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Boundary value problems for a second–order elliptic equation in unbounded domains

Abstract. We investigate the behaviour of weak solutions to the boundary value problems for the second order elliptic linear equation in a neighborhood of infinity. The exponent of the decreasing rate of solutions at infinity has been exactly calculated.

1. Introduction

Let $B_1(\mathcal{O})$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$ with center at the origin $\mathcal{O}$ and $G \subset \mathbb{R}^n \setminus B_1(\mathcal{O})$ be an unbounded cone – like domain with the smooth boundary ∂G. We consider the following elliptic boundary value problem

$$\begin{cases} \mathcal{L}[u] \equiv \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}) + b^i(x)u_{x_i} + c(x)u = f(x), & x \in G, \\ \mathcal{B}[u] \equiv \alpha(x)\dfrac{\partial u}{\partial \nu} + \dfrac{1}{|x|}\gamma\left(\dfrac{x}{|x|}\right)u = g(x), & x \in \partial G, \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases} \qquad (L)$$

(summation over repeated indices from 1 to n is understood); here:

$$\alpha(x) = \begin{cases} 0, & \text{if } x \in \mathcal{D}, \\ 1, & \text{if } x \notin \mathcal{D} \end{cases}$$

and $\mathcal{D} \subseteq \partial G$ is the part of the boundary ∂G, where the Dirichlet boundary condition is posed; $\frac{\partial}{\partial \nu} = a^{ij}(x)\cos(\overrightarrow{n}, x_i)\frac{\partial}{\partial x_j}$, where $\overrightarrow{n}$ denotes the unit outward with respect to G normal to ∂G. Thus, if $\mathcal{D} = \partial G$ then we have the Dirichlet problem, if $\mathcal{D} = \emptyset$ then $\alpha(x) = 1$ and we have the Robin problem and if $\mathcal{D} \subset \partial G$ then we have the mixed problem.

A few mathematicians have considered boundary value problems for linear elliptic equations in unbounded domains (see for example [12], [13], [14]). Such problems have applications to mechanics of inhomogeneous media [7].

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The main advantage of this article is the best estimation of solutions for minimal smooth coefficients in n–dimensional domains. The theoretical results are confirmed by examples.

We introduce the following notations:

- S^{n-1}: unit sphere in $\mathbb{R}^n$;
- (r,ω), $\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$: spherical coordinates of $x \in \mathbb{R}^n$:

$$\begin{aligned} x_1 &= r\cos\omega_1, \\ x_2 &= r\cos\omega_2\sin\omega_1, \\ &\vdots \\ x_{n-1} &= r\cos\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1, \\ x_n &= r\sin\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1. \end{aligned}$$

- $\mathcal{C}$: rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$;
- $\partial\mathcal{C}$: lateral surface of $\mathcal{C}: \{x_1 = r\cos\frac{\omega_0}{2}\}$;
- Ω: a domain on the unit sphere S^{n-1} with a smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathcal{C}$ with the sphere S^{n-1};
- $\partial\Omega = \partial\mathcal{C} \cap S^{n-1}$;
- $G_a^b = \{(r,\omega)|\ a < r < b;\ \omega \in \Omega\} \cap G$: layer in $\mathbb{R}^n$;
- $\Gamma_a^b = \{(r,\omega)|\ a < r < b;\ \omega \in \partial\Omega\} \cap \partial G$: lateral surface of layer G_a^b;
- $G_d = G_d^\infty$; $\Gamma_d = \Gamma_d^\infty$, $d \gg 1$;
- $\Omega_d = G \cap \{|x| = d\}$; $d \gg 1$.

We use the standard function spaces: $C^k(\overline{G})$ with the norm $|u|_{k,G}$, the Lebesgue space $L_p(G)$, $p \geq 1$ with the norm $\|u\|_{p,G}$, the Sobolev space $W^{k,p}(G)$ with the norm $\|u\|_{p,k;G}$. We define the weighted Sobolev spaces $V_{p,\alpha}^k(G)$ for an integer $k \geq 0$ and a real α as the closure of $C_0^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{V_{p,\alpha}^k(G)} = \left(\int\limits_G \sum_{|\beta|=0}^k r^{\alpha+p(|\beta|-k)}|D^\beta u|^p\,dx\right)^{\frac{1}{p}}$$

and $V_{p,\alpha}^{k-\frac{1}{p}}(\partial G)$ as the space of traces of functions φ, given on ∂G, with the norm

$$\|\varphi\|_{V_{p,\alpha}^{k-\frac{1}{p}}(\partial G)} = \inf\|\Phi\|_{V_{p,\alpha}^k(G)},$$

where the infimum is taken over all functions Φ such that $\Phi|_{\partial G} = \varphi$ in the sense of traces. $W_{\mathrm{loc}}^{k,p}(G)$ is a local space which consists of functions belonging to $W^{k,p}(G')$ for all $G' \subset G$. We denote

$$W^k(G) \equiv W^{k,2}(G), \qquad W_{\mathrm{loc}}^k(G) \equiv W_{\mathrm{loc}}^{k,2}(G),$$
$$\overset{\circ}{W}{}_\alpha^k(G) \equiv V_{2,\alpha}^k(G), \qquad \overset{\circ}{W}{}_\alpha^{k-\frac{1}{2}}(\partial G) \equiv V_{2,\alpha}^{k-\frac{1}{2}}(\partial G).$$

Definition 1.1
Function $u(x)$ is called a *weak* solution of the problem (L) provided that $u(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}^1_0(G)$ and satisfies the integral identity

$$\int\limits_G \{a^{ij}(x)u_{x_j}\eta_{xi} - b^i(x)u_{x_i}\eta(x) - c(x)u\eta(x)\}\,dx + \int\limits_G f(x)\eta(x)\,dx + \int\limits_{\partial G} \alpha(x)\Big\{\frac{1}{r}\gamma(\omega)u(x) - g(x)\Big\}\eta(x)\,ds = 0 \qquad (II)$$

for all functions $\eta(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}^1_0(G)$ such that $\lim_{|x|\to\infty}\eta(x) = 0$.

Remark 1.2
In the Dirichlet boundary condition case we assume, without loss of generality, that

$$g\big|_{\partial G\cap D} = 0 \implies u\big|_{\partial G\cap D} = 0.$$

Let $M_0 = \max_{x\in\overline{G}}|u(x)|$ be known. It is also assumed that there exists $d \gg 1$ such that Γ_d is a conic surface, i.e.,

$$\Gamma_d = \Big\{(r,\omega)|\ x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^n x_i^2;\ r \in (d,\infty),\ |\omega_1| = \frac{\omega_0}{2},\ \omega_0 \in (0,2\pi)\Big\}.$$

Lemma 1.3
Let $u(x)$ be a weak solution of (L). For any function $\eta(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}^1_0(G)$ with $\lim_{|x|\to\infty}\eta(x) = 0$ equality

$$\begin{aligned}\int\limits_{G_R} \{a^{ij}(x)u_{x_j}\eta_{x_i} &+ (f(x) - b^i(x)u_{x_i} - c(x)u)\eta(x)\}\,dx \\ &= -\int\limits_{\Omega_R} a^{ij}(x)u_{x_j}\eta(x)\cos(r,x_i)\,d\Omega_R \\ &\quad + \int\limits_{\Gamma_R} \alpha(x)\Big(g(x) - \frac{1}{r}\gamma(\omega)u(x)\Big)\eta(x)\,ds\end{aligned} \qquad (II)_{loc}$$

holds for a.e. $R \in (d,\infty)$, $d \gg 1$.

Proof. Let $\chi_R(x)$ be the characteristic function of the set G_R. We consider the integral identity (II), replacing $\eta(x)$ by $\eta(x)\chi_R(x)$. As a result we obtain

$$\begin{aligned}\int\limits_{G_R} &\{a^{ij}(x)u_{x_j}\eta_{x_i} + (f(x) - b^i(x)u_{x_i} - c(x)u)\eta(x)\}\,dx \\ &= -\int\limits_{G_R} a^{ij}(x)u_{x_j}\eta(x)\chi_{x_i}\,dx + \int\limits_{\Gamma_R} \alpha(x)\Big(g(x) - \frac{1}{r}\gamma(\omega)u(x)\Big)\eta(x)\,ds.\end{aligned}$$

Let $\delta(r-R)$ be the Dirac distribution lumped on the sphere $r=R$. Using formula (7') of subsection 3 §1 chapt. 3 [10]

$$\chi_{x_i} = \frac{x_i}{r}\delta(r-R),$$

we get (see Example 4 of subsection §1 chapt. 3 [10])

$$\begin{aligned}-\int\limits_{G_R} a^{ij}(x)u_{x_j}\eta(x)\chi_{x_i}\,dx &= -\int\limits_{G_R} a^{ij}(x)u_{x_j}\eta(x)\frac{x_i}{r}\delta(r-R)\,dx\\ &= -\int\limits_{\Omega_R} a^{ij}(x)u_{x_j}\eta(x)\cos(r,x_i)\,d\Omega_R.\end{aligned}$$

The lemma is proved.

Let following **conditions** be fulfilled:

(a) *condition of the uniform ellipticity:*

$$\begin{aligned}&\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2 \qquad \forall x\in\overline{G},\ \forall\xi\in\mathbb{R}^n;\\ &\nu,\mu = const > 0 \qquad \text{and} \qquad \lim_{|x|\to\infty} a^{ij}(x) = \delta_i^j,\end{aligned}$$

where δ_i^j *is the Kronecker symbol;*

(b) *inequalities*

$$\sqrt{\sum_{i,j=1}^n |a^{ij}(x)-\delta_i^j|^2} \le \mathcal{A}\left(\frac{1}{|x|}\right),\quad |x|\left(\sum_{i=1}^n |b^i(x)|^2\right)^{\frac12} + |x|^2|c(x)| \le \mathcal{A}\left(\frac{1}{|x|}\right)$$

hold for $a^{ij}(x)\in C^0(\overline{G})$, $b^i(x)\in L_p(G)$, $c(x), f(x)\in L_{\frac{p}{2}}(G)\cap L_2(G)$; $p>n$, $x\in G_d$, *where* $\mathcal{A}(t)$, $t\ge 0$ *is a monotonically increasing, nonnegative function, continuous at zero and* $\lim_{r\to\infty}\mathcal{A}(\frac1r)=0$;

(c) $c(x)\le 0$ *in* G; $\gamma(\omega)$ *is a positive bounded piecewise smooth function on* $\partial\Omega$ *such that* $\gamma(\omega)\ge\gamma_0>0$;

(d) *there exist numbers* $f_1\ge 0$, $g_1\ge 0$, $s>0$ *such that*

$$|f(x)|\le f_1|x|^{-s-2}, \qquad |g(x)|\le g_1|x|^{-s-1}.$$

Our main result is the following theorem. Let

$$\lambda_- = \frac{2-n-\sqrt{(n-2)^2+4\vartheta}}{2}, \tag{1.1}$$

where ϑ is the smallest positive eigenvalue of the problem (EVP) (see section 2.2).

Theorem 1.4

Let u be a weak solution of the problem (L). If assumptions (a)–(d) *are satisfied with $\mathcal{A}(t)$ Dini–continuous at zero, then there are $d \gg 1$ and a constant $C > 0$ depending only on n, s, λ_-, M_0, f_1, g_1, γ_0, ν, μ, t, p, $\int_0^1 \frac{\mathcal{A}(t)}{t}\,dt$, $\|\sum_{i=1}^n |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}$ such that for all $x \in G_d$*

$$|u(x)| \le C_0\left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}}g_1\right)\cdot\begin{cases}|x|^{\lambda_-}, & \text{if } s > -\lambda_-;\\ |x|^{\lambda_-}\ln|x|, & \text{if } s = -\lambda_-;\\ |x|^{-s}, & \text{if } 0 < s < -\lambda_-.\end{cases} \tag{1.2}$$

Suppose, in addition, that

$$a^{ij}(x) \in C^1(G),\ \gamma(\omega) \in C^1(\partial G),\ f(x) \in V^0_{p,2p-n}(G),\ g(x) \in V^{1-\frac{1}{p}}_{p,2p-n}(\partial G);\ p > n$$

and there is a number

$$\tau_s := \sup_{R<\infty} R^s\|g\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^R_{R/2})}. \tag{1.3}$$

Then for all $x \in G_d$

$$|\nabla u(x)| \le C_1\left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}}g_1 + \tau_s\right)\cdot\begin{cases}|x|^{\lambda_- -1}, & \text{if } s > -\lambda_-;\\ |x|^{\lambda_- -1}\ln|x|, & \text{if } s = -\lambda_-;\\ |x|^{-s-1}, & \text{if } 0 < s < -\lambda_-.\end{cases} \tag{1.4}$$

Furthermore, if $u \in V^2_{p,2p-n}(G)$, then

$$\|u\|_{V^2_{p,2p-n}(G_R)} \le C_2\left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}}g_1 + \tau_s\right)\cdot\begin{cases}R^{\lambda_-}, & \text{if } s > -\lambda_-;\\ R^{\lambda_-}\ln R, & \text{if } s = -\lambda_-;\\ R^{-s}, & \text{if } 0 < s < -\lambda_-.\end{cases} \tag{1.5}$$

2. Preliminaries

2.1. Auxiliary formulae

Let us recall some well known formulae related to the spherical coordinates $(r, \omega_1, \dots, \omega_{n-1})$:

- $dx = r^{n-1}drd\Omega$,
- $d\Omega_R = R^{n-1}d\Omega$,
- $d\Omega = J(\omega)d\omega$ denotes the $(n-1)$–dimensional area element of the unit sphere,
- $J(\omega) = \sin^{n-2}\omega_1 \sin^{n-3}\omega_2 \dots \sin\omega_{n-2}$,

- $d\omega = d\omega_1 \dots d\omega_{n-1}$;
- ds denotes the $(n-1)$–dimensional area element on ∂G;
- $d\sigma$ denotes the $(n-2)$–dimensional area element on $\partial\Omega$;
- $ds = r^{n-2} dr d\sigma$;
- $|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}|\nabla_\omega u|^2$, where $|\nabla_\omega u|$ is the projection of the vector ∇u onto the tangent plane to the unit sphere at the point ω,
- $|\nabla_\omega u|^2 = \sum_{i=1}^{n-1} \frac{1}{q_i}(\frac{\partial u}{\partial \omega_i})^2$, where $q_1 = 1$, $q_i = (\sin\omega_1 \dots \sin\omega_{i-1})^2$, $i \geq 2$,
- $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_\omega u$,
- $\Delta_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \omega_i}(\frac{J(\omega)}{q_i}\frac{\partial u}{\partial \omega_i})$, the Beltrami–Laplace operator.

$C = C(\dots)$, $c = c(\dots)$ denote the constants depending only on the quantities appearing in parentheses. In what follows, the same letters C, c will (generally) be used to denote different constants depending on the same set of arguments.

By means of the direct calculation we obtain

Lemma 2.1

$$x_i \cos(\vec{n}, x_i)|_{\Gamma_d} = 0, \qquad d \gg 1. \tag{2.1}$$

2.2. Auxiliary inequalities

We need some statements and inequalities.

The eigenvalue problem: Let $\Omega \subset S^{n-1}$ with a smooth boundary $\partial\Omega$ be the intersection of the cone $\mathcal{C}$ with the unit sphere S^{n-1}. Let $\overrightarrow{\nu}$ be the exterior normal to $\partial\mathcal{C}$ at points of $\partial\Omega$. Let $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function. We consider the eigenvalue problem for the Laplace–Beltrami operator $\triangle_\omega$ on the unit sphere

$$\begin{cases} \triangle_\omega \psi + \vartheta\psi = 0, & \omega \in \Omega; \\ \alpha(\omega)\dfrac{\partial\psi}{\partial\overrightarrow{\nu}} + \gamma(\omega)\psi(\omega) = 0, & \omega \in \partial\Omega, \end{cases} \tag{EVP}$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has a non-zero weak solutions (eigenfunctions).

Definition 2.2

Function ψ is called a *weak* solution of the problem (EVP) provided that $\psi \in C^0(\overline{\Omega}) \cap W^1(\Omega)$ and satisfies the integral identity

$$\int_\Omega \left\{ \frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} - \vartheta\psi\eta \right\} d\Omega + \int_{\partial\Omega} \alpha(\omega)\gamma(\omega)\psi\eta \, d\sigma = 0$$

for all $\eta(x) \in C^0(\overline{\Omega}) \cap W^1(\Omega)$.

Remark 2.3
We observe that $\vartheta = 0$ is **not** an eigenvalue of (EVP). In fact, setting $\eta = \psi$ and $\vartheta = 0$ we have

$$\int\limits_{\Omega} |\nabla_\omega \psi|^2 \, d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)|\psi|^2 \, d\sigma = 0 \implies \psi \equiv 0,$$

since $\gamma(\omega) > 0$, if $\alpha(\omega) = 1$, and $\psi|_{\partial\Omega} = 0$, if $\alpha(\omega) = 0$.

Now, let us introduce the following functionals on $C^0(\overline{\Omega}) \cap W^1(\Omega)$:

$$F[\psi] = \int\limits_{\Omega} |\nabla_\omega \psi|^2 \, d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)|\psi|^2 \, d\sigma,$$

$$G[\psi] = \int\limits_{\Omega} \psi^2 \, d\Omega,$$

$$H[\psi] = \int\limits_{\Omega} \left\langle |\nabla_\omega \psi|^2 - \vartheta\psi^2 \right\rangle d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)|\psi|^2 \, d\sigma$$

and the corresponding bilinear forms

$$\mathcal{F}(\psi, \eta) = \int\limits_{\Omega} \frac{1}{q_i} \frac{\partial\psi}{\partial\omega_i} \frac{\partial\eta}{\partial\omega_i} \, d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)\psi\eta \, d\sigma, \qquad \mathcal{G}(\psi, \eta) = \int\limits_{\Omega} \psi\eta \, d\Omega.$$

We introduce also the set $K = \{\psi \in W^1(\Omega) | \; G[\psi] = 1\}$. Since $K \subset W^1(\Omega)$, $F[\psi]$ is bounded from below for $\psi \in K$. We denote the greatest lower bound of $F[\psi]$ for this family by ϑ: $\inf_{\psi \in K} F[\psi] = \vartheta$.

Theorem 2.4 (Theorem of subsection 4, Section 2.5, p.123 [11])
Let $\Omega \subset S^{n-1}$ be a bounded domain with a smooth boundary $\partial\Omega$. Let $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function. There exist $\vartheta > 0$ and a function $\psi \in K$ such that

$$\mathcal{F}(\psi, \eta) - \vartheta\mathcal{G}(\psi, \eta) = 0 \qquad \textit{for arbitrary } \eta \in W^1(\Omega).$$

In particular $F[\psi] = \vartheta$. In addition, on Ω, ψ has continuous derivatives of arbitrary order, satisfies the equation $\triangle_\omega \psi + \vartheta\psi = 0$, $\omega \in \Omega$ as well as the boundary conditions of (EVP) in the weak sense (for details see the Remark on pp. 121–122 [11]).

Next from the variational principle we obtain *the Friedrichs–Wirtinger type inequality*:

Theorem 2.5
Let ϑ be the smallest positive eigenvalue of the problem (EVP) (it exists according to Theorem 2.4). Let $\Omega \subset S^{n-1}$ and $\psi \in W^1(\Omega)$ satisfy the boundary condition

of (EVP) in the weak sense. Let $\gamma(\omega) \in C^0(\partial\Omega)$ be a positive bounded piecewise smooth function. Then

$$\vartheta \int\limits_{\Omega} \psi^2(\omega)\, d\Omega \le \int\limits_{\Omega} |\nabla_\omega \psi(\omega)|^2\, d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)\psi^2(\omega)\, d\sigma \tag{2.2}$$

with sharp constant ϑ.

Proof. By approximation arguments, it is clearly sufficient to consider the above described functionals $F[\psi]$, $G[\psi]$, $H[\psi]$ on $C^0(\overline{\Omega}) \cap W^1(\Omega)$. We will find the minimum of the functional $F[\psi]$ on the set K. For this we investigate the minimization of the functional $H[\psi]$ on all functions $\psi(\omega)$, for which the integrals exist and which satisfy the boundary condition from (EVP) in the weak sense. We use formally the Lagrange multipliers and get the Euler equation from the condition $\delta H[\psi] = 0$. By the calculation of the first variation δH, we have

$$\begin{aligned}
\delta H[\psi] &= \delta\Bigg(\int\limits_{\Omega} \Big\{ \sum_{i=1}^{N-1} \frac{1}{q_i}\Big(\frac{\partial\psi}{\partial\omega_i}\Big)^2 - \vartheta\psi^2 \Big\}\, d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)\psi^2\, d\sigma \Bigg) \\
&= -2\int\limits_{\Omega} \sum_{i=1}^{N-1} \frac{\partial}{\partial\omega_i}\Big(\frac{J(\omega)}{q_i}\frac{\partial\psi}{\partial\omega_i}\Big)\cdot\delta\psi\, d\omega - 2\vartheta\int\limits_{\Omega} \psi\cdot\delta\psi\, d\Omega \\
&\quad + 2\int\limits_{\partial\Omega} \frac{\partial\psi}{\partial\overrightarrow{\nu}}\cdot\delta\psi\, d\sigma + 2\int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)\psi\cdot\delta\psi\, d\sigma \\
&= -2\int\limits_{\Omega} (\triangle_\omega\psi + \vartheta\psi)\cdot\delta\psi\, d\Omega + 2\int\limits_{\partial\Omega} \Big\{\frac{\partial\psi}{\partial\overrightarrow{\nu}} + \alpha(\omega)\gamma(\omega)\psi\Big\}\cdot\delta\psi\, d\sigma.
\end{aligned}$$

Hence, because of $\delta H[\psi] = 0$ for all $\delta\psi \in C^0(\overline{\Omega}) \cap W^1(\Omega)$, it follows the eigenvalue problem (EVP). Conversely, let ϑ, $\psi(\omega)$ be a weak solution of the eigenvalue problem (EVP). From the definition of the weak eigenfunction with $\eta = \psi(\omega)$ we have

$$0 = F[\psi] - \vartheta G[\psi] \stackrel{(\text{by} K)}{=} F[\psi] - \vartheta \implies \vartheta = F[u].$$

Hence, the required minimum is the least eigenvalue of the eigenvalue problem (EVP). The existence of a function $\psi \in K$ such that $F[\psi] \le F[v]$ for all $v \in K$ follows from Theorem 2.4.

By definition (1.1) the Friedrichs–Wirtinger inequality may be written now in the following form

$$\lambda_-(\lambda_- + n - 2)\int\limits_{\Omega} \psi^2(\omega)\, d\Omega \le \int\limits_{\Omega} |\nabla_\omega\psi|^2\, d\Omega + \int\limits_{\partial\Omega} \alpha(\omega)\gamma(\omega)\psi^2(\omega)\, d\sigma, \tag{2.3}$$

for all $\psi(\omega) \in W^1(\Omega)$ satisfying the boundary condition of (EVP), $\gamma(\omega) \in C^0(\partial\Omega)$, $\gamma(\omega) \ge \gamma_0$.

THEOREM 2.6 (**Hardy inequality** (THEOREM 330 [4]))
Let $p > 1$, $\alpha > p - 1$. Then

$$\int_d^\infty r^{\alpha-p}|u(r)|^p\,dr \le \frac{p^p}{|\alpha+1-p|^p}\int_d^\infty r^\alpha\left|\frac{\partial u}{\partial r}\right|^p dr \tag{2.4}$$

for any function $u(r)$ absolutely continuous in $[d,\infty)$ and vanishing at infinity. The constant is the best possible.

COROLLARY 2.7
If $u \in \overset{\circ}{W}{}^1_{\alpha-2}(G_R)$, $\alpha > 4 - n$ and $\lim_{|x|\to\infty} u(x) = 0$, then

$$\int_{G_d} r^{\alpha-4}u^2(x)\,dx \le \frac{4}{(4-N-\alpha)^2}\int_{G_d} r^{\alpha-2}|\nabla u(x)|^2\,dx. \tag{2.5}$$

Proof. Replace α by $\alpha+n-3$ and put $p = 2$ in (2.4). Integration of the result over Ω yields inequality (2.5).

THEOREM 2.8 (**Hardy–Friedrichs–Wirtinger type inequality)**
Let $u \in C^0(\overline{G}) \cap \overset{\circ}{W}{}^1_{\alpha-2}(G)$ and $u(\cdot,\omega)$ satisfy the boundary condition from (EVP) in the weak sense. Let λ_- be as above in (1.1) and $\gamma(\omega)$, $\omega \in \partial\Omega$ be a non-negative bounded piecewise smooth function. Then

$$\int_{G_d} r^{\alpha-4}u^2\,dx \le H(\lambda_-,n,\alpha)\left\{\int_{G_d} r^{\alpha-2}|\nabla u|^2\,dx + \int_{\Gamma_d} r^{\alpha-3}\alpha(x)\gamma(\omega)u^2(x)ds\right\},$$
$$H(\lambda_-,n,\alpha) = \left[\left(\frac{\alpha+n-4}{2}\right)^2 + \lambda_-(\lambda_-+n-2)\right]^{-1}, \qquad \alpha \ge 4-n \tag{2.6}$$

provided that integrals on the right hand side are finite.

Proof. Multiplying the inequality (2.3) by $r^{n-5+\alpha}$ and integrating over $r \in (d,\infty)$ we obtain for any α,

$$\int_{G_d} r^{\alpha-4}u^2\,dx \le \frac{1}{\lambda_-(\lambda_-+n-2)}\left\{\int_{G_d} r^{\alpha-2}\frac{1}{r^2}|\nabla_\omega u|^2\,dx + \int_{\Gamma_d} r^{\alpha-3}\gamma(\omega)u^2(x)\alpha(x)\,ds\right\}. \tag{2.7}$$

Inequality (2.6) follows from (2.7) with $\alpha = 4-n$. Now, let $\alpha > 4-n$. We denote $\lim_{|x|\to\infty} u(x) = A$ and show that $A = 0$. In fact, the representation $A = u(x) - (u(x) - A)$, by the Cauchy inequality, yields $2Au(x) \le \varepsilon A^2 + \frac{1}{\varepsilon}u^2$ for all $\varepsilon > 0$ and therefore $u^2(x)+|u(x)-A|^2 = 2u^2(x)-2Au(x)+A^2 \ge (1-\varepsilon)A^2+(2-\frac{1}{\varepsilon})u^2(x) = \frac{1}{2}A^2$, if we choose $\varepsilon = \frac{1}{2}$. Thus we obtain $\frac{1}{2}A^2 \le |u(x)|^2 + |u(x)-A|^2$. Multiplying this inequality by $r^{\alpha-4}$ and integrating it over G_d we obtain

$$\frac{1}{2}A^2\int_{G_d} r^{\alpha-4}\,dx \le \int_{G_d} r^{\alpha-4}u^2(x)\,dx + \int_{G_d} r^{\alpha-4}v^2(x)\,dx, \tag{2.8}$$

where $v(x) = u(x) - A$, hence $\lim_{|x|\to\infty} v(x) = 0$. By (2.7) and from our assumption, the first integral on the right hand side of (2.8) is finite and the second integral from on the right hand side is finite in virtue of Corollary 2.7. Therefore the right hand side of (2.8) is finite, at the same time the integral in the left hand side

$$\int\limits_{G_d} r^{\alpha-4}\,dx = \operatorname{meas}\Omega \int\limits_{d}^{\infty} r^{\alpha+n-5}\,dr = \infty,$$

because of $\alpha > 4-n$. The assumption $A \neq 0$ contradicts to (2.8). Thus $A = 0$. Therefore we can use the Corollary 2.7. Adding inequalities (2.7), (2.5) we get the desired relation (2.6).

Lemma 2.9

Let G_d be the conical domain and $\nabla u(R,\cdot) \in L_2(\Omega)$ for a.e. $R \in (d,\infty)$. Let

$$U(R) = \int\limits_{G_R} r^{2-n}|\nabla u|^2\,dx + \int\limits_{\Gamma_R} r^{1-n}\gamma(\omega)u^2(x)\alpha(x)\,ds < \infty. \tag{2.9}$$

Then

$$\int\limits_{\Omega}\left(Ru\frac{\partial u}{\partial r} + \frac{n-2}{2}u^2\right)\bigg|_{r=R} d\Omega \geq -\frac{R}{2\lambda_-}U'(R), \tag{2.10}$$

where λ_- is defined by (1.1).

Proof. Writing $U(R)$ in spherical coordinates

$$U(R) = \int\limits_{R}^{\infty} r\int\limits_{\Omega}\left(u_r^2 + \frac{1}{r^2}|\nabla_\omega u|^2\right)d\Omega\,dr + \int\limits_{R}^{\infty}\frac{1}{r}\bigg(\int\limits_{\partial\Omega}\alpha(x)\gamma(\omega)|u|^2\,d\sigma\bigg)\,dr$$

and differentiating with respect to R we obtain

$$U'(R) = -\int\limits_{\Omega}\left(R\Big(\frac{\partial u}{\partial r}\Big)^2 + \frac{1}{R}|\nabla_\omega u|^2\right)\bigg|_{r=R} d\Omega - \frac{1}{R}\int\limits_{\partial\Omega}\alpha(R,\omega)\gamma(\omega)u^2(R,\omega)\,d\sigma. \tag{2.11}$$

Moreover, by the Cauchy inequality, we have $Ru\frac{\partial u}{\partial r} \geq -\frac{\varepsilon}{2}u^2 - \frac{1}{2\varepsilon}R^2(\frac{\partial u}{\partial r})^2$ for all $\varepsilon > 0$. Then

$$\int\limits_{\Omega}\left(Ru\frac{\partial u}{\partial r} + \frac{n-2}{2}u^2\right)\bigg|_{r=R} d\Omega \geq \frac{n-2-\varepsilon}{2}\int\limits_{\Omega}u^2\bigg|_{r=R} d\Omega - \frac{R^2}{2\varepsilon}\int\limits_{\Omega}\Big(\frac{\partial u}{\partial r}\Big)^2\bigg|_{r=R} d\Omega.$$

Now we take into account that $\lambda_- < 0$ and $(\lambda_- + n - 2) < 0$. Then choosing $\varepsilon = -\lambda_-$ we obtain, by the Friedrichs–Wirtinger inequality (2.3),

$$\int\limits_{\Omega}\left(Ru\frac{\partial u}{\partial r} + \frac{n-2}{2}u^2\right)\bigg|_{r=R} d\Omega$$

$$\geq \frac{n-2-\varepsilon}{2\lambda_-(\lambda_-+n-2)}\int_\Omega |\nabla_\omega u|^2\, d\Omega - \frac{R^2}{2\varepsilon}\int_\Omega \left(\frac{\partial u}{\partial r}\right)^2 d\Omega$$
$$+\frac{n-2-\varepsilon}{2\lambda_-(\lambda_-+n-2)}\int_{\partial\Omega} \alpha(x)\gamma(\omega)u^2(\varrho,\omega)\, d\sigma$$
$$= -\frac{R}{2\lambda_-}U'(R).$$

We need also the well known inequality:

$$\int_{\partial G} v^2\, ds \leq \int_G \left(\delta|\nabla v|^2 + \frac{1}{\delta}c_0 v^2\right) dx, \qquad \forall v(x)\in W^{1,2}(G),\ \forall\delta>0; \tag{2.12}$$

and the Sobolev inequality (see (2.19), §2, chapt. II [6])

$$\|u\|^2_{L_{\frac{2p}{p-2}}(G)} \leq \varepsilon\|\nabla u\|^2_{L_2(G)} + c_\varepsilon(p,n,G)\|u\|^2_{L_2(G)}, \qquad p>n,\ \forall\varepsilon>0. \tag{2.13}$$

2.3. Cauchy problem for differential inequality

THEOREM 2.10

Let $U(R)$ be a monotonically decreasing, nonnegative differentiable function defined on $[d,\infty)$, $d\gg 1$ and satisfy the problem

$$\begin{cases} U'(R)+\mathcal{P}(R)U(R)-\mathcal{Q}(R)\leq 0, & R>d,\\ U(d)\leq U_0, \end{cases} \tag{CP}$$

where $\mathcal{P}(R)$, $\mathcal{Q}(R)$ are nonnegative continuous functions defined on $[d,\infty)$ and U_0 is a constant. Then

$$U(R)\leq U_0\exp\left(-\int_d^R \mathcal{P}(s)\, ds\right) + \int_d^R \mathcal{Q}(t)\exp\left(-\int_t^R \mathcal{P}(s)\, ds\right) dt. \tag{2.14}$$

Proof. Multiplying the differential inequality (CP) by the integrating factor $\exp(\int_d^t P(s)\, ds)$ and integrating the result over t from d to R we get

$$\int_d^R U'(t)\exp\left(\int_d^t P(s)\, ds\right) dt + \int_d^R P(t)U(t)\exp\left(\int_d^t P(s)\, ds\right) dt$$
$$-\int_d^R Q(t)\exp\left(\int_d^t P(s)\, ds\right) dt \leq 0.$$

Integration by parts in the first term yields

$$U(t)\exp\left(\int_d^t P(s)\, ds\right)\Big|_{t=d}^{t=R} - \int_d^R U(t)\exp\left(\int_d^t P(s)\, ds\right)P(t)\, dt$$

$$+\int_d^R P(t)U(t)\exp\Big(\int_d^t P(s)\,ds\Big)\,dt - \int_d^R Q(t)\exp\Big(\int_d^t P(s)\,ds\Big)\,dt \le 0.$$

Hence, by the assumption $U(d) \le U_0$, we finally obtain the desired estimate (2.14).

3. Global integral estimates

First we shall obtain a global estimate for the Dirichlet integral.

THEOREM 3.1
Let $u(x)$ be a weak solution of problem (L). Let assumptions (a)–(c) *be fulfilled. Suppose, in addition, that $\int_{\partial G} rg^2\,ds < \infty$. Then the inequality*

$$\begin{aligned}\int_G |\nabla u|^2\,dx &+ \int_{\partial G} \frac{\gamma(\omega)}{r}\alpha(x)u^2(x)\,ds\\ &\le C\Big\{\int_G u^2(x)\,dx + \int_G f^2(x)\,dx + \frac{1}{\gamma_0}\int_{\partial G} rg^2(x)\alpha(x)\,ds\Big\}\end{aligned} \tag{3.1}$$

holds, where the constant $C > 0$ depends only on p, n, $\mathrm{meas}(G\setminus G_d)$, *$\nu$, d, $\|\sum_{i=1}^n |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}$.*

Proof. Putting in (II) $\eta(x) = u(x)$, by assumptions (a), (c), we get

$$\begin{aligned}\nu\int_G |\nabla u|^2\,dx &+ \int_{\partial G} \frac{\gamma(\omega)}{r}\alpha(x)u^2(x)\,ds\\ &\le \int_G b^i(x)uu_{x_i}\,dx + \int_{\partial G} |u||g(x)|\alpha(x)\,ds + \int_G |u||f(x)|\,dx.\end{aligned} \tag{3.2}$$

Now, by the representation $G = G_d \cup (G\setminus G_d)$, and by application of the Hölder, the Cauchy and the Sobolev (see (2.13)) inequalities, we obtain:

$$\begin{aligned}&\int_{G\setminus G_d} b^i(x)uu_{x_i}\,dx\\ &\le \frac{\varepsilon\nu}{2}\int_{G\setminus G_d}|\nabla u|^2\,dx + \frac{1}{2\varepsilon\nu}\int_{G\setminus G_d}\sum_{i=1}^n |b^i(x)|^2|u|^2\,dx \qquad (3.3)\\ &\le \frac{\varepsilon\nu}{2}\int_{G\setminus G_d}|\nabla u|^2\,dx + \frac{1}{2\varepsilon\nu}\Big\{\int_{G\setminus G_d}\Big(\sum_{i=1}^n |b^i(x)|^2\Big)^{\frac{p}{2}}dx\Big\}^{\frac{2}{p}}\cdot\Big(\int_{G\setminus G_d}|u|^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{p}}\\ &\le \frac{\varepsilon\nu}{2}\int_{G\setminus G_d}|\nabla u|^2\,dx + \frac{1}{2\varepsilon\nu}\Big\|\sum_{i=1}^n |b^i(x)|^2\Big\|_{L_{\frac{p}{2}}(G)}\cdot\int_{G\setminus G_d}(\delta|\nabla u|^2 + c_\delta u^2\,)dx\\ &\qquad\qquad \forall\delta,\varepsilon > 0;\ p > n.\end{aligned}$$

Further, by assumption (b),

$$\int_{G_d} b^i(x)uu_{x_i}\,dx \le \frac{\mathcal{A}(\frac{1}{d})}{d}\int_{G_d} |u||\nabla u|\,dx \le \frac{\mathcal{A}(\frac{1}{d})}{2d}\int_{G_d}(|u|^2+|\nabla u|^2)\,dx \tag{3.4}$$

and by the Cauchy inequality

$$\int_G |u||f(x)|\,dx \le \frac{1}{2}\int_G |u|^2\,dx + \frac{1}{2}\int_G |f|^2\,dx. \tag{3.5}$$

As a result from (3.2)–(3.5) we obtain

$$\begin{aligned}
&\nu\int_G |\nabla u|^2\,dx + \int_{\partial G}\gamma(\omega)r^{-1}\alpha(x)u^2\,ds \\
&\quad\le \frac{1}{2}\left(\frac{\delta}{\varepsilon\nu^2}\left\|\sum_{i=1}^n |b^i(x)|^2\right\|_{L_{\frac{p}{2}}(G)} + \frac{\mathcal{A}(\frac{1}{d})}{\nu d} + \varepsilon\right)\nu\int_G |\nabla u|^2\,dx \\
&\quad+ \frac{1}{2}\left(1+\frac{c_\delta}{\varepsilon\nu}\left\|\sum_{i=1}^n |b^i(x)|^2\right\|_{L_{\frac{p}{2}}(G)} + \frac{\mathcal{A}(\frac{1}{d})}{d}\right)\int_G |u|^2\,dx \\
&\quad+ \int_{\partial G}|u||g(x)|\alpha(x)\,ds + \frac{1}{2}\int_G f^2\,dx.
\end{aligned} \tag{3.6}$$

Choosing $\varepsilon = 1$, $\delta = \frac{\nu^2}{2\|\sum_{i=1}^n |b^i(x)|^2\|_{\frac{p}{2}(G)}}$, from (3.6) we have

$$\begin{aligned}
&\nu\left(\frac{1}{4} - \frac{\mathcal{A}(\frac{1}{d})}{2\nu d}\right)\int_G |\nabla u|^2\,dx + \int_{\partial G}\gamma(\omega)r^{-1}\alpha(x)u^2\,ds \\
&\quad\le C\left(\left\|\sum_{i=1}^n |b^i(x)|^2\right\|_{L_{\frac{p}{2}}(G)}, \nu, d\right)\int_G |u|^2\,dx \\
&\quad+ \int_{\partial G}|u||g(x)|\alpha(x)\,ds + \frac{1}{2}\int_G f^2\,dx.
\end{aligned} \tag{3.7}$$

By assumption (b), we can choose d big enough such that $\frac{\mathcal{A}(\frac{1}{d})}{d} < \frac{\nu}{2}$. Then

$$\begin{aligned}
&\int_G |\nabla u|^2\,dx + \int_{\partial G}\gamma(\omega)r^{-1}u^2\,ds \\
&\quad\le C\left(\left\|\sum_{i=1}^n |b^i(x)|^2\right\|_{L_{\frac{p}{2}}(G)}, \nu, d\right)\left\{\int_G |u|^2\,dx + \int_{\partial G}|u||g(x)|\,ds + \int_G f^2\,dx\right\}.
\end{aligned} \tag{3.8}$$

Now, by the Cauchy inequality, in virtue of assumption (c), we obtain

$$\begin{aligned}\int\limits_{\partial G} |u||g(x)|\,ds &= \int\limits_{\partial G}\left(\sqrt{\frac{\gamma(\omega)}{r}}|u|\right)\left(\sqrt{\frac{r}{\gamma(\omega)}}|g(x)|\right)ds\\ &\le \frac{1}{2}\int\limits_{\partial G}\frac{\gamma(\omega)}{r}u^2\,ds + \frac{1}{2\gamma_0}\int\limits_{\partial G} rg^2(x)\,ds.\end{aligned} \tag{3.9}$$

Finally, from (3.8) and (3.9) we get the desired inequality (3.1).

Theorem 3.2

Let $u(x)$ be a weak solution of problem (L) and λ_- be as above in (1.1). *Let assumptions* (a)–(c) *be satisfied. Suppose, in addition, that $\int_{\partial G} r^{\alpha-1}g^2\,ds < \infty$, $f(x) \in \mathring{W}^0_\alpha(G)$, where*

$$4 - n \le \alpha < 4 - n - 2\lambda_-. \tag{3.10}$$

Then $u(x) \in \mathring{W}^1_{\alpha-2}(G)$ and

$$\begin{aligned}&\int\limits_G (r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)\,dx + \int\limits_{\partial G} r^{\alpha-3}\gamma(\omega)u^2\alpha(x)\,ds\\ &\le C\left\{\int\limits_G u^2\,dx + \int\limits_G r^\alpha f^2\,dx + \frac{1}{\gamma_0}\int\limits_{\partial G}\alpha(x)r^{\alpha-1}g^2(x)\,ds\right\},\end{aligned} \tag{3.11}$$

where the constant C depends only on p, n, $\mathrm{meas}(G \setminus G_d)$, μ, d, α, ν, c_0, λ_-, $\|\sum_{i=1}^n |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}$.

Proof. Setting in (II), $\eta = r^{\alpha-2}u(x)$, with regard to

$$\eta_{x_i} = r^{\alpha-2}u_{x_i} + (\alpha-2)r^{\alpha-4}x_i u(x),$$

we obtain

$$\begin{aligned}&\int\limits_G a^{ij}u_{x_j}[r^{\alpha-2}u_{x_i} + (\alpha-2)r^{\alpha-4}x_i u(x)]\,dx\\ &\qquad - \int\limits_G [b^i u_{x_i} + c(x)u(x) - f(x)]r^{\alpha-2}u(x)\,dx\\ &= \int\limits_{\partial G}\left(g - \frac{1}{r}\gamma(\omega)u\right)ur^{\alpha-2}\alpha(x)\,ds.\end{aligned} \tag{3.12}$$

By assumption (a), we have

$$\int\limits_G r^{\alpha-2}a^{ij}u_{x_j}u_{x_i}\,dx = \int\limits_G r^{\alpha-2}|\nabla u|^2\,dx + \int\limits_G (a^{ij}(x) - \delta_i^j)r^{\alpha-2}u_{x_i}u_{x_j}\,dx;$$

$$\int\limits_G r^{\alpha-4}x_i a^{ij}uu_{x_j}\,dx = \int\limits_G u_{x_i}r^{\alpha-4}x_i u\,dx + \int\limits_G (a^{ij}(x) - \delta_i^j)u_{x_j}r^{\alpha-4}x_i u\,dx.$$

Then integrating by parts, we obtain

$$\int\limits_G r^{\alpha-4} x_i u u_{x_i}\, dx = \frac{1}{2}\int\limits_G r^{\alpha-4} x_i \frac{\partial u^2}{\partial x_i}\, dx = \frac{-(n+\alpha-4)}{2}\int\limits_G r^{\alpha-4}u^2\, dx + \frac{1}{2}\int\limits_{\partial G} u^2 r^{\alpha-4} x_i \cdot \cos(\overrightarrow{n}, x_i)\, ds.$$

Using (3.12) we can rewrite the latter relation in the form

$$\begin{aligned}
&\int\limits_G r^{\alpha-2}|\nabla u|^2\, dx + \int\limits_{\partial G} \gamma(\omega) u^2 r^{\alpha-3}\alpha(x)\, ds \\
&\quad = -\int\limits_G (a^{ij}(x) - \delta_i^j) r^{\alpha-2} u_{x_i} u_{x_j}\, dx \\
&\qquad + (2-\alpha)\int\limits_G (a^{ij}(x) - \delta_i^j) u_{x_j} r^{\alpha-4} u x_i\, dx \\
&\qquad + \frac{\alpha-2}{2}(n+\alpha-4)\int\limits_G r^{\alpha-4}u^2\, dx + \int\limits_G [b^i u_{x_i} + c(x)u(x) - f(x)] r^{\alpha-2} u(x)\, dx \\
&\qquad + \int\limits_{\partial G} r^{\alpha-2} g u \alpha(x)\, ds + \frac{2-\alpha}{2}\int\limits_{\partial G} r^{\alpha-4} u^2 x_i \cos(\overrightarrow{n}, x_i)\, dx.
\end{aligned} \tag{3.13}$$

Now we use the representation $G = G_d \cup (G \setminus G_d)$. It follows from assumptions and the Cauchy inequality that

1)
$$\int\limits_{G_d} (a^{ij}(x) - \delta_i^j) r^{\alpha-2} u_{x_i} u_{x_j}\, dx \le \mathcal{A}\Big(\frac{1}{d}\Big)\int\limits_{G_d} r^{\alpha-2}|\nabla u|^2\, dx$$
$$\int\limits_{G\setminus G_d} (a^{ij}(x) - \delta_i^j) r^{\alpha-2} u_{x_i} u_{x_j}\, dx \le (1+\mu)\max(1, d^{\alpha-2})\int\limits_{G\setminus G_d} |\nabla u|^2\, dx;$$

2)
$$\int\limits_{G_d} (a^{ij}(x) - \delta_i^j) u_{x_j} r^{\alpha-4} u x_i\, dx \le \frac{1}{2}\mathcal{A}\Big(\frac{1}{d}\Big)\int\limits_{G_d} (r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)\, dx,$$
$$\int\limits_{G\setminus G_d} (a^{ij}(x) - \delta_i^j) u_{x_j} r^{\alpha-4} u x_i dx \le \frac{(1+\mu)\max(1, d^{\alpha-3})}{2}\|u\|^2_{W^{1,2}(G)};$$

3)
$$\int\limits_{G_d} b^i u_{x_i} r^{\alpha-2} u(x)\,dx \le \frac{1}{2}\mathcal{A}\Big(\frac{1}{d}\Big)\int\limits_{G_d}\Big(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2\Big)\,dx,$$
$$\int\limits_{G\setminus G_d} b^i u_{x_i} r^{\alpha-2} u(x)\,dx \le \max(1,d^{\alpha-2})\int\limits_{G\setminus G_d}\sqrt{\sum_{i=1}^n |b^i(x)|^2}|u||\nabla u|$$
$$\le \frac{1}{2}\max(1,d^{\alpha-2})\Big\{\int\limits_{G\setminus G_d}|\nabla u|^2\,dx + \Big\|\sum_{i=1}^n |b^i|^2\Big\|_{L_{\frac{p}{2}}(G)} \cdot \int\limits_{G\setminus G_d}(|\nabla u|^2 + c_1u^2)\,dx\Big\}, \quad p>n,$$

(see (3.3));

4)
$$\int\limits_G f(x) r^{\alpha-2} u(x)\,dx \le \frac{\delta}{2}\int\limits_G r^{\alpha-4}u^2\,dx + \frac{1}{2\delta}\int\limits_G r^\alpha f^2\,dx,$$

5)
$$\int\limits_{\partial G} r^{\alpha-2}|u||g|\alpha(x)\,ds$$
$$\le \frac{\varepsilon}{2}\int\limits_{\partial G} r^{\alpha-3}\gamma(\omega)u^2\alpha(x)\,ds + \frac{1}{2\varepsilon\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2(x)\alpha(x)\,ds, \qquad \forall\varepsilon>0$$

(see (3.9)).

Further, (2.1) and (2.12) yields
$$\frac{2-\alpha}{2}\int\limits_{\partial G} r^{\alpha-4}u^2 x_i\cos(\overrightarrow{n},x_i)\,ds \le c(\alpha,d)\int\limits_G(|\nabla u|^2+u^2)\,dx.$$

In fact, by representation $\partial G = \Gamma_d \cup (\partial G\setminus\Gamma_d)$, we have
$$\frac{2-\alpha}{2}\int\limits_{\partial G} r^{\alpha-4}u^2 x_i\cos(\overrightarrow{n},x_i)\,ds$$
$$= \frac{2-\alpha}{2}\int\limits_{\partial G\setminus\Gamma_d} r^{\alpha-4}u^2 x_i\cos(\overrightarrow{n},x_i)\,ds + \frac{2-\alpha}{2}\int\limits_{\Gamma_d} r^{\alpha-4}u^2 x_i\cos(\overrightarrow{n},x_i)\,ds$$
$$\le c(\alpha)\int\limits_{\partial G\setminus\Gamma_d} r^{\alpha-3}u^2\,ds \le c(\alpha,d)\int\limits_{\partial G\setminus\Gamma_d} u^2\,ds.$$

Hence, (3.13) means that

$$
\begin{aligned}
U \equiv & \int\limits_G r^{\alpha-2}|\nabla u|^2\,dx + \int\limits_{\partial G} \gamma(\omega)u^2 r^{\alpha-3}\alpha(x)\,ds \\
\leq & \frac{2-\alpha}{2}(4-n-\alpha)\int\limits_{G_d} r^{\alpha-4}u^2 dx + \Big[\mathcal{A}\Big(\frac{1}{d}\Big)+\frac{\varepsilon}{2}\Big]\int\limits_{G_d} r^{\alpha-4}u^2\,dx \\
& + C_1\bigg(\mu, d, \alpha, \bigg\|\sum_{i=1}^n |b^i(x)|^2\bigg\|_{L_{\frac{p}{2}}(G)}, n\bigg)\cdot\int\limits_G (|\nabla u|^2+u^2)\,dx \qquad (3.14)\\
& + 2\mathcal{A}\Big(\frac{1}{d}\Big)\int\limits_{G_d} r^{\alpha-2}|\nabla u|^2\,dx + \frac{\varepsilon}{2}\int\limits_{\partial G}\gamma(\omega)u^2 r^{\alpha-3}\alpha(x)\,ds \\
& + \frac{1}{2\varepsilon}\int\limits_G r^\alpha f^2\,dx + \frac{1}{2\varepsilon\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2(x)\alpha(x)\,ds \qquad \forall\varepsilon>0.
\end{aligned}
$$

Now we consider two cases: 1) $2<\alpha<4-n-2\lambda_-$ and 2) $4-n\leq\alpha\leq 2$.

1) case $2<\alpha<4-n-2\lambda_-$.

In this case $\frac{(2-\alpha)(4-n-\alpha)}{2}>0$. Therefore, applying the Hardy–Wirtinger inequality (2.6), we have

$$
\frac{2-\alpha}{2}(4-n-\alpha)\int\limits_{G_d} r^{\alpha-4}u^2\,dx \leq \frac{2-\alpha}{2}(4-n-\alpha)H(\lambda_-,n,\alpha)U. \qquad (3.15)
$$

It is easily to verify that $\frac{2-\alpha}{2}(4-n-\alpha)H(\lambda_-,n,\alpha)<1$ for α which satisfies the inequality (3.10). Hence, applying once more the inequality (2.6) we get

$$
\begin{aligned}
& K(\lambda_-,n,\alpha)U \\
& \leq \Big(3\mathcal{A}\Big(\frac{1}{d}\Big)+\varepsilon\Big)HU + C_1\bigg(\mu,d,\alpha,\bigg\|\sum_{i=1}^n |b^i|^2\bigg\|_{L_{\frac{p}{2}}(G)},n\bigg)\int\limits_G(|\nabla u|^2+u^2)\,dx \\
& \quad + \frac{1}{2\varepsilon}\int\limits_G r^\alpha f^2\,dx + \frac{1}{2\varepsilon\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2(x)\alpha(x)\,ds, \qquad \forall\varepsilon>0;
\end{aligned}
$$

and

$$
K(\lambda_-,n,\alpha) = 1 - \frac{(2-\alpha)(4-n-\alpha)}{2}H(\lambda_-,n,\alpha) > 0.
$$

The property of the function $\mathcal{A}$ implies that if $\varepsilon=\frac{1}{4H}K(\lambda_-,n,\alpha)$ and $d\gg 1$, then $\mathcal{A}(\frac{1}{d})\leq\frac{1}{4H}K(\lambda_-,n,\alpha)$ holds. Thus, if we use the inequality (3.1) from Theorem 3.1, we obtain

$$
\begin{aligned}
& \int\limits_G r^{\alpha-2}|\nabla u|^2\,dx + \int\limits_{\partial G}\gamma(\omega)u^2 r^{\alpha-3}\alpha(x)\,ds \\
& \qquad \leq C\bigg\{\int\limits_G u^2\,dx + \int\limits_G r^\alpha f^2\,dx + \frac{1}{\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2(x)\alpha(x)\,ds\bigg\}, \qquad (3.16)
\end{aligned}
$$

where $C = const(p, n, \mu, \nu, d, \alpha, \| \sum_{i=1}^{n} |b^i|^2 \|_{L_{\frac{p}{2}}(G)}, n)$. Now by the Hardy–Friedrichs–Wirtinger inequality (2.6), from (3.16) we get the desired estimate (3.11).

2) case $4 - n \le \alpha \le 2$.

In this case we have $\frac{(2-\alpha)(4-n-\alpha)}{2} \le 0$. Therefore we can neglected the first integral on the right hand side in (3.14). Repeating arguments from case 1), we get again the estimate (3.11).

4. Local integral estimates

4.1. Local estimate near infinity

The weak solution of the problem (L) is locally bounded at infinity. More precisely, we have

THEOREM 4.1
Let $u(x)$ be a weak solution of the problem (L). Let assumptions (a)–(c) *be satisfied. Suppose, in addition, that $g(x) \in L_\infty(\partial G)$. Then the inequality*

$$\sup_{x \in G^{2R}_{R\varkappa}} |u(x)| \le \frac{C}{(\varkappa - 1)^{\frac{\tilde{n}}{t}}} \left\{ R^{\frac{-n}{t}} \|u\|_{t, G^{2R}_R} + R^{2(1-\frac{n}{p})} \|f\|_{\frac{p}{2}, G^{2R}_R} + R\|g\|_{\infty, \Gamma^{2R}_R} \right\} \tag{4.1}$$

holds for any $t > 0$, $p > \tilde{n} \begin{cases} \ge n \text{ for } n \ge 3 \\ > 2 \text{ for } n = 2 \end{cases}$, $\varkappa \in (1, 2)$ and $R > d$, where the constant $C > 0$ depends only on n, ν, μ, p and $\| \sum_{i=1}^{n} |b^i(x)|^2 \|_{L_{\frac{p}{2}}(G)}$.

Proof. We apply the Moser iteration method. First we assume that $t \ge 2$. We consider the integral identity (II) and make the coordinate transformation $x = Rx'$. Let G' be the image of G and $\partial G'$ be the image of ∂G. We have $dx = R^n dx'$, $ds = R^{n-1} ds'$. In addition, introduce

$$v(x') = u(Rx'), \qquad \mathcal{F}(x') = R^2 f(Rx'), \qquad \mathcal{G}(x') = Rg(Rx'). \tag{4.2}$$

Then (II) means that

$$\begin{aligned} &\int_{G'} \{a^{ij}(Rx') v_{x'_j} \eta_{x'_i} - Rb^i(Rx') v_{x'_i} \eta(x') - R^2 c(Rx') v(x') \eta(x')\} \, dx' \\ &\quad + \int_{\partial G'} \frac{1}{|x'|} \gamma(\omega) v(x') \alpha(Rx') \eta(x') \, ds' \\ &= \int_{\partial G'} \mathcal{G}(x') \alpha(Rx') \eta(x') \, ds' - \int_{G'} \mathcal{F}(x') \eta(x') \, dx' \end{aligned} \tag{II'}$$

for all $\eta(x') \in C^0(\overline{G'}) \cap W^1(G')$. Define the quantity k as follows

$$k = k(R) = \nu^{-1} (\|\mathcal{F}\|_{\frac{p}{2}, G^\infty_1} + \|\mathcal{G}\|_{\infty, \Gamma^\infty_1}) \tag{4.3}$$

and set

$$\overline{v}(x') = |v(x')| + k. \tag{4.4}$$

Along similar lines

$$\begin{gathered}|\mathcal{F}|\overline{v} = \frac{1}{k}|\mathcal{F}|\cdot k\overline{v} = \frac{1}{k}|\mathcal{F}|(\overline{v}-|v|)\cdot\overline{v} = \frac{1}{k}|\mathcal{F}|\cdot\overline{v}^2 - \frac{1}{k}|\mathcal{F}|\cdot|v|\overline{v} \le \frac{1}{k}|\mathcal{F}|\cdot\overline{v}^2;\\ |\mathcal{G}|\overline{v} \le \tfrac{1}{k}|\mathcal{G}|\cdot\overline{v}^2.\end{gathered} \tag{4.5}$$

The test function in the integral identity (II') is choosen as

$$\eta(x') = \zeta^2(|x'|)v\overline{v}^{t-2}(x'),$$

where $\zeta(|x'|) \in C_0^\infty([1,2])$ is a non-negative function to be further specified. By the chain and the product rules η is a proper test function in (II') and also

$$\eta_{x'_i} = \overline{v}^{t-2}v_{x'_i}\zeta^2(|x'|) + (t-2)\overline{v}^{t-3}|v|v_{x'_i}\zeta^2(|x'|) + 2\zeta\zeta_{x'_i}v\overline{v}^{t-2}(x').$$

Hence, by substitution in (II') using $c(Rx') \le 0$ in G', $v \le |v| \le \overline{v}$ and $t \ge 2$, we obtain

$$\begin{aligned}&\int\limits_{G_1^2} a^{ij}(Rx')v_{x'_i}v_{x'_j}\overline{v}^{t-2}\zeta^2(|x'|)\,dx' + \int\limits_{\Gamma_1^2}\frac{1}{|x'|}\gamma(\omega)v^2\overline{v}^{t-2}(x')\alpha(Rx')\zeta^2(|x'|)\,ds'\\ &\quad\le R\int\limits_{G_1^2}|b^i(Rx')v_{x'_i}|\overline{v}^{t-1}(x')\zeta^2(|x'|)\,dx' + 2\int\limits_{G_1^2}|a^{ij}(Rx')\zeta_{x'_i}v_{x'_j}|\overline{v}^{t-1}\zeta(|x'|)\,dx'\\ &\qquad+\int\limits_{\Gamma_1^2}\mathcal{G}(x')\overline{v}^{t-1}(x')\zeta^2(|x'|)\,ds' - \int\limits_{G_1^2}|\mathcal{F}(x')|\overline{v}^{t-1}(x')\zeta^2(|x'|)\,dx'.\end{aligned}$$

By the ellipticity condition, the assumption (c) and with regard to (4.5), we have

$$\begin{aligned}&\int\limits_{G_1^2}\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx'\\ &\quad\le\int\limits_{G_1^2}\Bigg(2\mu|\nabla' v|\cdot|\nabla'\zeta|\cdot\overline{v}^{t-1}\zeta(|x'|)\\ &\quad+R\Bigg(\sum_{i=1}^n|b^i(x)|^2\Bigg)^{\frac{1}{2}}|\nabla' v|\cdot\overline{v}^{t-1}\zeta^2(|x'|) + \frac{1}{k}|\mathcal{F}(x')|\cdot\overline{v}^t\zeta^2(|x'|)\Bigg)\,dx'\\ &\quad+\frac{1}{k}\|\mathcal{G}\|_{\infty,\partial G}\cdot\int\limits_{\Gamma_1^2}\overline{v}^t(x')\zeta^2(|x'|)\,ds'.\end{aligned} \tag{4.6}$$

We estimate every term by the Cauchy inequality for any $\varepsilon > 0$:

$$\begin{aligned}2\mu|\nabla' v|\cdot|\nabla'\zeta|\cdot\overline{v}^{t-1}\zeta &\le 2\big(\sqrt{\nu}|\nabla' v|\cdot\overline{v}^{\frac{t}{2}-1}\zeta\big)\cdot\left(\frac{\mu}{\sqrt{\nu}}\overline{v}^{\frac{t}{2}}|\nabla'\zeta|\right)\\ &\le\varepsilon\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2 + \frac{\mu^2}{\varepsilon\nu}\overline{v}^t|\nabla'\zeta|^2;\end{aligned}$$

$$
\begin{aligned}
&R\Big(\sum_{i=1}^{n}|b^i(x)|^2\Big)^{\frac{1}{2}}\cdot|\nabla' v|\cdot\overline{v}^{t-1}\zeta^2\\
&\qquad\le\big(\sqrt{\nu}|\nabla' v|\cdot\overline{v}^{\frac{t}{2}-1}\big)\cdot\Big(\frac{1}{\sqrt{\nu}}\overline{v}^{\frac{1}{2}}\cdot R\Big(\sum_{i=1}^{n}|b^i(x)|^2\Big)^{\frac{1}{2}}\Big)\zeta^2\\
&\qquad\le\frac{\varepsilon}{2}\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2+\frac{1}{2\varepsilon\nu}R^2\sum_{i=1}^{n}|b^i(x)|^2\cdot\overline{v}^t\zeta^2.
\end{aligned}
$$

In order to estimate integrals over the boundary we apply the inequality (2.12). Then (4.6) yields

$$
\begin{aligned}
&\int_{G_1^2}\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx'\\
&\qquad\le\frac{3}{2}\varepsilon\int_{G_1^2}\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx' \qquad (4.7)\\
&\qquad+\int_{G_1^2}\Big\{\frac{\mu^2}{\varepsilon\nu}\overline{v}^t\cdot|\nabla'\zeta|^2+\Big(\frac{1}{2\varepsilon\nu}R^2\sum_{i=1}^{n}|b^i(x)|^2+\frac{1}{k}|\mathcal{F}(x')|\Big)\cdot\overline{v}^t\zeta^2(|x'|)\Big\}\,dx'\\
&\qquad+\frac{1}{k}\|\mathcal{G}\|_{\infty,\Gamma_1^2}\cdot\int_{G_1^2}\Big(\delta|\nabla'(\zeta\overline{v}^{\frac{t}{2}})|^2+\frac{1}{\delta}c_0\overline{v}^t\zeta^2\Big)\,dx', \qquad \forall\varepsilon,\delta>0.
\end{aligned}
$$

The relations

$$
|\nabla'(\zeta\overline{v}^{\frac{t}{2}})|^2\le 2(\zeta^2|\nabla'(\overline{v}^{\frac{t}{2}})|^2+\overline{v}^t|\nabla'\zeta|^2),\qquad |\nabla'(\overline{v}^{\frac{t}{2}})|^2=\frac{t^2}{4}\overline{v}^{t-2}|\nabla' v|^2 \qquad (4.8)
$$

imply inequality

$$
|\nabla'(\zeta\overline{v}^{\frac{t}{2}})|^2\le\frac{t^2}{2}\overline{v}^{t-2}|\nabla' v|^2\zeta^2+2\overline{v}^t|\nabla'\zeta|^2. \qquad (4.9)
$$

Using (4.8), (4.9) and choosing $\varepsilon=\frac{1}{3}$, we find that

$$
\begin{aligned}
&\frac{1}{2}\int_{G_1^2}\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx'\\
&\qquad\le\frac{\delta t^2}{2\nu}\cdot\frac{\|\mathcal{G}\|_{\infty,\Gamma_1^2}}{k}\cdot\int_{G}\nu|\nabla' v|^2\cdot\overline{v}^{t-2}\zeta^2(|x'|)\,dx' \qquad (4.10)\\
&\qquad+\int_{G_1^2}\Big(\frac{3R^2}{2\nu}\sum_{i=1}^{n}|b^i(x)|^2+\frac{|\mathcal{F}(x')|+c_0\delta^{-1}\|\mathcal{G}\|_{\infty,\Gamma_1^2}}{k}\Big)\cdot\overline{v}^t\zeta^2(|x'|)\,dx'\\
&\qquad+\int_{G_1^2}\Big\{\frac{3\mu^2}{\nu}+\frac{2\delta}{k}\|\mathcal{G}\|_{\infty,\Gamma_1^2}\Big\}\,\overline{v}^t\cdot|\nabla'\zeta|^2\,dx', \qquad \forall\delta\in(0,1].
\end{aligned}
$$

We choose now $\delta = \frac{1}{2t^2}$; by the definition of the number k we can rewrite the last inequality (4.10) in the following form

$$\int\limits_{G_1^2} \nu|\nabla' v|^2 \cdot \overline{v}^{t-2}\zeta^2(|x'|)\,dx'$$
$$\leq 8c_0\nu t^2 \int\limits_{G_1^2} \zeta^2(|x'|)\overline{v}^t(x')\,dx' + \left(\frac{12\mu^2}{\nu} + \frac{4\nu}{t^2}\right) \cdot \int\limits_{G_1^2} |\nabla'\zeta|^2\overline{v}^t(x')\,dx'$$
$$+ 4\int\limits_{G_1^2} \left(\frac{3R^2}{2\nu}\sum_{i=1}^n |b^i(x)|^2 + \frac{|\mathcal{F}(x')|}{k}\right) \cdot \overline{v}^t\zeta^2(|x'|)\,dx'.$$

But, by (4.8), the last means

$$\int\limits_{G_1^2} \nu|\nabla'(\overline{v}^{\frac{t}{2}})|^2\zeta^2(|x'|)\,dx'$$
$$\leq 2c_0\nu t^4 \int\limits_{G_1^2} \zeta^2(|x'|)\overline{v}^t(x')\,dx' + \left(\frac{3\mu^2t^2}{\nu} + \nu\right) \cdot \int\limits_{G_1^2} |\nabla'\zeta|^2\overline{v}^t(x')\,dx'$$
$$+ t^2\int\limits_{G_1^2} \left(\frac{3R^2}{2\nu}\sum_{i=1}^n |b^i(x)|^2 + \frac{|\mathcal{F}(x')|}{k}\right) \cdot \overline{v}^t\zeta^2(|x'|)\,dx'.$$

Since $t \geq 2$, the above inequality can be rewritten in the form

$$\int\limits_{G_1^2} \nu|\nabla'(\overline{v}^{\frac{t}{2}})|^2\zeta^2(|x'|)\,dx'$$
$$\leq C_1t^4 \int\limits_{G_1^2} \nu(|\nabla'\zeta|^2 + \zeta^2(|x'|))\overline{v}^t\,dx' \tag{4.11}$$
$$+ C_2t^2\int\limits_{G_1^2} \nu\left(R^2\sum_{i=1}^n |b^i(x)|^2 + \frac{|\mathcal{F}(x')|}{k}\right) \cdot \overline{v}^t\zeta^2(|x'|)\,dx',$$

where the constants C_1, C_2 depend only on c_0, ν, μ and are independent on t. Substitution of

$$w(x') = \sqrt{\nu} \cdot \overline{v}^{\frac{t}{2}}(x') \tag{4.12}$$

into (4.11) yields

$$\int\limits_{G_1^2} |\nabla' w|^2\zeta^2(|x'|)\,dx'$$
$$\leq C_1t^4 \int\limits_{G_1^2} (|\nabla'\zeta|^2 + \zeta^2(|x'|))w^2(x')\,dx' \tag{4.13}$$

$$+ C_2 t^2 \int\limits_{G_1^2} \left(R^2 \sum_{i=1}^{n} |b^i(x)|^2 + \frac{|\mathcal{F}(x')|}{k} \right) \cdot w^2(x') \zeta^2(|x'|)\, dx'.$$

On iteration process can now be deduced on (4.13). By the Sobolev imbedding theorem (see Theorem 13, subsection 4.7, chapter 4 [1]), we have

$$\|\zeta w\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} \le C^* \int\limits_{G_1^2} ((|\nabla' \zeta|^2 + \zeta^2) w^2(x') + \zeta^2(|x'|) |\nabla' w|^2)\, dx', \tag{4.14}$$

where $\widetilde{n} \ge n$ for $n \ge 3$ and $\widetilde{n} > 2$ for $n = 2$ and C^* depends only on $\widetilde{n}$. Using the Hölder inequality for integrals

$$\int\limits_{G_1^2} |F(x')| \cdot w^2(x') \zeta^2(x')\, dx' \le \|F\|_{\frac{p}{2}, G} \cdot \|w\zeta\|^2_{\frac{2p}{p-2}, G_1^2}, \qquad p > 2, \tag{4.15}$$

we get from (4.13)–(4.15) that

$$\begin{aligned} \|\zeta w\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} &\le C_3 t^4 \int\limits_{G_1^2} (|\nabla' \zeta|^2 + \zeta^2(|x'|)) w^2(x')\, dx' \\ &\quad + C_4 t^2 \left\| R^2 \sum_{i=1}^{n} |b^i(x)|^2 + \frac{\mathcal{F}(x')|}{k} \right\|_{\frac{p}{2}, G_1^2} \cdot \|w\zeta\|^2_{\frac{2p}{p-2}, G_1^2}, \qquad p > \widetilde{n}. \end{aligned} \tag{4.16}$$

By the interpolation inequality for L_p–norms (see (7.10), §7.1 [3])

$$\|\zeta w\|_{\frac{2p}{p-2}, G_1^2} \le \varepsilon \|\zeta w\|_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} + \varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}} \|\zeta w\|_{2, G_1^2}, \qquad p > \widetilde{n} > 2,\ \forall \varepsilon > 0. \tag{4.17}$$

It follows from (4.17)–(4.17) that

$$\begin{aligned} \|\zeta w\|_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} &\le t\sqrt{C_4} \left\| R^2 \sum_{i=1}^{n} |b^i(x)|^2 + \frac{|\mathcal{F}(x')|}{k} \right\|^{\frac{1}{2}}_{\frac{p}{2}, G_1^2} \left(\varepsilon \|w\zeta\|_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} + \varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}} \|\zeta w\|_{2, G_1^2} \right) \\ &\quad + t^2 \sqrt{C_3} \cdot \|(\zeta + |\nabla' \zeta|) w\|_{2, G_1^2}, \qquad p > \widetilde{n},\ \forall \varepsilon > 0. \end{aligned}$$

Choosing $\varepsilon = \frac{1}{2t\sqrt{C_4}} \left\| R^2 \sum_{i=1}^{n} |b^i(x)|^2 + \frac{|\mathcal{F}(x')|}{k} \right\|^{-\frac{1}{2}}_{\frac{p}{2}, G_1^2}$ we obtain

$$\|\zeta w\|_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} \le C t^{\frac{p}{p-\widetilde{n}}} \|(\zeta + |\nabla' \zeta|) w\|_{2, G_1^2}, \qquad 2\widetilde{n} \ge p > \widetilde{n} > 2, \tag{4.18}$$

where C depends only on c_0, n, $\widetilde{n}$, ν, μ, p, $\| \sum_{i=1}^{n} |b^i(x)|^2 \|_{\frac{p}{2}, G_1^2}$ and is independent of t. Recalling the definition (4.12) of w, we finally establish from (4.18) the inequality:

$$\|\zeta \cdot \overline{v}^{\frac{t}{2}}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2}, G_1^2} \le C t^{\frac{p}{p-\widetilde{n}}} \|(\zeta + |\nabla' \zeta|) \cdot \overline{v}^{\frac{t}{2}}\|_{2, G_1^2}, \qquad 2\widetilde{n} \ge p > \widetilde{n} > 2. \tag{4.19}$$

This inequality can now be iterated to yield the desired estimate.

For any $\varkappa \in (1,2)$ we define sets

$$G'_{(j)} \equiv G^2_{\varkappa-(\varkappa-1)2^{-j}}, \qquad j = 0,1,2,\ldots.$$

It is easy to verify that

$$G^2_{\varkappa} \equiv G'_{(\infty)} \subset \ldots \subset G'_{(j+1)} \subset G'_{(j)} \subset \ldots \subset G'_{(0)} \equiv G^2_1.$$

Now we consider the sequence of cut-off functions $\zeta_j(x') \in C^\infty(G'_{(j)})$ such that

$$\begin{aligned}
&0 \le \zeta_j(x') \le 1 \text{ in } G'_{(j)} \qquad \text{and} \qquad \zeta_j(x') \equiv 1 \text{ in } G'_{(j+1)};\\
&\zeta_j(x') \equiv 0 \qquad \text{for } 1 < |x'| < \varkappa - 2^{-j}(\varkappa-1);\\
&|\nabla'\zeta_j| \le \frac{2^{j+1}}{\varkappa-1} \qquad \text{for } \varkappa - 2^{-j}(\varkappa-1) < |x'| < \varkappa - 2^{-j-1}(\varkappa-1).
\end{aligned}$$

We define the number sequence

$$t_j = t\Big(\frac{\widetilde{n}}{\widetilde{n}-2}\Big)^j, \qquad j = 0,1,2,\ldots.$$

Now we rewrite the inequality (4.19) replacing $\zeta(|x'|)$ by $\zeta_j(x')$ and t by t_j; then taking the t_j-th root we obtain

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \Big(\frac{C}{\varkappa-1}\Big)^{\frac{2}{t_j}} \cdot 4^{\frac{j}{t_j}} \cdot (t_j)^{\frac{2p}{p-\widetilde{n}}\cdot\frac{1}{t_j}} \|\overline{v}\|_{t_j,G'_{(j)}}.$$

After iteration, we find that

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \left\{\frac{Ct^{\frac{p}{p-\widetilde{n}}}}{\varkappa-1}\cdot\Big(\frac{\widetilde{n}}{\widetilde{n}-2}\Big)^{\frac{p}{p-\widetilde{n}}}\right\}^{2\sum_{j=0}^\infty \frac{1}{t_j}} \cdot 4^{\sum_{j=0}^\infty \frac{j}{t_j}} \cdot \|\overline{v}\|_{t,G^2_1}. \tag{4.20}$$

It is worth noting that the series $\sum_{j=0}^\infty \frac{j}{t_j}$ convergens by the d'Alembert ratio test; the series $\sum_{j=0}^\infty \frac{1}{t_j} = \frac{\widetilde{n}}{2t}$ is calculated as a geometric series. Therefore from (4.20) we obtain that

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \frac{C}{(\varkappa-1)^{\frac{\widetilde{n}}{t}}}\|\overline{v}\|_{t,G^2_1}.$$

Therefore, letting $j \to \infty$, we have

$$\sup_{x'\in G^2_\varkappa} |\overline{v}(x')| \le \frac{C}{(\varkappa-1)^{\frac{\widetilde{n}}{t}}}\|\overline{v}\|_{t,G^2_1}.$$

Basing on the definition (4.4) of the function $\overline{v}(x')$ and on the number k defined by (4.3), we obtain

$$\sup_{x'\in G^2_\varkappa} |v(x')| \le \frac{C}{(\varkappa-1)^{\frac{\widetilde{n}}{t}}}(\|v\|_{t,G^2_1} + \|\mathcal{F}\|_{\frac{p}{2},G^2_1} + \|\mathcal{G}\|_{\infty,\Gamma^2_1}).$$

Returning to variables x, u, introduced by (4.2), we obtain the required estimate (4.1) for $t \geq 2$.

Let now $0 < t < 2$. We consider (4.1) with $t = 2$:

$$\sup_{x \in G^{2R}_{R\varkappa}} |u(x)| \leq \frac{C}{(\varkappa - 1)^{\frac{\tilde{n}}{2}} R^{\frac{n}{2}}} \|u\|_{2,G^{2R}_R} + K(R), \tag{4.21}$$

where

$$K(R) = \frac{C}{(\varkappa - 1)^{\frac{\tilde{n}}{2}}} \Big\{ R^{2(1-\frac{n}{p})} \|f\|_{\frac{p}{2}, G^{2R}_R} + R\|g\|_{\infty, \Gamma^{2R}_R} \Big\}.$$

The Young inequality with $q = \frac{2}{t}$ and $q' = \frac{2}{2-t}$, can be written as

$$\begin{aligned} &\frac{C}{(\varkappa - 1)^{\frac{\tilde{n}}{2}} R^{\frac{n}{2}}} \|u\|_{2,G^{2R}_R} \\ &\quad = \frac{C}{(\varkappa - 1)^{\frac{\tilde{n}}{2}} R^{\frac{n}{2}}} \Big(\int\limits_{G^{2R}_R} u^t \cdot u^{2-t}\, dt \Big)^{\frac{1}{2}} \\ &\quad \leq \Big(\sup_{G^{2R}_R} |u(x)| \Big)^{1-\frac{t}{2}} \cdot \frac{C}{(\varkappa - 1)^{\frac{\tilde{n}}{2}} R^{\frac{n}{2}}} \|u\|^{\frac{t}{2}}_{t,G^{2R}_R} \\ &\quad \leq \frac{1}{2} \sup_{G^{2R}_R} |u(x)| + \frac{C_1}{(\varkappa - 1)^{\frac{\tilde{n}}{t}} R^{\frac{n}{t}}} \|u\|_{t,G^{2R}_R}. \end{aligned} \tag{4.22}$$

Let us define the function $\psi(s) = \sup_{x \in G^{2R}_s} |u(x)|$. Then it follows from (4.21)–(4.22), that

$$\psi(R\varkappa) \leq \frac{1}{2}\psi(R) + \frac{C_1}{(\varkappa - 1)^{\frac{\tilde{n}}{t}} R^{\frac{n}{t}}} \|u\|_{t,G^{2R}_R} + K(R), \qquad \varkappa \in (1, 2). \tag{4.23}$$

Further we apply the following statement:

Proposition 4.2 (see Lemma 4.1 in Chapter 2 [2])
Let $\psi(s)$ be a bounded non-negative function defined on the interval $[T_0, T_1]$, where $T_1 > T_0 \geq 1$. Suppose that for any $T_0 \leq s < \sigma < T_1$ the function $\psi(s)$ satisfies

$$\psi(\sigma) \leq \delta\psi(s) + \frac{A}{(\sigma - s)^\alpha} + B, \tag{4.24}$$

where $\delta \in (0, 1)$, A, B and α are non-negative constants. Then

$$\psi(d) \leq C\Big[\frac{A}{(d - \varrho)^\alpha} + B\Big], \qquad T_0 \leq \varrho < d < T_1, \tag{4.25}$$

where C depends only on α, δ.

Substitution of $d = R\varkappa$, $\varrho = R$, $\delta = \frac{1}{2}$, $\alpha = \frac{\tilde{n}}{t}$, $A = C_1\|u\|_{t,G^{2R}_R} \cdot R^{-\frac{\tilde{n}-n}{t}}$, $B = K(R)$ and use of (4.23) yields the required estimate (4.1) in the case $0 < t < 2$.

The proof of theorem 4.1 is complete.

4.2. Local integral weighted estimates

Theorem 4.3

Let $u(x)$ be a weak solution of problem (L) and λ_- be as above in (1.1). Let assumptions (a)–(d) with $\mathcal{A}(t)$ Dini–continuous at zero be satisfied. Then $\overset{\circ}{W}{}^1_{2-n}(G)$ and there exist $d \gg 1$ and a constant $C > 0$ depending only on n, s, λ_-, ω_o and on $\int_0^1 \frac{\mathcal{A}(r)}{r}\,dr$ such that for all $R > d$

$$\int\limits_{G_R} \left(r^{2-n}|\nabla u|^2 + r^{-n}u^2\right) dx + \int\limits_{\Gamma_R} r^{1-n}\gamma(\omega)u^2(x)\alpha(x)\,ds$$
$$\leq C\left(\|u\|^2_{2,G} + f_1^2 + \frac{1}{\gamma_0}g_1^2\right)\cdot \begin{cases} R^{2\lambda_-}, & \text{for } s > -\lambda_-; \\ R^{2\lambda_-}\ln^2 R, & \text{for } s = -\lambda_-; \\ R^{-2s}, & \text{for } 0 < s < -\lambda_-. \end{cases} \tag{4.26}$$

Proof. It follows from Theorem 3.2 that $u(x)$ belongs to $\overset{\circ}{W}{}^1_{2-n}(G)$, so it is enough to prove the estimate (4.26). Substitution of $\eta(x) = r^{2-n}u(x)$ in $(II)_{loc}$, and the definition (2.9) yield

$$U(R) = -R\int\limits_{\Omega} u(x)\frac{\partial u}{\partial r}\bigg|_{r=R} d\Omega - \int\limits_{\Omega_R} r^{2-n}u(x)(a^{ij}(x) - \delta_i^j)u_{x_j}\cos(r, x_i)\,d\Omega_R$$
$$+ \int\limits_{\Gamma_R} r^{2-n}u(x)g(x)\alpha(x)\,ds$$
$$+ \int\limits_{G_R} \Big\{ - r^{2-n}(a^{ij}(x) - \delta_i^j)u_{x_i}u_{x_j} \tag{4.27}$$
$$+ (n-2)r^{-n}u(x)a^{ij}(x)x_iu_{x_j} + r^{2-n}u(x)b^i(x)u_{x_i}$$
$$+ r^{2-n}c(x)u^2(x) - r^{2-n}u(x)f(x)\Big\}\,dx.$$

Now we transform the integrals on the right part of (4.27)

$$(n-2)\int\limits_{G_R} r^{-n}u(x)a^{ij}(x)x_iu_{x_j}\,dx$$
$$= \frac{n-2}{2}\int\limits_{G_R} r^{-n}x_i\frac{\partial u^2}{\partial x_i}\,dx + (n-2)\int\limits_{G_R} r^{-n}u(x)(a^{ij}(x) - \delta_i^j)x_iu_{x_j}\,dx.$$

Application of the Gauss–Ostogradskiy divergence theorem and use of condition $\lim_{|x|\to\infty} u(x) = 0$ yield

$$\int\limits_{G_R} r^{-n}x_i\frac{\partial u^2}{\partial x_i}\,dx = \lim_{\widetilde{R}\to\infty}\int\limits_{G_R^{\widetilde{R}}} r^{-n}x_i\frac{\partial u^2}{\partial x_i}\,dx$$
$$= -\int\limits_{G_R} u^2(x)\frac{\partial}{\partial x_i}(x_ir^{-n})\,dx - R^{-n}\int\limits_{\Omega_R} u^2(x)x_i\cos(r, x_i)\,d\Omega_R$$

$$+ \widetilde{R}^{-n} \int\limits_{\Omega_{\widetilde{R}}} u^2(x) x_i \cos(r, x_i)\, d\Omega_{\widetilde{R}} + \int\limits_{\Gamma_R} r^{-n} u^2(x) x_i \cos(n, x_i)\, ds.$$

We have

$$\begin{aligned}\lim_{\widetilde{R}\to\infty} \widetilde{R}^{-n} \int\limits_{\Omega_{\widetilde{R}}} u^2(x) x_i \cos(r, x_i)\, d\Omega_{\widetilde{R}} &\le \lim_{\widetilde{R}\to\infty} \int\limits_{\Omega} u^2(\widetilde{R}, \omega)\, d\Omega \\ &\le \operatorname{meas}\Omega \lim_{\widetilde{R}\to\infty} \cdot \sup_{\Omega} u^2(\widetilde{R}, \omega) \\ &= 0.\end{aligned}$$

Hence, by Lemma 2.1

$$x_i \cos(r, x_i)|_{\Omega_R} = R, \qquad x_i \cos(n, x_i)|_{\Gamma_R} = 0, \qquad R \gg 1,$$

we have

$$\frac{n-2}{2} \int\limits_{G_R} r^{-n} x_i \frac{\partial u^2}{\partial x_i}\, dx = -\frac{n-2}{2} \int\limits_{\Omega} u^2(x)\, d\Omega.$$

It follows from Lemma 2.9 and $c(x) \le 0$ that

$$\begin{aligned}U(R) \le \frac{R}{2\lambda_-} U'(R) &- \int\limits_{\Omega_R} r^{2-n} u(x) (a^{ij}(x) - \delta_i^j) u_{x_j} \cos(r, x_i)\, d\Omega_R \\ &+ \int\limits_{\Gamma_R} r^{2-n} u(x) g(x) \alpha(x)\, ds \\ &+ \int\limits_{G_R} \Big\{ - r^{2-n} (a^{ij}(x) - \delta_i^j) u_{x_i} u_{x_j} + (n-2) r^{-n} u(x) \\ &\qquad \times (a^{ij}(x) - \delta_i^j) x_i u_{x_j} + r^{2-n} u(x) b^i(x) u_{x_i} - r^{2-n} u(x) f(x) \Big\}\, dx.\end{aligned}$$

Application of the assumption (b) to (4.27) yields

$$\begin{aligned}U(R) \le \frac{R}{2\lambda_-} U'(R) &+ R\mathcal{A}\Big(\frac{1}{R}\Big) \int\limits_{\Omega} |u||\nabla u|\, d\Omega \\ &+ \int\limits_{\Gamma_R} r^{2-n} |u(x)||g(x)| \alpha(x)\, ds \qquad (4.28) \\ &+ c_1(n) \mathcal{A}\Big(\frac{1}{R}\Big) \int\limits_{G_R} (r^{2-n} |\nabla u|^2 + r^{1-n} |u||\nabla u|)\, dx + \int\limits_{G_R} r^{2-n} |u(x)||f(x)|\, dx.\end{aligned}$$

We shall obtain an upper bound for each integral on the right hand side. First, applying the Cauchy inequality and (2.3), (2.6), we have

$$R \int\limits_{\Omega} |u||\nabla u|\, d\Omega \le \frac{1}{2} \int\limits_{\Omega} (R^2 |\nabla u|^2 + |u|^2)\, d\Omega \le -c_2(\lambda_-, n) R U'(R);$$

$$\int\limits_{G_R} r^{1-n} |u||\nabla u|\, dx \le \int\limits_{G_R} (r^{2-n} |\nabla u|^2 + r^{-n} |u|^2)\, dx \le c_3(\lambda_-, n) U(R).$$

Further, for all $\delta > 0$:

$$
\begin{aligned}
&\int\limits_{\Gamma_R} r^{2-n}|u||g|\alpha(x)\,ds \\
&\qquad = \int\limits_{\Gamma_R} \alpha(x)\Big(r^{\frac{1-n}{2}}\sqrt{\gamma(\omega)}|u|\Big)\Big(r^{\frac{3-n}{2}}\frac{1}{\sqrt{\gamma(\omega)}}|g|\Big)\,ds \\
&\qquad \le \frac{\delta}{2}\int\limits_{\Gamma_R} r^{1-n}\gamma(\omega)|u|^2\alpha(x)\,ds + \frac{1}{2\delta\gamma_0}\int\limits_{\Gamma_R} r^{3-n}|g|^2\alpha(x)\,ds;
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
\int\limits_{G_R} r^{2-n}|u(x)||f(x)|\,dx &\le \frac{\delta}{2}\int\limits_{G_R} r^{-n}|u|^2\,dx + \frac{1}{2\delta}\int\limits_{G_R} r^{4-n}|f|^2\,dx \\
&\le \delta c_4(\lambda_-, n)U(R) + \frac{1}{2\delta}\int\limits_{G_R} r^{4-n}|f|^2\,dx
\end{aligned}
\tag{4.30}
$$

in virtue of the Hardy–Friedrichs–Wirtinger inequality (2.6). Thus from (4.28)–(4.30) we get

$$
\begin{aligned}
&\Big[1 - c_5(n,\lambda_-)\Big(\delta + \mathcal{A}\Big(\frac{1}{R}\Big)\Big)\Big]U(R) \\
&\quad \le \frac{R}{2\lambda_-}\Big(1 + c_6(\lambda_-, n)\mathcal{A}\Big(\frac{1}{R}\Big)\Big)U'(R) \\
&\qquad + \frac{1}{2\delta}\Big\{\int\limits_{G_R} r^{4-n}|f|^2\,dx + \frac{1}{\gamma_0}\int\limits_{\Gamma_R} r^{3-n}|g|^2\,ds\Big\}, \qquad \forall\delta > 0.
\end{aligned}
\tag{4.31}
$$

Using the condition (d), we obtain

$$
\int\limits_{G_R} r^{4-n}|f|^2\,dx + \frac{1}{\gamma_0}\int\limits_{\Gamma_R} r^{3-n}|g|^2\,ds \le \frac{1}{2s}c_0\Big(f_1^2 + \frac{1}{\gamma_0}g_1^2\Big)\cdot R^{-2s},
$$

where c_0 depends only on $\operatorname{meas}\Omega$, $\operatorname{meas}\partial\Omega$. Thus, (4.31) implies the differential inequality (CP) with

$$
\begin{aligned}
\mathcal{P}(R) &= -\frac{\frac{2\lambda_-}{R}\cdot[1 - c_5(n,\lambda_-)(\delta + \mathcal{A}(\frac{1}{R}))]}{1 + c_6(\lambda_-)\mathcal{A}(\frac{1}{R})}, \qquad \forall\delta > 0; \\
\mathcal{Q}(R) &= -\frac{\frac{\lambda_-}{s}c_0(G)(f_1^2 + \frac{1}{\gamma_0}g_1^2)\cdot\delta^{-1}R^{-2s-1}}{1 + c_6(\lambda_-)\mathcal{A}(\frac{1}{R})}, \qquad \forall\delta > 0; \\
U_0 &= C\Big\{\int\limits_{G}(u^2 + r^{4-n}f^2(x))\,dx + \frac{1}{\gamma_0}\int\limits_{\partial G} r^{3-n}g^2(x)\alpha(x)\,ds\Big\}.
\end{aligned}
$$

Here, (3.11) is used with $\alpha = 4 - n$.

1) $s > -\lambda_-$
Choosing $\delta = R^{-\varepsilon}$, for all $\varepsilon > 0$, we obtain the problem (CP) with

$$\mathcal{P}(s) = -\frac{2\lambda_-}{s} \cdot \left[1 - \frac{c_7 \mathcal{A}(\frac{1}{s}) + s^{-\varepsilon}}{1 + c_6 \mathcal{A}(\frac{1}{s})}\right], \qquad \forall \delta > 0;$$
$$\mathcal{Q}(R) = -\frac{\frac{\lambda_-}{s} c_0(G)(f_1^2 + \frac{1}{\gamma_0} g_1^2) \cdot R^{-2s-1+\varepsilon}}{1 + c_6(\lambda_-)\mathcal{A}(\frac{1}{R})}.$$

Since

$$-\mathcal{P}(s) \leq \frac{2\lambda_-}{s} - \frac{2\lambda_-}{s}\Big(c_7 \mathcal{A}\Big(\frac{1}{s}\Big) + s^{-\varepsilon}\Big), \qquad \forall \delta > 0;$$
$$\mathcal{Q}(R) \leq -\frac{\lambda_-}{s} c_0(G)\bigg(f_1^2 + \frac{1}{\gamma_0} g_1^2\bigg) \cdot R^{-2s-1+\varepsilon},$$

hence we have

$$-\int_d^R \mathcal{P}(s)\,ds \leq 2\lambda_- \ln\Big(\frac{R}{d}\Big) - 2\lambda_- \int_d^\infty \frac{s^{-\varepsilon} + c_7\mathcal{A}(\frac{1}{s})}{s}\,ds$$
$$\Longrightarrow \exp\bigg(-\int_d^R \mathcal{P}(s)\,ds\bigg) \leq \Big(\frac{R}{d}\Big)^{2\lambda_-} \exp\bigg(-2\lambda_- \int_d^\infty \frac{s^{-\varepsilon} + c_7\mathcal{A}(\frac{1}{s})}{s}\,ds\bigg)$$
$$= K_0\Big(\frac{R}{d}\Big)^{2\lambda_-},$$

where

$$K_0 = \exp\Big(-2\lambda_- \frac{d^{-\varepsilon}}{\varepsilon}\Big) \cdot \exp\bigg\{-2\lambda_- c_7 \int_d^\infty \frac{\mathcal{A}(\frac{1}{s})}{s}\,ds\bigg\}.$$

We observe that $\int_d^\infty \frac{\mathcal{A}(\frac{1}{s})}{s}\,ds$ is finite, because setting $t = \frac{1}{s}$ yields

$$\int_d^\infty \frac{\mathcal{A}(\frac{1}{s})}{s}\,ds = \int_0^{\frac{1}{d}} \frac{\mathcal{A}(t)}{t}\,dt < \infty.$$

Thus we get: $U_0 \exp(-\int_d^R P(s)\,ds) \leq K_0 U_0 R^{2\lambda_-}$.
We have also

$$\int_d^R \mathcal{Q}(t) \exp\bigg(-\int_t^R \mathcal{P}(s)\,ds\bigg)\,dt$$
$$\leq -\frac{\lambda_-}{s} c_0(G)\bigg(f_1^2 + \frac{1}{\gamma_0} g_1^2\bigg) R^{2\lambda_-} K_0 \int_R^r t^{-2s-2\lambda_-+\varepsilon-1}\,dt$$
$$\leq -\frac{\lambda_- c_0}{s}\bigg(f_1^2 + \frac{1}{\gamma_0} g_1^2\bigg) \cdot \frac{1}{s+\lambda_-} \cdot K_0 R^{\lambda_- - s},$$

since $s > -\lambda_-$ and we can choose $\varepsilon = s + \lambda_-$.

Now we apply Theorem 2.10: then from (2.14) by virtue of deduced inequalities and with regard to (2.6) for $\alpha = 4-n$, we obtain the statement of 4.3 for $s > -\lambda_-$.

2) $s = -\lambda_-$

Considering δ as a positive function $\delta(R) > 0$ of R we obtain

$$\mathcal{P}(R) = -\frac{2\lambda_-(1-\delta(R))}{R(1+c_6\mathcal{A}(\frac{1}{R}))} + c_5\frac{\mathcal{A}(\frac{1}{R})}{1+c_6\mathcal{A}(\frac{1}{R})}\cdot\frac{2\lambda_-}{R}$$
$$\geq -\frac{2\lambda_-(1-\delta)}{R(1+c_6\mathcal{A}(\frac{1}{R}))} + \frac{2\lambda_- c_5\mathcal{A}(\frac{1}{R})}{R}.$$

We have

$$-\frac{2\lambda_-(1-\delta)}{R(1+c_6\mathcal{A}(\frac{1}{R}))} = -\frac{2\lambda_-(1-\delta)}{R}\left[1-\frac{c_6\mathcal{A}(\frac{1}{R})}{1+c_6\mathcal{A}(\frac{1}{R})}\right]$$
$$\geq -\frac{2\lambda_-(1-\delta)}{R} + \frac{2\lambda_-(1-\delta)c_6\mathcal{A}(\frac{1}{R})}{R}$$
$$\geq -\frac{2\lambda_-(1-\delta)}{R} + \frac{2\lambda_- c_6\mathcal{A}(\frac{1}{R})}{R}$$
$$\Longrightarrow\ -P(R) \leq \frac{2\lambda_-(1-\delta)}{R} - \frac{2\lambda_- c_7\mathcal{A}(\frac{1}{R})}{R}.$$

If we choose $\delta(R) = -\frac{1}{2\lambda_- \ln R}$, then we obtain

$$-\mathcal{P}(s) \leq \frac{2\lambda_-}{s} + \frac{1}{s\ln s} - \frac{2\lambda_- c_7\mathcal{A}(\frac{1}{s})}{s}$$
$$\Longrightarrow\ -\int_d^R \mathcal{P}(s)\,ds \leq 2\lambda_-\ln\frac{R}{d} + \ln(\ln s)\Big|_d^R - 2\lambda_- c_7\int_d^R \frac{\mathcal{A}(\frac{1}{s})}{s}\,ds.$$

We get

$$\exp\left(-\int_d^R \mathcal{P}(s)\,ds\right) \leq \left(\frac{R}{d}\right)^{2\lambda_-}\cdot\frac{\ln R}{\ln d}\exp\left(c_8\int_d^\infty \frac{\mathcal{A}(\frac{1}{s})}{s}\,ds\right) \leq K_1\frac{\ln R}{\ln d}\cdot\left(\frac{R}{d}\right)^{2\lambda_-},$$

where $K_1 = \exp(c_8\int_0^{\frac{1}{d}}\frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma)$. Further, because of $s = -\lambda_-$, we have

$$\mathcal{Q}(R) = c_0\left(f_1^2 + \frac{1}{\gamma_0}g_1^2\right)\frac{R^{2\lambda_- -1}\cdot\delta^{-1}}{1+c_6\mathcal{A}(\frac{1}{R})} \leq c_9\left(f_1^2+\frac{1}{\gamma_0}g_1^2\right)\cdot R^{2\lambda_- -1}\ln R$$
$$\Longrightarrow\ \int_d^R \mathcal{Q}(t)\exp\left(-\int_t^R \mathcal{P}(s)ds\right)dt$$
$$\leq c_9\left(f_1^2+\frac{1}{\gamma_0}g_1^2\right)K_1 R^{2\lambda_-}\ln R\int_d^R t^{2\lambda_- -1}t^{-2\lambda_-}\,dt$$
$$\leq K_1c_9\left(f_1^2+\frac{1}{\gamma_0}g_1^2\right)R^{2\lambda_-}\ln^2 R, \qquad \text{since } d \gg 1.$$

Now we apply Theorem 2.10 and from (2.14) by virtue of deduced inequalities we obtain

$$U(R) \le K_1 c_9 \Big(U_0 + f_1^2 + \frac{1}{\gamma_0} g_1^2 \Big) R^{2\lambda_-} \ln^2 R.$$

Thus we have proved the statement of 4.3 for $s = -\lambda_-$.

3) $0 < s < -\lambda_-$

For any positive δ, we have

$$\begin{aligned} P(R) &= -\frac{2\lambda_-}{R} \Big\{ (1-\delta) - \frac{(c_5 + c_6 - c_6\delta)\mathcal{A}(\frac{1}{R})}{1 + c_6 \mathcal{A}(\frac{1}{R})} \Big\} \\ &= -\frac{2\lambda_-(1-\delta)}{R} + \frac{2\lambda_-}{R} \cdot \frac{(c_5 + c_6 - c_6\delta)\mathcal{A}(\frac{1}{R})}{1 + c_6 \mathcal{A}(\frac{1}{R})} \end{aligned}$$

$$\begin{aligned} \Longrightarrow\ -\int_d^R P(s)\,ds &= 2\lambda_-(1-\delta)\ln\frac{R}{d} - 2\lambda_-(c_5 + c_6 - \delta)\int_d^R \frac{1}{s} \cdot \frac{\mathcal{A}(\frac{1}{s})}{1 + c_6\mathcal{A}(\frac{1}{s})}\,ds \\ &\le 2\lambda_-(1-\delta)\ln\frac{R}{d} - 2\lambda_-(c_5 + c_6)\int_d^\infty \frac{1}{s}\mathcal{A}\Big(\frac{1}{s}\Big)\,ds \end{aligned}$$

$$\Longrightarrow\ \exp\Big(-\int_d^R \mathcal{P}(s)\,ds \Big) \le K_2 \Big(\frac{R}{d}\Big)^{2\lambda_-(1-\delta)},$$

where $K_2 = \exp(-2\lambda_-(c_5 + c_6)\int_0^1 \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma)$ and

$$\exp\Big(-\int_t^R \mathcal{P}(s)\,ds \Big) \le K_2 \Big(\frac{R}{t}\Big)^{2\lambda_-(1-\delta)}, \qquad \forall \delta > 0.$$

For $\delta \in (0, \frac{s+\lambda_-}{\lambda_-})$, we obtain

$$\begin{aligned} &\int_d^R \mathcal{Q}(t) \exp\Big(-\int_t^R \mathcal{P}(s)\,ds \Big)\,dt \\ &\quad \le -\frac{\lambda_-}{s} \cdot c_0 \Big(f_1^2 + \frac{1}{\gamma_0} g_1^2 \Big) K_2 \cdot \delta^{-1} R^{2\lambda_-(1-\delta)} \int_R^r \frac{t^{-2\lambda_-(1-\delta)-2s-1}}{1 + c_6\mathcal{A}(\frac{1}{t})}\,dt \qquad (4.32) \\ &\quad \le -\frac{\lambda_-}{s} K_2 c_0 \Big(f_1^2 + \frac{1}{\gamma_0} g_1^2 \Big) \cdot \delta^{-1} R^{2\lambda_-(1-\delta)} \cdot \frac{R^{-2\lambda_-(1-\delta)-2s} - d^{-2\lambda_-(1-\delta)-2s}}{-2\lambda_-(1-\delta) - 2s} \\ &\quad \le \frac{c_0 K_2 \lambda_-}{2\delta s(s + \lambda_- - \delta\lambda_-)} \Big(f_1^2 + \frac{1}{\gamma_0} g_1^2 \Big) \cdot R^{-2s}. \end{aligned}$$

Now we apply Theorem 2.10 and we use of (2.14),

$$U(R) \le U_0 K_2 \Big(\frac{R}{d}\Big)^{2\lambda_-(1-\delta)} + \frac{c_0 K_2 \lambda_-}{2\delta s(s + \lambda_- - \delta\lambda_-)} \Big(f_1^2 + \frac{1}{\gamma_0} g_1^2 \Big) R^{-2s}$$

$$\le C(\lambda_-, d, s)K_2\left(U_0 + f_1^2 + \frac{1}{\gamma_0}g_1^2\right)R^{-2s},$$

since $R^{2\lambda_-(1-\delta)} < R^{-2s}$ for $\delta \in (0, \frac{s+\lambda_-}{\lambda_-})$.
Thus Theorem is proved for $0 < s < -\lambda_-$.

5. The power modulus of continuity near infinity for weak solutions

Proof of Theorem 1.4.
We define the function

$$\psi(R) = \begin{cases} R^{\lambda_-}, & \text{if } s > -\lambda_-; \\ R^{\lambda_-}\ln R, & \text{if } s = -\lambda_-; \\ R^{-s}, & \text{if } 0 < s < -\lambda_-. \end{cases}$$

Theorem 4.1 devoted to the local bound of the weak solution modulus, we have

$$\sup_{G^{2R}_{3/2}R} |u(x)| \le C\Big\{R^{\frac{-n}{2}}\|u\|_{2,G^{2R}_R} + R^{2(1-\frac{n}{p})}\|f\|_{\frac{p}{2},G^{2R}_R} + R\|g\|_{\infty,\Gamma^{2R}_R}\Big\}, \tag{5.1}$$

where $C = C(n, \nu, \mu, p, \|\sum_{i=1}^n |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)})$ and $2n \ge p > n \ge 2$. Now by Theorem 4.3 we have

$$\begin{aligned} R^{\frac{-n}{2}}\|u\|_{2,G^{2R}_R} &\le 2^{\frac{n}{2}}\left(\int_{G^{2R}_R} r^{-n}u^2(x)\,dx\right)^{\frac{1}{2}} \\ &\le C\left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}}g_1\right)\psi(R). \end{aligned} \tag{5.2}$$

Further, by the assumption (d), we obtain

$$R^{2(1-\frac{n}{p})}\|f\|_{\frac{p}{2},G^{2R}_R} + R\|g\|_{\infty,\Gamma^{2R}_R} \le c\left(f_1 + \frac{1}{\sqrt{\gamma_0}}g_1\right)\psi(R). \tag{5.3}$$

It follows from (5.1)–(5.3) that

$$\sup_{G^{2R}_{3/2}R} |u(x)| \le C\left\{\|u\|_{2,G} + f_1 + \frac{1}{\gamma_0}g_1\right\}\psi(R).$$

Putting now $|x| = \frac{7}{4}R$ we finally obtain the desired estimate (1.2).

Now we consider two sets $G^{2R}_{R/4}$ and $G^{R}_{R/2} \subset G^{2R}_{R/4}$. Changing the variables $x = Rx'$ and $u(Rx') = \psi(R)v(x')$. We see that the function $v(x')$ satisfies the problem

$$\begin{cases} \dfrac{\partial}{\partial x'_i}(a^{ij}(Rx')v_{x'_j}) + Rb^i(Rx')v_{x'_i} + R^2c(Rx')v = \dfrac{R^2}{\psi(R)}f(Rx'), & x \in G^1_{1/4}; \\ \alpha(Rx')\dfrac{\partial v}{\partial \nu'} + \dfrac{1}{|x'|}\gamma(\omega)v(x') = \dfrac{R}{\psi(R)}g(Rx), & x \in \Gamma^1_{1/4}. \end{cases} \tag{L''}$$

By the Sobolev Imbedding Theorem we have

$$\sup_{x'\in G^1_{1/2}} |\nabla' v(x')| \le c\|v\|_{W^{2,p}(G^1_{1/2})}, \qquad p>n. \tag{5.4}$$

By the local L^p–a priori estimate [8, 9], for the solution of equation (L'') inside the domain $G^2_{1/4}$ and near smooth portions of the boundary $\Gamma^2_{1/4}$ we have

$$\begin{aligned} &\|u\|_{W^{2,p}(G^1_{1/2})} \\ &\quad \le c\left\{ \frac{R^2}{\psi(R)}\|f\|_{L^p(G^2_{1/4})} + \frac{R}{\psi(R)}\|g\|_{W^{1-\frac{1}{p},p}(\Gamma^2_{1/4})} + \|v\|_{L^p(G^2_{1/4})}\right\}. \end{aligned} \tag{5.5}$$

Return to the variable x. It follows from (5.4) and (5.5), that

$$\begin{aligned} &\sup_{G^R_{R/2}} |\nabla u| \\ &\quad \le cR^{-1}\left\{ R^{\frac{-n}{p}}\|u\|_{L^p(G^{2R}_{R/4})} + R^{2-\frac{n}{p}}\|f\|_{p,G^{2R}_{R/4}} + R^{2-\frac{n}{p}}\|g\|_{V^{1-\frac{1}{p}}_{p,0}(\Gamma^{2R}_{R/4})}\right\} \end{aligned}$$

and

$$\begin{aligned} &R^{2-\frac{n}{p}}\|u\|_{V^2_{p,0}(G^R_{R/2})} \\ &\quad \le c\left\{ R^{\frac{-n}{p}}\|u\|_{L^p(G^{2R}_{R/4})} + R^{2-\frac{n}{p}}\|f\|_{p,G^{2R}_{R/4}} + R^{2-n/p}\|g\|_{V^{1-\frac{1}{p}}_{p,0}(\Gamma^{2R}_{R/4})}\right\} \end{aligned}$$

or

$$\sup_{G^R_{R/2}} |\nabla u| \le cR^{-1}\left\{ |u|_{0,G^{2R}_{R/4}} + \|f\|_{V^0_{p,2p-N}(G^{2R}_{R/4})} + \|g\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^{2R}_{R/4})}\right\}$$

and

$$\|u\|_{V^2_{p,2p-n}(G^R_{R/2})} \le c\left\{ |u|_{0,G^{2R}_{R/4}} + \|f\|_{V^0_{p,2p-N}(G^{2R}_{R/4})} + \|g\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^{2R}_{R/4})}\right\}.$$

Hence, because of (1.2), (1.3) and the assumption (d), (1.4) and (1.5) holds.

6. Examples

We present examples that show that the conditions of Theorem 1.4 (in particular the Dini condition for the function $\mathcal{A}(\frac{1}{r})$) are essential for their validity. Suppose that $n=2$, the domain G lies inside the corner

$$G_d = \left\{(r,\omega)|\ r>d;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2};\ \omega_0 \in (0,\pi)\right\}.$$

We denote

$$\Gamma^{\pm}_d = \left\{(r,\omega)|\ r>d,\ \omega = \pm\frac{\omega_0}{2}\right\}$$

and we put

$$\gamma(\omega)\big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = const > 0 \quad \text{and} \quad \alpha(x)\big|_{\Gamma^\pm_d} = \alpha_\pm \in \{0,1\}.$$

I. We consider the following problem

$$\begin{cases} \Delta u = 0, \qquad x \in G_d; \\ \Big(\alpha_\pm \dfrac{\partial u}{\partial n} + \dfrac{1}{r}\gamma_\pm u\Big)\Big|_{\Gamma_d^\pm} = 0. \end{cases}$$

We verify that the function $u(r,\omega) = r^{\lambda_-}\psi(\omega)$, $\lambda_- < 0$ is a solution of our problem if λ_-^2 is the least positive eigenvalue of the problem

$$\begin{cases} \psi'' + \lambda_-^2\psi = 0, \qquad \omega \in (-\frac{\omega_0}{2}, \frac{\omega_0}{2}); \\ (\pm\alpha_\pm\psi' + \gamma_\pm\psi)\Big|_{\omega=\pm\frac{\omega_0}{2}} = 0 \end{cases}$$

and $\psi(\omega)$ is a regular eigenfunction associated with λ_-^2.
The solution of our equation has the form $\psi(\omega) = A\cos(\omega\lambda_-) + B\sin(\omega\lambda_-)$. In order to find A, B from the boundary conditions we obtain the system:

$$\begin{cases} \big(\gamma_+\cos\frac{\omega_0\lambda_-}{2} - \lambda_-\alpha_+\sin\frac{\omega_0\lambda_-}{2}\big)A + \big(\gamma_+\sin\frac{\omega_0\lambda_-}{2} + \lambda_-\alpha_+\cos\frac{\omega_0\lambda_-}{2}\big)B = 0, \\ \big(\gamma_-\cos\frac{\omega_0\lambda_-}{2} - \lambda_-\alpha_-\sin\frac{\omega_0\lambda_-}{2}\big)A - \big(\gamma_-\sin\frac{\omega_0\lambda_-}{2} + \lambda_-\alpha_-\cos\frac{\omega_0\lambda_-}{2}\big)B = 0. \end{cases}$$

Since $A^2 + B^2 \neq 0$, the system determinant must be equal to zero; this means that $\lambda_- < 0$ is defined via the transcendence equation

$$\begin{aligned} (\lambda_-^2\alpha_-\alpha_+ - \gamma_+\gamma_-)\cdot\sin(\omega_0\lambda_-) - \lambda_-(\alpha_+\gamma_- + \alpha_-\gamma_+)\cdot\cos(\omega_0\lambda_-) = 0 \\ \Longrightarrow\ \tan(\omega_0\lambda_-) = \tfrac{\lambda_-(\alpha_+\gamma_- + \alpha_-\gamma_+)}{\lambda_-^2\alpha_+\alpha_- - \gamma_+\gamma_-}. \end{aligned} \tag{6.1}$$

Then we find the eigenfunction

$$\psi(\omega) = \lambda_-\alpha_+\cos\Big[\lambda_-\Big(\omega - \frac{\omega_0}{2}\Big)\Big] - \gamma_+\sin\Big[\lambda_-\Big(\omega - \frac{\omega_0}{2}\Big)\Big]. \tag{6.2}$$

Now we investigate some particular cases of the boundary conditions. Such equations were systematically studied in [7].

Dirichlet problem: $\alpha_\pm = 0$, $\gamma_\pm = 1$.

Equation (6.1) becomes $\tan(\omega_0\lambda_-) = 0$. Hence, $\lambda_- = -\frac{\pi}{\omega_0}$ and the corresponding eigenfunction $\psi(\omega) = \cos(\frac{\pi\omega}{\omega_0})$.

Neumann problem: $\gamma_\pm = 0$, $\alpha_\pm = 1$.
Equation (6.1) becomes $\tan(\omega_0\lambda_-) = 0$. Hence, $\lambda_- = -\frac{\pi}{\omega_0}$ and the corresponding eigenfunction $\psi(\omega) = \sin(\frac{\pi\omega}{\omega_0})$.

Mixed problem: $\alpha_+ = 1$ $\alpha_- = 0$; $\gamma_+ = 0$, $\gamma_- = 1$.

Equation (6.1) becomes $\cos(\omega_0\lambda_-) = 0$. Hence, $\lambda_- = -\frac{\pi}{2\omega_0}$ and the corresponding eigenfunction $\psi(\omega) = \cos(\frac{\pi\omega}{2\omega_0} - \frac{\pi}{4})$.

Robin problem: $\alpha_\pm = 1$, $\gamma_+ \neq 0$, $\gamma_- \neq 0$.

In this case we obtain the largest eigenvalue as the largest negative root of the transcendence equation $\tan(\omega_0\lambda_-) = \frac{\lambda_-(\gamma_-+\gamma_+)}{\lambda_-^2-\gamma_+\gamma_-}$ and the corresponding eigenfunction

$$\psi(\omega) = \lambda_- \cos\left[\lambda_-\left(\omega - \frac{\omega_0}{2}\right)\right] - \gamma_+ \sin\left[\lambda_-\left(\omega - \frac{\omega_0}{2}\right)\right].$$

In particular, if $\gamma_+ = \gamma_-$, we get $\psi(\omega) = \cos(\omega\lambda_-^*)$, where λ_-^* is the largest negative root of the transcendence equation $\tan(\frac{\omega_0\lambda_-}{2}) = \frac{\gamma}{\lambda_-}$. We denote $-\lambda_- = \lambda > 0$. Then we have $\tan(\frac{\omega_0\lambda}{2}) = \frac{\gamma}{\lambda}$ and $0 < -\lambda_-^* < \frac{\pi}{\omega_0} \implies -\frac{\pi}{\omega_0} < \lambda_-^* < 0$ (see Figure 1).

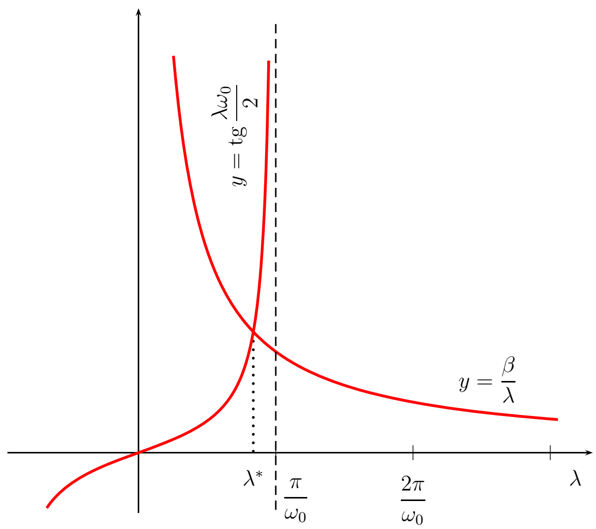

Figure 1

II. The function

$$u(r,\omega) = r^{\lambda_-} \ln^{\frac{-\lambda_-+1}{-\lambda_--1}}\left(\frac{1}{r}\right) \cdot \psi(\omega)$$

with $\lambda_- < -1$ and $\psi(\omega)$ defined by (6.1)–(6.2) is a solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}) + b^i(x)u_{x_i} = 0, & x \in G_d; \\ \left(\alpha(x)\dfrac{\partial u}{\partial n} + \dfrac{1}{r}\gamma_\pm u\right)\Big|_{\Gamma_d^\pm} = 0, & \gamma_\pm > 0 \end{cases}$$

in the domain G_d, where

$$a^{11}(x) = 1 - \frac{2}{1+\lambda_-} \cdot \frac{x_2^2}{r^2 \ln\frac{1}{r}},$$

$$a^{22}(x) = 1 - \frac{2}{1+\lambda_-} \cdot \frac{x_1^2}{r^2 \ln \frac{1}{r}},$$
$$a^{12}(x) = a^{21}(x) = \frac{2}{1+\lambda_-} \cdot \frac{x_1 x_2}{r^2 \ln \frac{1}{r}};$$
$$\lim_{|x|\to\infty} a^{ij}(x) = \delta_i^j, \qquad (i,j = 1,2);$$

and

$$b^1 = -\frac{1}{r}\mathcal{A}\Big(\frac{1}{r}\Big)\cos\omega, \qquad b^2 = -\frac{1}{r}\mathcal{A}\Big(\frac{1}{r}\Big)\sin\omega.$$

In the domain G_d^∞ for $d > \exp(\frac{2}{-\lambda_- -1})$, equation is uniformly elliptic with the ellipticity constant $\mu = 1$, $\nu = 1 + \frac{2}{(\lambda_- +1)\ln d}$. Further, $\mathcal{A}(\frac{1}{r}) = \frac{2}{\lambda_- +1}\ln^{-1}(\frac{1}{r})$, i.e., the function $\mathcal{A}(r)$ does not satisfy the Dini condition at zero. Moreover, $a^{ij}(x)$ are continuous at the infinity. This example shows that the condition of Theorem 1.4 about Dini-continuity of the leading coefficients of the (L) are essential. It also illustrates the precision of the assumption of Theorem 1.4.

III. The function

$$u(r,\omega) = r^{\lambda_-}\psi(\omega)\ln r$$

with $\lambda_- < 0$ and $\psi(\omega)$ defined by (6.1)–(6.2) is a solution of the problem

$$\begin{cases} \Delta u = 2\lambda_- r^{\lambda_- -2}\psi(\omega), & x \in G_d; \\ \Big(\alpha(x)\dfrac{\partial u}{\partial n} + \dfrac{1}{r}\gamma_\pm u\Big)\Big|_{\Gamma_d^\pm} = 0, & \gamma_\pm > 0 \end{cases}$$

in the domain G_d. All assumptions of Theorem 1.4 are fulfiled with $s = -\lambda_-$. This example shows the precision of the assumption for the right hand side of (L) in Theorem 1.4.

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References

[1] V.I. Burenkov, *Sobolev Spaces on Domains*, Teubner–Texte zur Mathematik [Teubner Texts in Mathematics] 137, B.G. Teubner Verlagsgesellschaft, Stuttgart, 1998.

[2] Ya-Zhe Chen, Lan-Cheng Wu, *Second order elliptic equations and elliptic systems*, Translations of Mathematical Monographs, **174** (1998), American Mathematical Society, Providence, RI, 246p.

[3] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224 Springer–Verlag, Berlin–New York, 1977 (Revised Third Printing, 1998).

[4] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, University Press, Cambridge, 1952.

[5] V.A. Kozlov, V.G. Maz'ya, J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, vol. **52**, American Mathematical Society, Providence, RI, 1997.

[6] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear elliptic equations*, Academic Press, New York–London, 1968.

[7] V. Mityushev, P. Adler, *Darcy flow around a two-dimensional lens*, J. Phys. A. **39**, (2006), 3545–3560.

[8] Z.G. Sheftel, *Estimates in L_p of solutions of elliptic equations with discontinuous coefficients and satisfying general boundary conditions and conjugacy conditions*, Soviet Math. Dokl. **4** (1963), 321–324.

[9] Z.G. Sheftel, *Energy inequalities and general boundary problems for elliptic equations with discontinuous coefficients*, Sibirsk. Mat. Ž. **6** (1965), 636–668.

[10] G.E. Shilov, I.M. Gel'fand, *Generalized functions*, vol. I. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964.

[11] S.L. Sobolev, *Some applications of functional analysis in mathematical physics*, Translations of Mathematical Monographs, vol. **90**, American Mathematical Society, Providence, RI, 1991.

[12] T.S. Soboleva, *Boundary value problems for a second-order elliptic equation in unbounded domains*, Sibirsk. Mat. Ž. **19** (1978), no. 5, 1184–1195, 1216.

[13] T.S. Soboleva, *Some properties of second order elliptic equations in unbounded domains*, Uspehi Mat. Nauk **31** (1976), no. 5 (191), 253–254.

[14] M. Troisi, *A regularity theorem for solutions of elliptic boundary value problems in unbounded domains in R^n*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **11** (1987), no.1, 127–137.

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