FOLIA 84

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica IX (2010)

*Yufeng Wang, Yanjin Wang*

# On schwarz-type boundary-value problems of polyanalytic equation on a triangle

**Abstract.** We will consider the Schwarz-type boundary-value problems (BVPs) of the polyanalytic equation on an isosceles orthogonal triangle. In contrast to [20], the expression of its unique solution is explicitly obtained by the different decomposition of polyanalytic functions.

## 1. Introduction

Various kinds of BVPs of partial differential equations (PDEs) have been solved by the Riemann-Hilbert technique [13], and BVPs of complex PDEs have been widely discussed by complex analytic methods, see for example [2–12, 14–19, 21]. The Schwarz-type BVP is one of a basic problem, which is closely connected with other type BVPs.

In [21], we have considered the Schwarz-type BVP of the nonhomogeneous Cauchy-Riemann equation on an isosceles orthogonal triangle with three vertices $0, 1, i$. Such an triangle domain is denoted as $\Delta$. The following result is obtained by the technique of plane parqueting used in [7].

Theorem 1.1 ([21])
*The Schwarz-type BVP of the nonhomogeneous Cauchy-Riemann equation*

$$\begin{cases} \partial_{\bar z} w(z) = f(z), & z \in \Delta,\ f \in L_p(\Delta; \mathbb{C}),\ p > 2, \\ [\operatorname{Re} w]^+(t) = \rho(t), & t \in \partial\Delta,\ \rho \in C(\partial\Delta; \mathbb{C}) \end{cases} \tag{1.1}$$

*is solvable and its solution can be represented as*

$$w(z) = S_\alpha[\rho](z) + A_\alpha[f](z) + i \operatorname{Im} w(\alpha), \qquad z \in \Delta, \tag{1.2}$$

AMS (2000) Subject Classification: 30E25, 30G30, 31A25, 45E05.

The first author is supported by NNSF of China (#10871150) and RFDP of Higher Eduction of China (#20060486001). The second author is supported by Tianyuan Fund of Mathematics (#10926188).

Volumes I-VII appeared as *Annales Academiae Paedagogicae Cracoviensis Studia Mathematica*.

*where $\alpha \in \Delta$ is a fixed constant and*

$$\begin{cases} S_\alpha[\rho](z) = \dfrac{1}{\pi i}\displaystyle\int\limits_{\partial\Delta} \rho(\zeta)\sum_{m,n}\Big[g_{m,n}(\zeta,z) - \dfrac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\overline{\alpha})}{2}\Big]\,d\zeta, \\ A_\alpha[f](z) = -\dfrac{1}{\pi}\displaystyle\int\limits_{\Delta}\{f(\zeta)G_\alpha(\zeta,z) - \overline{f(\zeta)}G_\alpha(\overline{\zeta},z)\}\,d\xi\,d\eta, \end{cases} \quad z\in\Delta \quad (1.3)$$

*with*

$$G_\alpha(\zeta,z) = \sum_{m,n}[g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\alpha)] \tag{1.4}$$

*and*

$$g_{m,n}(\zeta,z) = \frac{1}{\zeta - z - 2m - 2ni} + \frac{1}{\zeta + iz - (2m+1) - (2n+1)i} + \frac{1}{\zeta - iz - (2m+1) - (2n-1)i} + \frac{1}{\zeta + z - (2m+2) - 2ni},$$

*where the double series is uniformly convergent along the rectangles with center at the origin.*

The following result is obtained by Heinrich Begehr and Tatyana Vaitekhovich [8].

Theorem 1.2 (Theorem 2 in [8])
*The Schwarz problem*

$$\begin{aligned} &\partial_{\bar z} w = f \text{ in } \mathbb{D}^+, && f\in L_p(\mathbb{D}^+;\mathbb{C}),\ p>2,\\ &\operatorname{Re} w = \gamma \text{ on } \partial\mathbb{D}^+, && \gamma\in C(\partial\mathbb{D}^+;\mathbb{R}),\ \gamma(1)=\gamma(-1)=0,\\ &\frac{1}{\pi}\int_0^\pi \operatorname{Im} w(e^{i\varphi})\,d\varphi = c, && c\in\mathbb{R} \end{aligned}$$

*is uniquely solvable by*

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int\limits_{|\zeta|=1,\,\operatorname{Im}\zeta>0} \gamma(\zeta)\Big(\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\Big)\frac{d\zeta}{\zeta}\\ & + \frac{1}{\pi i}\int_{-1}^{1}\gamma(t)\Big(\frac{1}{t-z} - \frac{z}{1-zt}\Big)\,dt\\ & - \frac{1}{\pi}\int\limits_{\mathbb{D}^+}\Big\{f(\zeta)\Big[\frac{1}{\zeta-z} - \frac{z}{1-z\zeta}\Big] - \overline{f(\zeta)}\Big[\frac{1}{\overline{\zeta}-z} - \frac{z}{1-z\overline{\zeta}}\Big]\Big\}\,d\xi\,d\eta\\ & + ic, \qquad z\in\mathbb{D}^+, \end{aligned} \tag{1.5}$$

*where $\mathbb{D}^+$ is the upper half unit disc.*

The solutions (1.2) and (1.5) consist of three parts: the linear integral, the area integral and the free term. For example, in (1.2), $S_\alpha[\rho](z)$ is the linear integral, $A_\alpha[f](z)$ is the area integral and $i\mathrm{Im}\, w(\alpha)$ is the free term. For the solution (1.5), the free term $ic$ is determined by the function value on the half unit circumference. However, for the solution (1.2), the free term is determined by the function value at one point $\alpha \in \Delta$.

In [10], the Schwarz-type BVP of polyanlytic equation has been solved by the iteration. Basing on Theorem 1.1, we will investigate the Schwarz-type BVP of polyanalytic equation on the triangle domain $\Delta$. Because of the distinction of free terms, the Schwarz-type BVPs of the polyanalytic equation in the triangle $\Delta$ are different from the corresponding BVPs in [20].

The properties of polyanalytic functions have been exposed in [1]. The decomposition of polyanalytic functions or polyharmonic functions plays very important role in solving BVPs of higher order PDEs, see for example [2, 11, 16–18]. In this article, we will make use of the special decomposition of polyanalytic functions to solve the Schwarz-type BVP of the polyanalytic equation in the triangle $\Delta$, and the unique solution is explicitly obtained.

In what follows, $\alpha$ is always a fixed complex constant and $\alpha \in \Delta$, and the Cauchy-Riemann operator is

$$\partial_{\bar{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

## 2. Boundary behavior of the poly-Schwarz operator

Similarly to [20], we introduce the poly-Schwarz operator $S_{\alpha,n}$ on the triangle $\Delta$ as follows

$$\begin{aligned} S_{\alpha,n}&[\rho_0, \rho_1, \ldots, \rho_{n-1}](z) \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \frac{1}{2\pi i} \int\limits_{\partial\Delta} (\zeta - z + \overline{\zeta - z})^k \rho_k(\zeta)[G_\alpha(\zeta, z) + G_{\bar{\alpha}}(\zeta, z)]\, d\zeta, \quad (2.1) \\ &\qquad\qquad z \in \Delta, \end{aligned}$$

where $G_\alpha$ and $G_{\bar{\alpha}}$ are defined by (1.4), and kernel densities $\rho_0, \rho_1, \ldots, \rho_{n-1} \in C(\partial\Delta; \mathbb{R})$. Obviously $S_{\alpha,0} = S_\alpha$, where $S_\alpha$ is the Schwarz-type operator defined in (1.3).

Theorem 2.1
*If $\rho_0, \rho_1, \ldots, \rho_{n-1} \in C(\partial\Delta; \mathbb{R})$, then*

$$\frac{\partial^n S_{\alpha,n}[\rho_0, \rho_1, \ldots, \rho_{n-1}]}{\partial \overline{z}^n}(z) = 0, \qquad z \in \Delta, \tag{2.2}$$

*where the operator $S_{\alpha,n}$ is defined by* (2.1).

*Proof.* By (2.1), one has

$$
\begin{aligned}
S_{\alpha,n}&[\rho_0,\rho_1,\dots,\rho_{n-1}](z)\\
&=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\frac{1}{2\pi i}\int_{\partial\Delta}\sum_{\ell=0}^{k}\binom{k}{\ell}(\zeta+\overline{\zeta})^{\ell}(-z-\overline{z})^{k-\ell}\rho_k(\zeta)[G_\alpha(\zeta,z)+G_{\bar\alpha}(\zeta,z)]\,d\zeta\\
&=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(-z-\overline{z})^{k-\ell}\frac{1}{2\pi i}\int_{\partial\Delta}(\zeta+\overline{\zeta})^{\ell}\rho_k(\zeta)[G_\alpha(\zeta,z)+G_{\bar\alpha}(\zeta,z)]\,d\zeta\\
&=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(-z-\overline{z})^{k-\ell}S_{\alpha,0}[\rho_{k,\ell}](z),\qquad z\in\Delta
\end{aligned}\tag{2.3}
$$

with

$$\rho_{k,\ell}(\zeta)=(\zeta+\overline{\zeta})^{\ell}\rho_k(\zeta).\tag{2.4}$$

Hence

$$
\begin{aligned}
\frac{\partial^n S_{\alpha,n}[\rho_0,\rho_1,\dots,\rho_{n-1}]}{\partial\overline{z}^n}(z)&=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\Big[\frac{\partial^n}{\partial\overline{z}^n}(-z-\overline{z})^{k-\ell}\Big]S_{\alpha,0}[\rho_{k,\ell}](z)\\
&=0.
\end{aligned}
$$

This completes the proof.

Theorem 2.2
*If $\rho_0,\rho_1,\dots,\rho_{n-1}\in C(\partial\Delta;\mathbb{R})$, then*

$$\Big\{\mathrm{Re}\Big[\frac{\partial^{\ell}S_{\alpha,n}[\rho_0,\rho_1,\dots,\rho_{n-1}]}{\partial\overline{z}^{\ell}}\Big]\Big\}^{+}(t)=\rho_\ell(t),\quad t\in\partial\Delta,\ \ell=0,1,\dots,n-1,\tag{2.5}$$

*where the operator $S_{\alpha,n}$ is defined by* (2.1)*.*

*Proof.* When $\ell=0$,

$$
\begin{aligned}
\frac{\partial^{\ell}S_{\alpha,n}[\rho_0,\rho_1,\dots,\rho_{n-1}]}{\partial\overline{z}^{\ell}}(z)&=\frac{\partial^{0}S_{\alpha,n}[\rho_0,\rho_1,\dots,\rho_{n-1}]}{\partial\overline{z}^{0}}(z)\\
&=S_{\alpha,n}[\rho_0,\rho_1,\dots,\rho_{n-1}](z).
\end{aligned}
$$

By Theorem 3.3 in [21] and (2.3), one has

$$
\begin{aligned}
\{\mathrm{Re}[S_{\alpha,n}[\rho_0,\rho_1,\dots,\rho_{n-1}]]\}^{+}(t)&=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(-t-\overline{t})^{k-\ell}\rho_{k,\ell}(t)\\
&=\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}(t+\overline{t}-t-\overline{t})^{k}\rho_k(t)+\rho_0(t)\\
&=\rho_0(t),\qquad t\in\partial\Delta,
\end{aligned}\tag{2.6}
$$

where $\rho_{k,\ell}$ is defined by (2.4). When $\ell=1,2,\dots,n-1$, one easily gets

$$\begin{aligned}
&\frac{\partial^\ell S_{\alpha,n}[\rho_0,\rho_1,\ldots,\rho_{n-1}]}{\partial\overline{z}^\ell}(z)\\
&\quad=\sum_{k=\ell}^{n-1}\frac{(-1)^{k-\ell}}{(k-\ell)!}\frac{1}{2\pi i}\int\limits_{\partial\Delta}(\zeta-z+\overline{\zeta-z})^{k-\ell}\rho_k(\zeta)[G_\alpha(\zeta,z)+G_{\bar\alpha}(\zeta,z)]\,d\zeta \qquad (2.7)\\
&\quad=\sum_{k=\ell}^{n-1}\frac{(-1)^{k-\ell}}{(k-\ell)!}\sum_{j=0}^{k-\ell}\binom{k-\ell}{j}(-z-\overline{z})^{k-\ell-j}S_{\alpha,0}[\rho_{k,j}](z), \qquad z\in\Delta.
\end{aligned}$$

Similarly to the proof of the previous part, (2.5) is valid for $\ell=1,2,\ldots,n-1$.

Remark 2.3
By (2.3) and (2.7), if $\rho_0,\rho_1,\ldots,\rho_{n-1}\in C(\partial\Delta;\mathbb{R})$, then

$$\begin{aligned}
&\frac{\partial^\ell S_{\alpha,n}[\rho_0,\rho_1,\ldots,\rho_{n-1}]}{\partial\overline{z}^\ell}(\alpha)\\
&\quad=\sum_{k=\ell}^{n-1}\frac{(-1)^{k-\ell}}{(k-\ell)!}\sum_{j=0}^{k-\ell}\binom{k-\ell}{j}(-\alpha-\overline{\alpha})^{k-\ell-j}S_{\alpha,0}[\rho_{k,j}](\alpha), \quad \ell=0,1,2,\ldots,n-1
\end{aligned}$$

are real numbers since $S_{\alpha,0}[\rho_{k,\ell}](\alpha)\in\mathbb{R}$ according to [21].

## 3. Pompeiu-type operator on the triangle

In this section, the following area integral operator is introduced as in [20]

$$\begin{aligned}
&A_{\alpha,\ell}[f](z)\\
&\quad=\frac{(-1)^\ell}{\pi(\ell-1)!}\int\limits_{\Delta}(\zeta-z+\overline{\zeta-z})^{\ell-1}[f(\zeta)G_\alpha(\zeta,z)-\overline{f(\zeta)}G_\alpha(\overline{\zeta},z)]\,d\xi\,d\eta, \qquad (3.1)\\
&\qquad z\in\Delta,\ \ell=1,2,\ldots
\end{aligned}$$

with $f\in L_p(\Delta;\mathbb{C})$, $p>2$, where $G_\alpha$ is defined by (1.4). The operator $A_{\alpha,\ell}$ is called the Pompeiu-type operator here. When $\ell=1$, (3.1) is just

$$A_{\alpha,1}[f](z)=A_\alpha[f](z), \qquad z\in\Delta, \tag{3.2}$$

where $A_\alpha$ is defined in (1.3). We assume that $A_{\alpha,0}[f](z)=f(z)$, $z\in\Delta$ in the following. By Theorem 4.1 in [21], $A_{\alpha,1}[f]\in C(\partial\Delta;\mathbb{C})$ and

$$\frac{\partial}{\partial\overline{z}}A_{\alpha,1}[f](z)=A_{\alpha,0}[f](z), \qquad z\in\Delta. \tag{3.3}$$

Theorem 3.1
*If $f\in L_p(\Delta;\mathbb{C})$, $p>2$, then*

$$\frac{\partial}{\partial\overline{z}}A_{\alpha,\ell}[f](z)=A_{\alpha,\ell-1}[f](z), \qquad z\in\Delta,\ \ell=1,2,3,\ldots \tag{3.4}$$

*and*

$$\frac{\partial^n}{\partial \overline{z}^n} A_{\alpha,n}[f](z) = f(z), \qquad z \in \Delta, \tag{3.5}$$

*where $A_{\alpha,\ell}$ is the Pompeiu-type operator defined by* (3.1).

*Proof.* When $\ell = 1$, (3.4) is just (3.3). When $\ell > 1$, one has

$$A_{\alpha,\ell}[f](z) = \frac{(-1)^{\ell-1}}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} (-z-\overline{z})^{\ell-k-1} A_{\alpha,1}[f_k](z), \qquad z \in \Delta \tag{3.6}$$

with

$$f_k(\zeta) = (\zeta + \overline{\zeta})^k f(\zeta), \qquad k = 0, 1, \ldots, \ell-1. \tag{3.7}$$

Thus

$$\begin{aligned}
&\frac{\partial}{\partial \overline{z}} A_{\alpha,\ell}[f](z) \\
&\quad = \frac{(-1)^{\ell-1}}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} \Big\{ \Big( \frac{\partial}{\partial \overline{z}} (-z-\overline{z})^{\ell-k-1} \Big) A_{\alpha,1}[f_k](z) \\
&\qquad\qquad + (-z-\overline{z})^{\ell-k-1} \frac{\partial}{\partial \overline{z}} A_{\alpha,1}[f_k](z) \Big\} \\
&\quad = A_{\alpha,\ell-1}[f](z),
\end{aligned}$$

since $\frac{\partial}{\partial \overline{z}} A_{\alpha,1}[f_k](z) = 0$. Using (3.4) repeatedly, we easily gets (3.5).

Theorem 3.2
*If $f \in L_p(\Delta;\mathbb{C})$, $p > 2$, then $\{\operatorname{Re} A_{\alpha,\ell}[f]\}^+(t) = 0$, $t \in \partial\Delta$ for $\ell = 1, 2, 3, \ldots$, where $A_{\alpha,\ell}$ is defined by* (3.1).

*Proof.* This theorem directly follows from

$$\operatorname{Re}\{A_{\alpha,\ell}[f](z)\} = \frac{(-1)^{\ell-1}}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} (-z-\overline{z})^{\ell-k-1} \operatorname{Re}\{A_{\alpha,1}[f_k](z)\}, \qquad z \in \Delta$$

and

$$\{\operatorname{Re} A_{\alpha,1}[f_k]\}^+(t) = 0, \qquad t \in \partial\Delta,$$

where $f_k$ is defined by (3.7).

Remark 3.3
Since $G_\alpha(\zeta,\alpha) = G_\alpha(\overline{\zeta},\alpha) = 0$, one has

$$A_{\alpha,\ell}[f](\alpha) = 0, \qquad \ell = 1, 2, 3, \ldots.$$

## 4. Schwarz-type BVPs of polyanalytic equation on the triangle

Similarly to [20], by an appropriate decomposition of polyanalytic functions, the following lemma is obtained.

Lemma 4.1
*The homogeneous Schwarz-type BVP of the homogeneous polyanalytic equation*

$$\begin{cases} \partial_{\bar{z}}^n w(z) = 0, & z \in \Delta, \\ [\mathrm{Re}(\partial_{\bar{z}}^k w)]^+(t) = 0, & t \in \partial\Delta,\ k = 0, 1, \dots, n-1 \end{cases} \tag{4.1}$$

*is solvable and its solution can be represented as*

$$w(z) = \sum_{k=0}^{n-1} \frac{(z - \alpha + \overline{z} - \overline{\alpha})^k}{k!} ic_k \tag{4.2}$$

*with* $c_k = \mathrm{Im}(\partial_{\bar{z}}^k w)(\alpha) \in \mathbb{R}$ *for* $k = 0, 1, 2, \dots, n-1$.

*Proof.* The homogeneous polyanalytic equation $(\partial_{\bar{z}}^n w)(z) = 0$, $z \in \Delta$ implies that $w$ is a polyanalytic function. Hence $w$ can be uniquely decomposed as [11]

$$w(z) = \sum_{k=0}^{n-1} \frac{(z - \alpha + \overline{z} - \overline{\alpha})^k}{k!} \varphi_k(z), \qquad z \in \Delta, \tag{4.3}$$

where $\varphi_k$ is analytic on the triangle domain $\Delta$. Substituting (4.3) into the boundary conditions in (4.1), one has the boundary behaviors of analytic functions

$$[\mathrm{Re}\,\varphi_k]^+(t) = 0, \qquad t \in \partial\Delta,\ k = 0, 1, \dots, n-1, \tag{4.4}$$

and hence

$$\varphi_k(z) \equiv ic_k, \qquad c_k \in \mathbb{R},\ k = 0, 1, \dots, n-1$$

by Theorem 1.1. By the direct computation, $c_k = \mathrm{Im}(\partial_{\bar{z}}^k w)(\alpha)$, $k = 0, 1, \dots, n-1$.

In general, we have the following result.

Theorem 4.2
*The Schwarz-type BVP of the polyanalytic equation*

$$\begin{cases} (\partial_{\bar{z}}^n w)(z) = f(z), & z \in \Delta,\ f \in L_p(\Delta; \mathbb{C}),\ p > 2, \\ [\mathrm{Re}(\partial_{\bar{z}}^k w)]^+(t) = \rho_k(t), & t \in \partial\Delta,\ \rho_k \in C(\partial\Delta; \mathbb{R}),\ k = 0, 1, \dots, n-1, \\ \mathrm{Im}(\partial_{\bar{z}}^k w)(\alpha) = a_k, & k = 0, 1, \dots, n-1 \end{cases} \tag{4.5}$$

*is solvable and its unique solution can be written as*

$$\begin{aligned} w(z) = {} & S_{\alpha,n}[\rho_0, \rho_1, \dots, \rho_{n-1}](z) + A_{\alpha,n}[f](z) \\ & + \sum_{k=0}^{n-1} \frac{(z - \alpha + \overline{z} - \overline{\alpha})^k}{k!} ia_k, \qquad z \in \Delta, \end{aligned} \tag{4.6}$$

*where the operators* $S_{\alpha,n}, A_{\alpha,n}$ *are defined by* (2.1) *and* (3.1)*, respectively, and* $a_k$, $k = 0, 1, 2, \dots, n-1$ *are* $n$ *given real constants.*

*Proof.* Firstly, let

$$w_0(z) = S_{\alpha,n}[\rho_0, \rho_1, \dots, \rho_{n-1}](z) + A_{\alpha,n}[f](z), \qquad z \in \Delta,$$

where the operators $S_{\alpha,n}, A_{\alpha,n}$ are defined by (2.1) and (3.1), respectively. By Theorems 2.1 and 3.1,

$$\frac{\partial^n w_0}{\partial \overline{z}^n}(z) = f(z), \qquad z \in \Delta.$$

By Theorems 2.2 and 3.2, one has

$$\Big[\mathrm{Re}\frac{\partial^k w_0}{\partial \overline{z}^k}\Big]^+(t) = \rho_k(t), \qquad t \in \partial\Delta,\ k = 0, 1, \dots, n-1.$$

Hence $w_0$ is a special solution of the Schwarz-type BVP (4.5) of the polyanalytic equation. Secondly, let

$$w(z) = w_0(z) + w_1(z), \qquad z \in \Delta. \tag{4.7}$$

Substituting $w(z)$ defined by (4.7) into (4.5), one easily gets

$$\begin{cases} (\partial_{\overline{z}}^n w_1)(z) = 0, & z \in \Delta, \\ [\mathrm{Re}(\partial_{\overline{z}}^k w_1)]^+(t) = 0, & t \in \partial\Delta,\ k = 0, 1, \dots, n-1, \\ \mathrm{Im}(\partial_{\overline{z}}^k w_1)(\alpha) = a_k, & k = 0, 1, \dots, n-1 \end{cases} \tag{4.8}$$

which is just the homogeneous Schwarz-type BVP (4.1) of the polyanalytic function. The third formula in (4.8) is obtained by Remarks 2.3 and 3.3. Finally, by Lemma 4.1,

$$w_1(z) = \sum_{k=0}^{n-1} \frac{(z - \alpha + \overline{z} - \overline{\alpha})^k}{k!} i c_k$$

with $c_k = \mathrm{Im}(\partial_{\overline{z}}^k w)(\alpha)$ for $k = 0, 1, 2, \dots, n-1$. This completes the proof.

## Acknowledgements

This work is done while the first author is visiting Free University Berlin in Germany under the China State Scholarship Fund from October 2009 to October 2010. The first author is grateful to Professor Heinrich Begehr for his support and hospitality as well as many useful suggestions.

## References

[1] M.B. Balk, *Polyanalytic Functions*, Akademie Verlag, Berlin, 2001.

[2] H. Begehr, Jinyuan Du, Yufeng Wang, *A Dirichlet problem for polyharmonic function*, Ann. Math. Pura Appl. **187** (2008), 435–457.

[3] H. Begehr, G.N. Hile, *A hierarchy of integral operators*, Rocky Mountain J. Math. **27** (1997), 669–706.

[4] H. Begehr, A. Kumar, *Boundary value problem for inhomogeneous polyanalytic equation I*, Analysis, **25** (2005), 55–71.

[5] H. Begehr, *Boundary value problems in complex analysis I*. Bol. Asoc. Math. Venezolana, **12** (2005), 65–85; II. Bol. Asoc. Math. Venezolana, **12** (2005), 217–250.

[6] H. Begehr, *Complex Analytic Methods for Partial Differential Equation*, An introductory text. World Scientific, Singapore, 1994.

[7] H. Begehr, T. Vaitekhovich, *Green functions, reflections, and plane parqueting*, Preprint, FU Berlin, 2009.

[8] H. Begehr, T. Vaitekhovich, *Harmonic boundary value problems in the half disc and half ring*, Funct. Approx. Comment. Math. **40** (2009), 251–282.

[9] H. Begehr, C.J. Vanegas, *Iterated Newmann problem for higher order Poission equation*, Math. Nach. **279** (2006), 38–57.

[10] H. Begehr, D. Schmersau, *The Schwarz problem for polyanalytic functions*, Z. Anal. Anwendungen, **24** (2005), 341–351.

[11] Jinyuan Du, Yufeng Wang, *Riemann boundary value problems of polyanalytic functions and metaanalytic functions on the closed curves*, Complex Var. Theory Appl. **50** (2005), 521–533.

[12] Jinyuan Du, Yufeng Wang, *On boundary value problems of polyanalytic function on the real axis*, Complex Var. Theory Appl. **48** (2003), 527–542.

[13] A.S. Fokas, *A unified transform method for solving linear and certain nonlinear PDEs*, Proc. Roy. Soc. London Ser. A, **453** (1997), 1411–1443.

[14] Jianke Lu, *Boundary Value Problems for Analytic Functions*, World Scientific, Singapore, 1993.

[15] N.I. Muskhelishvili, *Singular Integral Equations*, 2nd ed., Noordhoff, Groningen, 1968.

[16] Yufeng Wang, Jinyuan Du, *Haseman boundary value problem of bianalytic functions with different shifts on the unit circumference*, J. Appl. Funct. Anal. **2** (2007), 147–158.

[17] Yufeng Wang, Jinyuan Du, *Hilbert boundary value problems of polyanalytic functions on the unit circumference*, Complex Var. Elliptic Equ. **51** (2006), 923–943.

[18] Yufeng Wang, Jinyuan Du, *On Riemann boundary value problem of polyanalytic function on the real axis*, Acta Math. Sci. Ser. B Engl. Ed. **24** (2004), 663–671.

[19] Yufeng Wang, *On modified Hilbert boundary-value problems of polyanalytic functions*, Math. Methods Appl. Sci. **32** (2009), 1415–1427.

[20] Yufeng Wang, *Schwarz-type boundary value problems of polyanalytic function on the half unit disc*, to appear.

[21] Yufeng Wang, Yanjin Wang, *Schwarz-type problem of nonhomogeneous Cauchy-Riemann equation on a triangle*, to appear.

*Yufeng Wang*
*School of Mathematics and Statistics, Wuhan University,*
*Wuhan* 430072,
*P.R.China*
*E-mail:* wh_yfwang@hotmail.com

*Yanjin Wang*
*Institute of Applied Physics and Computational Mathematics*
*P.O. Box 8009-15, Beijing, 100088,*
*P.R.China*
*E-mail:* wang_jasonyj2002@yahoo.com

*Received: 30 April 2010; final version: 9 June 2010;*
*available online: 23 July 2010.*