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Regulated functions and integrability

Abstract. Properties of functions defined on a bounded closed interval, weaker than continuity, have been considered by many mathematicians. Functions having both sides limits at each point are called regulated and were considered by J. Dieudonné [2], D. Fraňková [3] and others (see for example S. Banach [1], S. Saks [8]). The main class of functions we deal with consists of piece-wise constant ones. These functions play a fundamental role in the integration theory which had been developed by Igor Kluvanek (see Š. Tkacik [9]). We present an outline of this theory.

1. Regulated functions

Everybody familiar with basic calculus remembers properties of continuous functions defined on a bounded closed interval. Some of those properties can be extended to suitably discontinuous functions, namely to functions having the right and the left limits at each point; such functions are called regulated. We shall deal with a special class of regulated functions consisting of piece-wise constant functions.

From now on, I will denote a closed bounded interval $[a, b]$ of real numbers. All considered functions will be bounded and defined in the interval I.

A limit of a function is meant to be proper, i.e., different from $+\infty$ or $-\infty$.

Definition 1

A function $f\colon I \to \mathbb{R}$ is called regulated on I if f has the left-sided limit at every point of the interval I except the point a and f has the right-sided limit at every point of the interval I except the point b.

The idea of regulated functions can be spread out to functions defined in a subset of the interval I, namely to a set E, such that each point from the interval I is left-sided and right-sided accumulation point of the set E. Nevertheless we are not concerned to such approach.

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In this definition we do not require that the right-sided limit and the left-sided limit of the function at a point are equal. The picture below shows an example of a regulated function on I.

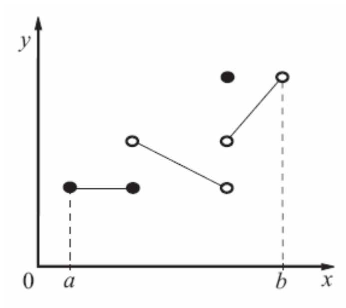

Figure 1

Important class of regulated functions consists of piece-wise constant ones.

Definition 2
A function $f\colon I \to \mathbb{R}$ is said to be a step function on I whenever there exist: a positive integer n, a sequence of points $(c_1, \ldots, c_n)$ such that

$$a = c_0 < c_1 < \ldots < c_{j-1} < c_j < \ldots < c_n = b$$

and the function f is constant on each interval (c_{j-1}, c_j), $j = 1, 2, \ldots, n$.

An example of a step function is shown in Figure 2.

It follows from the definition that if f is regulated on an interval I, then it is also regulated in each subinterval J ($J \subseteq I$).

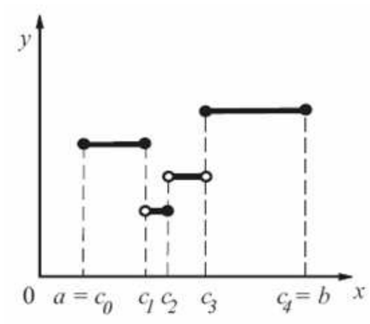

Figure 2

Although the next theorem is known (see [2] for example) we shall present an elementary proof of it.

This theorem states that a regulated function can be aproximated with arbitrary accuracy by a step function.

Theorem 1
Let $f\colon I \to \mathbb{R}$ be a regulated function and let ε be a positive number. Then there exists a step function g such that

$$|f(x) - g(x)| < \varepsilon$$

in each point x of the interval I.
If the function f is continuous on the interval I, then we can choose the function g to be right-continuous at each point of the interval $[a, b)$ or to be left-continuous at each point of the interval $(a, b]$.

Proof. Let Z be the set of all numbers z from the interval $I = [a, b]$ for which there exists a step function g_z such that

$$|f(x) - g_z(x)| < \varepsilon \tag{1}$$

for every $x \in [a, z]$. If the function f is continuous, then the step function g_z is assumed to be right-continuous at every point of the interval $[a, z)$. Our aim is to show that $b \in Z$. If it is so we take g_b for g.

We shall do that by showing that the supremum of the set Z belongs to Z and that it is equal to b. The set Z has a supremum because it is not empty (the number a surely belongs to it) and bounded from above (no element of the set Z is greater than b). Then, let $s = \sup Z$.

1. We prove that $s \in Z$.

If $s = a$, then $s \in Z$. So now assume that $a < s$. Then the function f has a left limit k at s and for a positive number ε there exists a number $c < s$ such that

$$|f(x) - k| < \varepsilon \tag{2}$$

for every $x \in (c, s)$. Since $c < s$, there exists a number $z \in Z$ such that $c < z$.

Let g_z be a step function such that (1) holds for every $x \in [a, z]$ and, if the function f is continuous in $[a, b]$, let g_z be left-continuous at every point of the interval $[a, z)$. Define the function g_s by letting $g_s(s) = f(s)$, provided s belongs to the domain of f, further $g_s(x) = k$ for every $x \in [z, s)$ and, finally, $g_s(x) = g_z(x)$ for every $x \in [a, z]$. Then g_s is a step function such that (1) holds for every $x \in [a, s]$ and if the function f happens to be continuous in $[a, b]$, then g_s is right-continuous at every point of the interval $[a, s)$. Hence, $s \in Z$.

2. We prove that $s = b$.

Assume to the contrary that $s < b$.

Since $s < b$, the function f has a right-sided limit k at s and there exists a number $d > s$ such that (2) holds for every $x \in (s, d)$. As $s \in Z$, there exists a step function g_s such that (1) holds for every $x \in [a, s]$ and g_s is left-continuous at every point of the interval $[a, s)$ in case when f is continuous in $[a, b]$. Let z be a number such that $s < z < d$. Let $g_z(x) = g_s(x)$ for every $x \in [a, s)$; let $g_z(s) = f(s)$; and let $g_z(x) = k$ for every $x \in (s, z)$. Then g_z is a step function such that (1) holds for every $x \in [a, z]$ and g_z is right-continuous at every point of the interval $[a, z)$ if the function f is continuous in $[a, b]$. Hence $z \in Z$, which is a contradiction since $s < z$ and $s = \sup Z$.

Similar arguments can be used for the case, when we want the function g to be left-continuous, simply apply the previous argument to the function $f(-x)$, when $x \in [-b,-a]$.

2. Examples

Example 1
During the first 19 weeks of the financial year, the wage of an employee was 186 Euro weekly. Than he was promoted and had 203,50 Euro weekly. A month before the end of the financial year, due to general salaries and wages increase, his wage was increased to 211,30 Euro weekly. This last month represents 4,4 working weeks (four full weeks and two working days, each representing 0,2 of a working week). Indicate how the weekly wage depends on time.

If we want to introduce a function indicating how the weekly wage of the employee depended on time we represent the year by the interval $[0,52]$, taking a week for a unit of time. The function f representing the dependence of the wage on time can then be defined in the following manner:

$$f(t) = \begin{cases} 186 & \text{for } t \in [0,19], \\ 203,50 & \text{for } t \in \left(19, 47\frac{3}{5}\right), \\ 211,30 & \text{for } t \in \left[47\frac{3}{5}, 52\right]. \end{cases}$$

If $\chi_A(t)$ is a characteristic function of the set A, then we have

$$f(t) = 186 \cdot \chi_{[0,19]}(t) + 203,50 \cdot \chi_{(19,47\frac{3}{5})}(t) + 211,30 \cdot \chi_{[47\frac{3}{5},52]}(t)$$

for every $t \in [0,52]$.

Now we can ask what was the average (mean) wage of that employee during the year or what was his total income from wages in that year? Clearly, his total income was

$$186 \cdot 19 + 203,50 \cdot (47,6 - 19) + 211,30 \cdot (52 - 47,6) = 10283,82$$

Euro. His average wage was

$$\frac{10283,82}{52} = 197,76$$

Euro per week (rounded to whole cents). In this example it is easy to see that the function f is a step function and it does not matter, if we use open or bounded intervals for calculating of the total income.

Here we defined $c_1 = 186$; $c_2 = 203,50$; $c_3 = 211,30$; $J_1 = [0,19]$, $J_2 = [19, 47\frac{3}{5}]$, $J_3 = [47\frac{3}{5}, 52]$. If the number $b - a = \lambda(J)$ is the length of the interval $J = [a,b]$, then the total income has the form

$$c_1\lambda(J_1) + c_2\lambda(J_2) + c_3\lambda(J_3) = \sum_{j=1}^{3} c_j\lambda(J_j).$$

This number is also the area of the set $S = \{(t,y) : \ t \in [0,52],\ 0 \le y \le f(t)\}$.

Therefore, it is possible to express the step function by the formula

$$f(x) = \sum_{j=1}^{n} c_j \chi_{J_j}(x)$$

for every x in an interval I, where n is a positive integer, c_j are arbitrary numbers and J_j some bounded intervals ($\bigcup_{j=1}^{n} J_j = I$) for every $j = 1, 2, 3, \ldots, n$. In each case, the number

$$\sum_{j=1}^{n} c_j \lambda(J_j)$$

is called the integral of the function f.

Example 2

Now, we try to calculate the area of the set

$$S = \{(x, y) : \; x \in I, \, 0 \le y \le f(x)\},$$

where f is some continuous and non-negative function in the (compact) interval I.

If the function f is not a step function in the interval I, then the set S is not equal to the union of finite number of rectangles. Nevertheless, with the exception of some points on the boundary, which may be disregarded when calculating the area, this set can be covered by an infinite sequence of non-overlapping rectangles as illustrated in Figure 3. The sum of the areas of these rectangles is equal to the area of S.

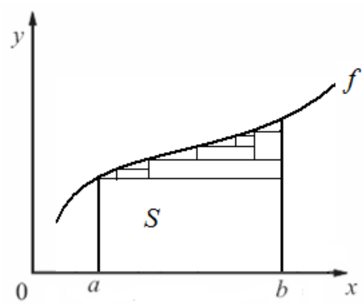

Figure 3

That is, there exist intervals $J_j \subset I$ and numbers c_j, $j = 1, 2, 3, \ldots$, such that

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x) \tag{3}$$

for every $x \in I$ and the area of set S is equal to the number

$$\sum_{j=1}^{\infty} c_j \lambda(J_j). \tag{4}$$

The class of functions to which the procedure can be applied is much larger than in the case when $c_j \ge 0$ for every $j = 1, 2, 3, \ldots$. In particular, we now may consider functions with both positive and negative values. Consequently, we can also calculate the integral (4) of a function f when it has an interpretation different from that of the area of a planar figure. Of course, if so desired, the integral of a function in an interval I can always be interpreted "geometrically" as a difference of the areas of the sets

$$S^+ = \{(x, y) : \ x \in I, \, 0 \leq y \leq f(x)\} \quad \text{and} \quad S^- = \{(x, y) : \ x \in I, \, f(x) \leq y \leq 0\}.$$

3. Definition of the integral

To obtain a workable definition of integral for a sufficiently large class of functions, it suffices to require the existence of the sum (4) and to note that this sum is then independent of the particular choice of the numbers c_j and intervals J_j, $j = 1, 2, 3, \ldots$, used in the representation (3) of the function f.

Definition 3
A function f is said to be integrable in the interval I whenever there exist numbers c_j and bounded intervals $J_j \subset I$, $j = 1, 2, 3, \ldots$, such that

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty \tag{5}$$

and the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for every $x \in I$ such that

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty. \tag{6}$$

Now we shall introduce the notions of a virtually primitive function. We shall use the term *a condition $\mathcal{P}$ is fulfilled nearly everywhere*. It means that the set of points for which the condition $\mathcal{P}$ is not fulfilled is at most countable.

Definition 4
A function F is said to be virtually primitive to a function f in an interval I, if the function F is continuous in the interval I and $F'(x) = f(x)$ nearly everywhere in I.

In this definition we do not require I to be a compact interval, it can be as well an unbounded interval.

We shall prove that if a function f is integrable in the interval I, then the sum (4) is the same for every choice of the numbers c_j and intervals J_j, $j = 1, 2, 3, \ldots$, satisfying the condition (5), such that (3) holds for every $x \in I$ for which the inequality (6) does hold.

The next three theorems, which are technical ones, are useful in the proof that the definition of the Kluvanek integral is correct.

Theorem 2

Let n be a positive integer, c_j non-negative numbers, J_j bounded subintervals of I, $j = 1, 2, 3, \ldots, n$, d_k non-negative numbers and K_k bounded intervals, $k = 1, 2, 3, \ldots$, such that

$$\sum_{j=1}^{n} c_j \chi_{J_j}(x) \leq \sum_{k=1}^{\infty} d_k \chi_{K_k}(x) \tag{7}$$

for every $x \in (-\infty, \infty)$. Then

$$\sum_{j=1}^{n} c_j \lambda(J_j) \leq \sum_{k=1}^{\infty} d_k \lambda(K_k). \tag{8}$$

Proof. It follows from the assumptions that a is a number not greater than the left end-point and b is a number not less than the right end-point of each of the intervals J_j, $j = 1, 2, 3, \ldots, n$. Let F_j be a function virtually primitive in $(-\infty, \infty)$ to the function $c_j \chi_{J_j}$ such that $F_j(a) = 0$, $j = 1, 2, 3, \ldots, n$, and G_k the function virtually primitive to $d_k \chi_{K_k}$ such that $G_k(a) = 0$, $k = 1, 2, 3, \ldots$. Since $c_j \lambda(J_j) = F_j(b)$, $j = 1, 2, 3, \ldots, n$, if we prove that

$$\sum_{j=1}^{n} F_j(b) \leq \sum_{k=1}^{\infty} G_k(b),$$

then (8) will follow.

Suppose to the contrary that

$$\sum_{k=1}^{\infty} G_k(b) < \sum_{j=1}^{n} F_j(b). \tag{9}$$

First note that $0 \leq G_k(x) \leq G_k(b)$ for every $x \in [a, b]$ and every $k = 1, 2, 3, \ldots$. Hence, by (9), the sequence of functions $\{G_k\}_{n=1}^{\infty}$ is uniformly convergent in the interval $[a, b]$. Let

$$F(x) = \sum_{j=1}^{n} F_j(x) \quad \text{and} \quad G(x) = \sum_{k=1}^{\infty} G_k(x)$$

for every $x \in [a, b]$. The functions $F_j(x)$, $j = 1, 2, 3, \ldots, n$, and $G_k(x)$, $k = 1, 2, 3, \ldots$, are continuous in the interval $[a, b]$. Therefore, the functions $F(x)$ and $G(x)$ are also continuous in the interval $[a, b]$ and, of course, $F(a) = G(a) = 0$. Let

$$k = \frac{F(b) - G(b)}{2(b-a)} \quad \text{and} \quad q = \frac{F(b) - G(b)}{2}.$$

By (9), $k > 0$ and $q > 0$. If $t \in (0, k)$, let

$$h_t(x) = F(x) - G(x) - t(x - a) - q$$

for every $x \in [a, b]$. Then, for every $t \in (0, k)$, h_t is a function continuous in the interval $[a, b]$ such that $h_t(a) < 0$ and $h_t(b) > 0$. Let $\xi(t)$ be its maximal root in the interval (a, b). That is $h_t(\xi(t)) = 0$ and $h_t(y) > 0$ for every $y \in (\xi(t), b)$.

The function $\xi(t), t \in (0,k)$, is (strictly) increasing, because if $0 < t < s < k$, then

$$h_s(\xi(t)) = h_s(\xi(t)) - h_t(\xi(t)) = (t-s)(\xi(t)-a) < 0$$

and, hence, the largest root, $\xi(s)$, of the function h_s is greater than $\xi(t)$. So, this function is injective. Since its domain, $(0,k)$, is not a countable set, the set of its values $\{\xi(t) :\ t \in (0,k)\}$ is not countable either. But the set of end-points of all intervals J_j, $j = 1,2,3,\dots,n$, and $K_k, k = 1,2,3,\dots$, is countable. So, there is a number $t \in (0,k)$ such that $\xi(t)$ is not an end-point of any of intervals J_j, $j = 1,2,3,\dots,n$, and $K_k, k = 1,2,3,\dots$. Let t be such a number and $x = \xi(t)$, the corresponding point of the interval (a,b). Then $h_t(x) = 0$ and $h_t(y) > 0$ for every $y \in (x,b)$. That is,

$$F(x) - G(x) = t(x-a) - q \quad \text{and} \quad F(y) - G(y) > t(y-a) - q$$

for every $y \in (x,b)$. Consequently,

$$\frac{F(y)-F(x)}{y-x} - \frac{G(y)-G(x)}{y-x} > t \tag{10}$$

for every $y \in (x,b]$.

On the other hand, since x is not an end-point of any of the intervals J_j and K_k, each function F_j and G_k is differentiable at x and $F_j'(x) = c_j\chi_{J_j}(x)$ for $j = 1,2,3,\dots,n$ and $G_k'(x) = d_k\chi_{K_k}(x)$ for $k = 1,2,3,\dots$. So, by (7),

$$F'(x) = \sum_{j=1}^{n} F_j'(x) \le \sum_{k=1}^{\infty} G_k'(x).$$

Since $t > 0$, there exists a positive integer m such that

$$F'(x) \le \sum_{k=1}^{\infty} G_k'(x) < \sum_{k=1}^{m} G_k'(x) + t.$$

Therefore,

$$\lim_{y\to x^+}\left(\frac{F(y)-F(x)}{y-x} - \sum_{k=1}^{m}\frac{G_k(y)-G_k(x)}{y-x}\right) < t.$$

From the properties of limits we have, that there exists a point y in the interval $[x,b]$ such that

$$\frac{F(y)-F(x)}{y-x} - \sum_{k=1}^{m}\frac{G_k(y)-G_k(x)}{y-x} < t. \tag{11}$$

Now, $G_k(y) - G_k(x) > 0$ for every $k = m+1, m+2, \dots$, because the functions G_k are non-decreasing. Hence,

$$\frac{G(y)-G(x)}{y-x} = \sum_{k=1}^{\infty}\frac{G_k(y)-G_k(x)}{y-x} \ge \sum_{k=1}^{m}\frac{G_k(y)-G_k(x)}{y-x}.$$

So, (11) contradicts (10).

THEOREM 3
Let c_j and d_j be non-negative numbers and let J_j and K_j be subintervals of I, $j = 1, 2, 3, \ldots$, such that

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) < \infty, \qquad \sum_{j=1}^{\infty} d_j \lambda(K_j) < \infty$$

and

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) \tag{12}$$

for every x for which

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) < \infty.$$

Then

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j). \tag{13}$$

Proof. Let ε be an arbitrary positive number. Let n be a positive integer such that

$$\sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \frac{\varepsilon}{2}.$$

Then

$$\sum_{j=1}^{n} c_j \chi_{J_j}(x) \leq \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) + \sum_{j=n+1}^{\infty} c_j \chi_{J_j}(x)$$

for every $x \in (-\infty, \infty)$ with no exception.

By Theorem 2,

$$\sum_{j=1}^{n} c_j \lambda(J_j) \leq \sum_{j=1}^{\infty} d_j \lambda(K_j) + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \sum_{j=1}^{\infty} d_j \lambda(K_j) + \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{\infty} c_j \lambda(J_j) &= \sum_{j=1}^{n} c_j \lambda(J_j) + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) \\
&< \sum_{j=1}^{\infty} d_j \lambda(K_j) + \frac{\varepsilon}{2} + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) \\
&< \sum_{j=1}^{\infty} d_j \lambda(K_j) + \varepsilon.
\end{aligned}$$

Because the inequality between the first and the last term holds for every positive ε, we have

$$\sum_{j=1}^{\infty} c_j\lambda(J_j) \leq \sum_{j=1}^{\infty} d_j\lambda(K_j).$$

The reverse inequality can be proved by a symmetric argument. Hence (13) holds.

Recall that nonnegative x^+ and nonpositive x^- parts of a number x are defined by

$$x^+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad x^- = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then: $x^+ \geq 0$, $x^- \geq 0$, $x = x^+ - x^-$ and $|x| = x^+ + x^-$ for any real number x.

Theorem 4
Let c_j and d_j be real numbers and let J_j and K_j, $j = 1, 2, \ldots,$ be subintervals of I such that

$$\sum_{j=1}^{\infty} |c_j|\lambda(J_j) < \infty, \qquad \sum_{j=1}^{\infty} |d_j|\lambda(K_j) < \infty. \tag{14}$$

If

$$\sum_{j=1}^{\infty} c_j\chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j\chi_{K_j}(x)$$

for every $x \in I$ for which

$$\sum_{j=1}^{\infty} |c_j|\chi_{J_j}(x) < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} |d_j|\chi_{K_j}(x) < \infty,$$

then

$$\sum_{j=1}^{\infty} c_j\lambda(J_j) = \sum_{j=1}^{\infty} d_j\lambda(K_j).$$

Proof. The conditions (14) imply:

$$\sum_{j=1}^{\infty} c_j^+\lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} c_j^-\lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} d_j^+\lambda(K_j) < \infty, \quad \sum_{j=1}^{\infty} d_j^-\lambda(J_j) < \infty.$$

From condition

$$\sum_{j=1}^{\infty} c_j\chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j\chi_{K_j}(x)$$

we have

$$\sum_{j=1}^{\infty} c_j^+\chi_{J_j}(x) - \sum_{j=1}^{\infty} c_j^-\chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j^+\chi_{K_j}(x) - \sum_{j=1}^{\infty} d_j^-\chi_{K_j}(x).$$

That is

$$\sum_{j=1}^{\infty} c_j^+ \chi_{J_j}(x) + \sum_{j=1}^{\infty} d_j^- \chi_{K_j}(x) = \sum_{j=1}^{\infty} d_j^+ \chi_{K_j}(x) + \sum_{j=1}^{\infty} c_j^- \chi_{J_j}(x)$$

for every x such that both sides represent a real number (not ∞). By Theorem 3

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) + \sum_{j=1}^{\infty} d_j^- \lambda(K_j) = \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) + \sum_{j=1}^{\infty} c_j^- \lambda(J_j),$$

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) - \sum_{j=1}^{\infty} c_j^- \lambda(J_j) = \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) - \sum_{j=1}^{\infty} d_j^- \lambda(K_j)$$

and

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

Now we are able to proceed with the definition of integral:

Definition 5
Let f be a function integrable in the interval I. Let c_j be numbers and let $J_j \subset I$ be intervals, $j = 1, 2, 3, \ldots$, satisfying the condition

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

such that the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for every $x \in I$ satisfying the condition

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty.$$

Then the number

$$\sum_{j=1}^{\infty} c_j \lambda(J_j)$$

is called the integral of f in the interval I; it will be denoted by $\int_I f(x)\,dx$.

Clearly, for every constant function $f(x) = \beta$ in the interval $[a, b]$ we have

$$\int_a^b f(x)\,dx = \beta(b-a).$$

4. Integration of regulated functions

The next theorem shows how to integrate regulated functions.

THEOREM 5
Let f be a regulated function in the interval $[a,b]$ $(a<b)$. Then f is integrable in this interval and

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any virtually primitive function to f in the interval $[a,b]$.

Proof. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a uniformly convergent sequence of step functions in the interval $[a,b]$ such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for every $x \in [a,b]$. This sequence exists from the theory of regulated and piecewise constant functions (see [5]). The functions $f_n(x)$ are bounded. Let

$$\beta_n = \sup\{|f_n(x)| : \ x \in I\}$$

for every $n = 1, 2, 3, \ldots$.

It follows from the uniform convergence of the sequence $\{f_n(x)\}_{n=1}^{\infty}$ that

$$\sum_{n=1}^{\infty} \beta_n < \infty. \tag{15}$$

For every $n = 1, 2, 3, \ldots$ we have

$$\int_a^b |f_n(x)|\,dx \le \int_a^b \beta_n\,dx = \beta_n(b-a).$$

From (15) we have $\sum_{n=1}^{\infty} \int_a^b |f_n(x)|\,dx < \infty$. The function f is integrable in the interval $[a,b]$ and

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx. \tag{16}$$

Let F_n be a function virtually primitive to the function f_n in the interval $[a,b]$ such that $F_n(a) = 0$ for $n = 1, 2, 3, \ldots$. The sum

$$F(x) = \sum_{n=1}^{\infty} F_n(x)$$

exists for every $x \in [a,b]$ and function F defined in this way is virtually primitive to f in $[a,b]$. Thus

$$\int_a^b f_n(x)\,dx = F_n(b) - F_n(a)$$

holds for every $n = 1, 2, 3, \ldots$. Hence by (16)

$$\int_a^b f_n(x)\,dx = \sum_{n=1}^{\infty} (F_n(b) - F_n(a)) = F(b) - F(a).$$

Since the difference of any two functions virtually primitive to f in $[a,b]$ is constant, the last equality holds for any function F virtually primitive to f in $[a,b]$.

5. Conclusions

Our aim was to provide an introduction to the theory of integral developed by Professor Igor Kluvanek during his stay at Flinders University in Adelaide (Australia). In his approach regulated functions play an important role (see I. Kluvánek [4]).

The definition of integral given in this article applies an idea of Archimedes. The most effective method for the calculation of integrals is the one which is based on differential calculus (see V.V. Mityushev, S.V. Rogosin [6] and W.F. Pfeffer [7]).

As everybody knows Dirichlet function (characteristic function of the set of rational numbers) is not integrable in Riemann sense. It is possible to show, that this function is integrable in the sense of I. Kluvanek and the value of this integral is zero. In fact, let $\mathbb{Q} \cap [a,b] = \{q_j : \ j \in \mathbb{N}\}$. Let further $J_{2j} = \{q_j\}$ and let J_{2j-1} be any subintervals of $[0,1]$. Hence the Dirichlet function $D\colon [0,1] \to \mathbb{R}$ can be represented in the form

$$D(x) = \sum_{j=1}^{\infty} c_j \cdot \chi_{J_j}(x),$$

where $c_{2j} = 1$ and $c_{2j-1} = 0$. Hence its integral equals 0.

Applying properties of this kind of integral it is possible to prove that integral of a regulated function f is an additive function of interval.

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