\infty$; $\chi(z)$ has a normal form at infinity with respect to columns and

$$\chi^+[\alpha(t)]=a(t)\chi^-(t),\qquad t\in\Gamma_1.$$

Mapping conformally D_2^+ and D_1^- into interior and exterior parts of the unit circle Γ respectively we get the same problem (2.1), where $\alpha(t)$ maps Γ onto Γ; the matrices $a(t),b(t)$ have the same properties. Consider the problem in case $\Gamma_1=\Gamma_2=\Gamma$.

After proving some useful propositions we get the following

Theorem 1
All solutions of the problem (2.1) *are given by the formulas*

$$\begin{aligned}\varphi[\alpha(z)]&=\chi_\alpha[\alpha(z)][Tf+h(z)+P(w_0(z))],\qquad && z\in D^+,\\ \varphi(z)&=\chi_\alpha(z)[Tf+h(z)+P(w_0(z))],\qquad && z\in D^-,\end{aligned}\tag{2.2}$$

where $P(z)$ is arbitrary polynomial vector and the vector $f \in L_p(\overline{D})$, $p > 2$, is a solution (unique) of the equation

$$Kf =: f(z) - q(z)\Pi f = g(z);$$
$$h(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{[\chi_\alpha^+(\alpha(t))]^{-1} b(t)}{t-z}\, dt,$$
$$g(z) = (g_1, \ldots, g_n) = q(z)h'(z) \in L_p(\overline{D^+}).$$

The solutions vanishing at infinity are given by the formulas (2.2), *where*

$$P(z) = (P_{\varkappa_1 - 1}, \ldots, P_{\varkappa_n - 1}), \qquad \varkappa_1 \geq \ldots \geq \varkappa_n$$

are the partial indices of the problem (2.1), *$P_j(z)$ is arbitrary polynomial of order j ($P_j(z) = 0$ if $j < 0$); if $0 \geq \varkappa_{s+1} \geq \ldots \geq \varkappa_n$, then the vector $b(t)$ has to satisfy the following conditions*

$$2i \iint\limits_{D} g_j(\zeta) L(\zeta^k)\, d\zeta\, d\eta = \int\limits_{\gamma} t^k \big\{ \big[\chi_\alpha^+(\alpha(t))\big]^{-1} b(t) \big\}_j\, dt,$$
$$j = s+1, \ldots, n; \; k = 0, \ldots, |\varkappa_j| - 1,$$

where $Lf = f(z) - \Pi(qf)$.

3. BVP of linear conjugation with shift for generalized analytic vectors

Define the classes for Q-holomorphic vectors. Let D^+, (D^-) be finite (infinite) domain which is bounded by a simple closed Liapunov smooth curve Γ.

Denote by $E_{s,p}(D,Q)$, $s \geq 0$, $p \geq 1$, $Q(z) = (q_{ik}) \in W^s_{p_0}(\mathbb{C})$, $p > 2$, (D is one of the domains $D^+, D^-, W^s_{p_0}(\mathbb{C})$ is a Sobolev space) the class of Q-holomorphic vectors $\Phi(z) = (\Phi_1, \ldots, \Phi_n)$ in the domain D satisfying the following conditions

$$\int\limits_{\delta_{kr}} \left| \frac{\partial^s \Phi_k}{\partial z^s} \right|^p |dz| \leq C, \qquad k = 1, \ldots, n,$$

where C is a constant, δ_{kr} is an image of the circle $|\xi| = r$, $r < 1$, while quasicon–formal mapping $\xi = \omega_k(S_k(z))$ of the unit circle $|\xi| < 1$ onto D, ω_k is an analytic function in the domain $S_k(D)$, S_k is a fundamental homeomorphism of the Beltrami equation

$$\partial_{\bar{z}} S - q_{kk}(z) \partial_z S = 0.$$

If D is infinite domain, then for the simplicity of notation we suppose that $W(\infty) = 0$ (remind that Q-holomorphic vectors are the analytic functions in vicinity of the point $z = \infty$, because $Q = 0$ at infinity). By $E_{s,p}(D,Q,S)$ denote the class of the vectors Φ, belonging to the class $E_{s,\lambda}(D,Q)$ for some $\lambda > 1$, for which the angular boundary values are belonging to $L_p(\Gamma, \rho)$,

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\nu_k}, \qquad t_k \in \Gamma, \; -1 < \nu_k < p-1, \; p > 1. \tag{3.1}$$

Let $Q^+(z)$ and $Q^-(z)$ are two given matrices, satisfying the conditions $Q^+ \in W^l_{p_0}(\mathbb{C})$, $Q^- \in W^m_{p_0}(\mathbb{C})$, $l, m \geq 0$, $p_0 > 2$. $Q^+, Q^- \in W^1_{p_0}(\mathbb{C})$, when $l = m = 0$. By $E^{\pm}_{l,m,p}(\Gamma, Q^{\pm}, \rho^{\pm})$ we denote the class of the vectors defined on cut along Γ plane, belonging to the class $E_{l,p}(D^+, Q^+, \rho^+)$ $(E_{m,p}(D^-, Q^-, \rho^-))$, in the domain $D^+(D^-)$, $E_{0,p}(D, Q) \equiv E_p(D, Q)$. Introduce the classes of the generalized analytic vectors, satisfying the equation of the form

$$Mw \equiv \partial_{\bar{z}} w - Q\partial_z w + Aw + B\overline{w} = 0; \tag{3.2}$$

in case of infinite domain we suppose, that Q, A, B are equal to zero at infinity. By $E_{l,p}(D, M)$, $l \geq 0$, $p \geq 1$, denote the class of the solutions of the equation (3.2) satisfying the conditions

$$\int\limits_{\delta_{kr}} \left| \frac{\partial^l w_k}{\partial z^l} \right|^p |dz| \leq C, \quad \int\limits_{\delta_{kr}} \left| \frac{\partial^s w_k}{\partial z^s} \right|^p |dz| \leq C, \qquad k = 1, \dots, n, \ s = 1, \dots, l-1,$$

the curve δ_{kr} is defined above, C is a constant; if D is infinite domain, then $w(\infty) = 0$. Here we consider, that $\Gamma \in H^m_a$, $Q(a) \in W^l_{p_0}(\mathbb{C})$, $p_0 > 2$, when $l = 0$, $Q \in W^1_{p_0}(\mathbb{C})$, $A, B \in L_\infty(\overline{D})$, $Q \in W^1_{p_0}(\mathbb{C})$, $A, B \in L_\infty(\overline{D})$, $A(z), B(z) \in H^{l-1}_a(\overline{D})$.

By $E_{l,p}(D, M, \rho)$, $l \geq 0$, $p > 1$, $\rho(t)$, is a function of the form (3.1), denote the class of the vectors $w(z)$, belonging to the class $E_{l,\lambda}(D, M)$, for some $\lambda > 1$, for which the angular boundary values of the vector $\frac{\partial^l w}{\partial z^l} \in L_p(\Gamma, \rho)$.

$E^{\pm}_{l,m,p}(\Gamma, M^{\pm}, \rho^{\pm})$ denotes the class of the vectors defined on the plane cut along Γ and belonging to the class $E_{l,p}(D^+, M^+, \rho^+)$ $[E_{m,p}(D^-, M^-, \rho^-)]$ in $D^+(D^-)$.

Consider the following boundary value problem:

Find a vector $\Phi(z) = (\Phi_1, \dots, \Phi_n)$ of the class $E^{\pm}_{l,m,p}(\Gamma, M^{\pm}, \rho^{\pm})$ satisfying boundary condition

$$\Phi^+[\alpha(t)] = a(t)\Phi^-(t) + b(t)\overline{\Phi^-(t)} + f(t), \qquad t \in \Gamma. \tag{3.3}$$

Γ is simple closed Liapunov curve, $\alpha(t)$ is function mapping Γ onto Γ in one-to-one manner, preserving the orientation, $a(t)$ and $b(t)$ are given piecewise-continuous $(n \times n)$-matrices on Γ, $\inf |\det \alpha(t)| > 0$, $f(t)$ is given vector of the class $L_p(\Gamma, \rho)$, $\rho = \rho^-(t)$ is a function (3.1), $\rho^+(t) = \prod_{k=1}^r |t - \alpha(t_k)|^{\nu_k}$.

The following proposition holds.

LEMMA

If the vector $\Phi(z) \in E^{\pm}_p(\Gamma, Q^{\pm}, \rho^{\pm})$, then it is uniquely representable in the form

$$\Phi(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int\limits_\Gamma V_+(\tau, z)\, d_Q + \tau_\mu[\beta(\tau)]', & z \in D^+, \\ \dfrac{1}{2\pi i} \displaystyle\int\limits_\Gamma V_-(\tau, z)\, d_Q - \tau_\mu(\tau), & z \in D^-, \end{cases} \tag{3.4}$$

where $\mu(t) \in L(\Gamma, \rho)$ *is a solution of Fredholm integral equation*

$$\begin{aligned} N_\mu \equiv \mu(t) + \frac{1}{2\pi i} \int_\Gamma \left[V_+(\alpha(\tau), \alpha(t))\, d_Q + \alpha(\tau) - V_-(\tau, t) d_Q - \tau\mu(\tau) \right] \\ = \Phi^+[\alpha(t)] - \Phi^-(t), \end{aligned}$$

$\beta(t)$ *is inverse of* $\alpha(t)$.

Substituting the representation (3.4) in the boundary condition (3.3) for the desired vector $\mu(t)$ we obtain the singular integral equation

$$\begin{aligned} \mathcal{L}\mu \equiv [l + \alpha(t_1)]\mu(t) + b(t)\overline{\mu(t)} + \frac{1}{\pi i} \int_\Gamma M_1(\tau, t)\mu(\tau)\, d\tau \\ + \frac{1}{\pi i} \int_\Gamma M_2(\tau, t)\overline{\mu(\tau)}\, d\tau = 2f(t), \end{aligned} \tag{3.5}$$

$$M_1(\tau, t)\mu(\tau) d\tau = [V_+(\alpha(\tau), \alpha(t)) d_Q + \alpha(\tau) - a(t) V_-(\tau, t) d_Q - \tau]\mu(\tau),$$

$$M_2(\tau, t)\overline{\mu(\tau)} d\tau = b(t)\overline{V_-(\tau, t)} d_Q - \overline{\tau\mu(\tau)}.$$

The following result holds.

Theorem 2
If the equation (3.5) *is Noetherian in the space* $L_p(\Gamma, \rho)$, *then the boundary problem* (3.3) *is Noetherian in* $E_p^\pm(\Gamma, Q^\pm, \rho^\pm)$; *the necessary and sufficient solvability conditions have the form*

$$\operatorname{Re} \int_\Gamma f(t)\psi(t)\, dt = 0, \qquad k = 1, \ldots, l',$$

where $\psi_k(t)$ *is a complete system of linearly independent solutions of conjugate homogeneous equation* $\mathcal{L}'\psi = 0$ *of the class* $L_q(\Gamma, \rho^{1-q})$, $q = \frac{p}{1+p}$; *the index of the problem* (3.3) *of the class* $E_p^\pm(\Gamma, Q^\pm, \rho^\pm)$ *is equal to the index of the equation* (3.5) *of the class* $L_p(\Gamma, \rho)$.

Consider now the problem (3.3) for generalized analytic vectors satisfying the equations of the form (3.2):

Find the vector $w(z) \in E_p^\pm(\Gamma, M^\pm, \rho^\pm)$ satisfying the boundary condition

$$w^+[\alpha(t)] = a(t) w^-(t) + b(t)\overline{w^-(t)} + f(t), \qquad t \in \Gamma. \tag{3.6}$$

We seek the solution of (3.6) by the formula (1.6) in the following form

$$\begin{aligned} w^\pm(z) = \Phi^\pm(z) + \iint_D \Gamma_1^\pm(z, t)\Phi^\pm(t)\, d\sigma_t \\ + \iint_D \Gamma_2^\pm(z, t)\overline{\Phi^\pm(t)}\, d\sigma_t + \sum_{k=1}^{N^\pm} c_k^\pm w_k^\pm(z), \end{aligned}$$

where $\Phi^{\pm}(z) \in E_p^{\pm}(\Gamma, Q^{\pm}, \rho^{\pm})$, $c_k^{\pm}(k = 1, \ldots, N^{\pm})$ are desired real numbers, $w_k^{\pm}(z)$ – solutions of the corresponding integral equations. The vectors $\Phi^{\pm}(t)$ have to satisfy the conditions

$$\operatorname{Im} \int_\Gamma \Phi^{\pm}(t)\, d_{Q^{\pm}} t \psi_k^{\pm}(t) = 0, \qquad k = 1, \ldots, N^{\pm},$$

where $\psi_k^{\pm}$ are the complete systems of the homogeneous conjugate equations. With respect to the vector $\Phi^{\pm}(z)$ we obtain the following boundary problem

$$\begin{aligned} \Phi^{+}\alpha(t_0)] &= a(t_0)\Phi^{-}(t_0) + b(t_0)\overline{\Phi^{-}(t_0)} + \mathcal{L}_{+}\Phi^{+} + \mathcal{L}_{-}\Phi^{-} + f_0(t_0), \\ f_0(t) &= f(t) + \sum_{k=1}^{N^-} c_k^{-}\left[a_k(t)w_k^{-}(t) + b_k(t)\overline{w_k^{-}(t)}\right] - \sum_{k=1}^{N^+} c_k^{+}[\alpha(t)]; \end{aligned} \tag{3.7}$$

the operators $\mathcal{L}_{+}$ and $\mathcal{L}_{-}$ are defined as follows

$$\begin{aligned} \mathcal{L}_{+}\Phi^{+} &= \iint_{D^+} \left[\Gamma_1^{+}(\alpha(t_0,t))\Phi^{+}(t) + \Gamma_2^{+}(\alpha(t_0,t))\overline{\Phi^{+}(t)}\right] d\sigma_t, \\ \mathcal{L}_{-}\Phi^{-} &= a(t_0)F(t_0) + b(t_0)\overline{F(t_0)}, \\ F(t_0) &= \iint_{D^-} \left[\Gamma_1^{-}(\alpha(t_0,t)\Phi^{-}(t) + \Gamma_2^{-}(t_0,t)\overline{\Phi^{-}(t)}\right] d\sigma_t. \end{aligned}$$

Substituting the representations (3.4) first in these formulas we obtain that the operators $\mathcal{L}_{+}$ and $\mathcal{L}_{-}$ are completely continuous operators in $L_p(\Gamma, \rho^{+}), L_p(\Gamma, \rho^{-})$ with respect to the angular boundary values $\Phi^{+}(\tau), \Phi^{-}(\tau)$ and then in (3.7) we get singular integral equation with respect to the vector $\mu(t)$

$$\Omega_\mu \equiv (\Omega_0 + \Omega_1)\mu + \Omega_2\overline{\mu} = 2\overline{f(t)} + \sum_{k=1}^{N} d_k \eta_k(t), \tag{3.8}$$

where Ω_0 is completely continuous operator, Ω_1 and Ω_2 are singular integral operators $\eta_k(t)$ are linearly independent continuous vectors, represented by $w_k^{\pm}(t)$, d_k $(k = 1, \ldots, N,\, N \le N^{+} + N^{-})$ are desired real constants. Besides (3.8) the vector μ has to satisfy the conditions

$$\operatorname{Im} \int_\Gamma \mu(t)\omega_k(t)\, dt = 0, \qquad k = 1, \ldots, N,$$

where $\omega_k(t)$ $(k = 1, \ldots, N)$ are linearly independent vectors, represented by $\psi_k^{\pm}(t)$.

The necessary and sufficient solvability conditions of the problem (3.6) in the class $E_p^{\pm}(\Gamma, M^{\pm}, \rho^{\pm})$ have the form (Prossdorf S. Some Classes of Singular Equations).

$$\operatorname{Re} \int_\Gamma f(t)Y_k(t)\, dt = 0, \qquad k = 1, \ldots, R, \tag{3.9}$$

where the linearly independent vectors $Y_k(t)$ $(k = 1, \ldots, R)$ belongingto the class

$L_p(\Gamma,\rho^{1-q})$, are representable by ψ^+,ψ^- and by the vectors composing the basis of subspace of the solutions of adjoint homogeneous equation $\Omega'\nu=0$. The index of the problem (3.6) is equal to

$$\varkappa + N - R, \tag{3.10}$$

where $\varkappa$ is the index of the operator Ω of the class $L_p(\Gamma,\rho)$. Actually $N=R$ in the formula (3.10).

Let $X^\pm$ be sets of the vectors $w^\pm(z)$ defined in the domains $D^\pm$ representable in the form

$$w^\pm(z)=\Phi^\pm(z)+h^\pm(z),\quad \Phi^\pm(z)\in E_p^\pm(\Gamma,Q^\pm,\rho^\pm),\quad h^\pm(z)\in H^\alpha(D^\pm).$$

This pair of sets coincides with the class $E_p^\pm(\Gamma,M^\pm,\rho^\pm)$. Introduce the norms

$$|w^\pm|_{X^\pm}=\inf\big\{|\phi^\pm|_{L_p(\Gamma,\rho^\pm)},|h^\pm|_{H^\alpha(D^\pm)}\big\},$$

$X^\pm$ are Banach spaces. Let $X=(X^+,X^-)$ be new Banach space with the norm $|w|_X=\max[|w^+|_{X^+},|w^-|_{X^-}]$. It is evident, that this norm in $E_p^\pm(\Gamma,M^\pm,\rho^\pm)$ is independent of $A^\pm,B^\pm$. Consider the set of the operators

$$M_\lambda^\pm w^\pm=\partial_{\bar z}w^\pm-Q^\pm\partial_z w^\pm+\lambda\big[A^\pm w^\pm+B^\pm\overline{w^\pm}\,\big],$$

where $\lambda\in[0,1]$. In order to calculate the index of the problem (3.6) we may take the differential operators of the form $=\partial_{\bar z}w^\pm-Q^\pm\partial_z w^\pm$ and for such operators the numbers N^+,N^- are equal to zero and hence $N=R$ in (3.10).

Theorem 3
The necessary and sufficient solvability conditions of the problem (3.6) *in the class* $E_p^\pm(\Gamma,M^\pm,\rho^\pm)$ *are the conditions* (3.9)*; the index of this problem is equal to the index* $\varkappa$ *of the operator* Ω.

Note that if the matrices $a(t)$ and $b(t)$ are continuous, then the index of any class is given by the formula

$$\varkappa=\frac{1}{\pi}[\arg\det\alpha(t)]_\Gamma.$$

References

[1] G. Akhalaia, G. Manjavidze, *Boundary value problems of the theory of generalized analytic vectors*, Complex methods for partial differential equations, vol. **6**, Kluwer Acad. Publ. Dordecht (1999), 57–97.

[2] N. Manjavidze, *Boundary value problems for analytic and generalized analytic functions on a cut plane*, Mem. Differential Equations Math. Phys. **33** (2004), 121–154.

[3] G. Manjavidze, *Boundary value problems for analytic and generalized functions*, Some Topics on Value Distribution and Differentiability in Complex and P-adic Analysis, vol. **11**, Science Press, Beijing (2008), 499–631.

[4] G. Manjavidze, N. Manjavidze, *Boundary value problems for generalized analytic functions*, J. Math. Sci. vol. **160**, 6, Springer (2009), 745–821.

G. Akhalaia
I. Vekua Institute of Applied Mathematics
of Iv. Javakhishvili Tbilisi State University
Department of Complex Analisys
2, University st.,Tbilisu 0186
Georgia
E-mail: giaakha@gmail.com

N. Manjavidze
Georgian Technical University
Department of Mathematics
77, Kostava st.,Tbilisi 0175
Georgia
E-mail: minomanjavidze@yahoo.com

Received: 30 April 2010; final version: 24 September 2010;
available online: 18 December 2010.