FOLIA 102

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica X (2011)

Krzysztof Żyjewski

Nonlocal Robin problem in a plane domain with a boundary corner point

Abstract. We investigate the behavior of weak solutions to the nonlocal Robin problem for linear elliptic divergence second order equations in a neighborhood of the boundary corner point. We find an exponent of the solution decreasing rate under the minimal assumptions on the problem coefficients.

1. Introduction

Our article is devoted to the linear elliptic divergence second order equations with the nonlocal Robin boundary condition in a plane bounded domain with a boundary corner point. The nonlocal condition means that the values of the unknown function u on the lateral side of a domain are connected with the values of u inside a domain. This problem appears often in different fields of physics and engineering. For example, nonlocal elliptic boundary value problems have important applications to the theory of diffusion processes, in the theory of turbulence etc. Various problems in this field have been studied by many mathematicians. We refer for the history of this problem and the extensive citation to [4, 11]. Questions of the solvability to nonlocal elliptic value boundary problems were considered by Skubachevskii [11]. In the same place there were obtained a priori estimates of solutions in the Sobolev spaces: both weighted and unweighted. All results in [11] relate to equations with *infinite–differentiable* coefficients. Gurevich [4] considered asymptotics of solutions for nonlocal elliptic problems for equations with *constant* coefficients in plane angles.

The aim of our article is the type $|u(x)| = O(|x|^\alpha)$ estimate of the weak solution modulus for our problem near an angular boundary point. A principal new feature of our work is the establishing of the weak solution decrease rate exponent under the consideration of *the minimal smoothness* required on the coefficients of the problem. Moreover, we derive global and local estimates for weighted and unweighted Dirichlet integrals applying different methods from those in [4, 11] that allows us to obtain more detailed and exact estimates of these integrals than previously known. We investigate the behavior of weak solutions for the considered

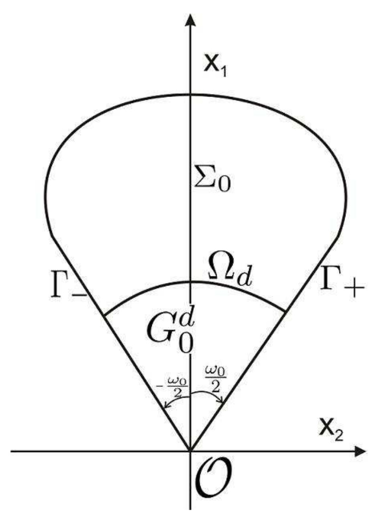
AMS (2000) Subject Classification: 35J25, 35J60, 35J85, 35B65.

problem in a neighborhood of the boundary corner point by integro–differential inequalities and Kondratiev's ring methods developed in [1]. For this purpose we use *the Friedrichs–Wirtinger type inequality* which is adapted to the our problem. All obtained results are new and distinguishes our work from cited above.

Setting of nonlocal problem. Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G = \overline{\Gamma}_+ \cup \overline{\Gamma}_-$ being a smooth curve everywhere except at the origin $\mathcal{O} \in \partial G$, where near the point $\mathcal{O}$ curves $\Gamma_\pm$ are lateral sides of an angle with the measure $\omega_0 \in [0, 2\pi)$ and the vertex at $\mathcal{O}$. Let $\Sigma_0 = G \cap \{x_2 = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$.

We will use the following notations:

- S^1: the unit circle in $\mathbb{R}^2$ centered at $\mathcal{O}$;
- (r, ω): the polar coordinates of $x = (x_1, x_2) \in \mathbb{R}^2$ with pole $\mathcal{O}$: $x_1 = r\cos\omega$, $x_2 = r\sin\omega$;
- $\mathcal{C}$: the angle $\{x_1 > r\cos\frac{\omega_0}{2};\ -\infty < x_2 < \infty\}$ with vertex $\mathcal{O}$;
- $\partial\mathcal{C}$: the lateral sides of $\mathcal{C}$: $x_1 = r\cos\frac{\omega_0}{2}$, $x_2 = \pm r\sin\frac{\omega_0}{2}$;
- Ω: an arc obtained by intersecting the angle $\mathcal{C}$ with S^1: $\Omega = \mathcal{C} \cap S^1$;
- $G_a^b = \{(r, \omega);\ 0 \le a < r < b;\ \omega \in \Omega\} \cap G$: a ring domain in $\mathbb{R}^2$;
- $\Gamma_{a\pm}^b = \{(r, \omega);\ 0 \le a < r < b;\ \omega = \pm\frac{\omega_0}{2}\} \cap \partial G$: the lateral sides of G_a^b;
- $G_d = G \setminus G_0^d$; $\Gamma_{d\pm} = \Gamma_\pm \setminus \Gamma_{0\pm}^d$, $d > 0$;
- $\Omega_\rho = G_0^d \cap \{|x| = \varrho\}$; $0 < \varrho < d$;
- $\operatorname{meas} G$: the Lebesgue measure of the set G.



We shall consider an elliptic equation with nonlocal boundary condition connecting the values of the unknown function u on the curve Γ_+ with its values of u on the Σ_0:

$$\begin{cases} \mathcal{L}[u] \equiv \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}) + b^i(x)u_{x_i} + c(x)u = f(x), & x \in G; \\ \mathcal{B}_+[u] \equiv \dfrac{\partial u}{\partial \nu} + \beta_+ \dfrac{u(x)}{|x|} + \dfrac{b}{|x|}u(\gamma(x)) = g(x), & x \in \Gamma_+; \\ \mathcal{B}_-[u] \equiv \dfrac{\partial u}{\partial \nu} + \beta_- \dfrac{u(x)}{|x|} = h(x), & x \in \Gamma_-; \end{cases} \qquad (L)$$

here:

- $\frac{\partial}{\partial \nu} = a^{ij}(x)\cos(\overrightarrow{n}, x_i)\frac{\partial}{\partial x_j}$ and $\overrightarrow{n}$ denotes the unit vector outwards with respect to G normal to $\partial G \setminus \mathcal{O}$ (summation over repeated indices from 1 to 2 is understood);
- γ is a diffeomorphism mapping of Γ_+ onto Σ_0; we assume that there exists $d > 0$ such that in the neighborhood Γ^d_{0+} of the point $\mathcal{O}$ the mapping γ is the rotation by the angle $-\frac{\omega_0}{2}$, that is $\gamma(\Gamma^d_{0+}) = \Sigma^d_0$.

Remark 1.1
We observe that

$$u(\gamma(x))|_{\Gamma^d_{0+}} = u(r, 0), \qquad 0 < r < d.$$

In fact, $\gamma(x) = \gamma(x_1, x_2) = \gamma(r\cos\frac{\omega_0}{2}, r\sin\frac{\omega_0}{2}) = (r, 0)$, because of in the neighborhood Γ^d_{0+} of the point $\mathcal{O}$ the mapping γ is the rotation by the angle $-\frac{\omega_0}{2}$.

We use also standard function spaces: $C^k(\overline{G})$ with the norm $|u|_{k,G}$, Lebesgue space $L_p(G)$, $p \geq 1$ with the norm $\|u\|_{p,G}$, the Sobolev space $W^{k,p}(G)$ with the norm $\|u\|_{p,k,(G)} = (\int_G \sum_{|\beta|=0}^k |D^\beta u|^p \, dx)^{\frac{1}{p}}$. We define the weighted Sobolev space: $V^k_{p,\alpha}(G)$ for integer $k \geq 0$ and real α as the closure of $C_0^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{V^k_{p,\alpha}(G)} = \left(\int\limits_G \sum_{|\beta|=0}^k r^{\alpha+p(|\beta|-k)} |D^\beta u|^p \, dx \right)^{\frac{1}{p}}.$$

We write $W^k(G)$ for $W^{k,2}(G)$ and $\overset{\circ}{W}{}^k_\alpha(G)$ for $V^k_{2,\alpha}(G)$.

Let us recall some well known formulae related to polar coordinates (r, ω) centered at the point $\mathcal{O}$:

- $dx = r dr d\omega$,
- $d\Omega_\rho = \rho d\omega$,
- ds denotes the length element on ∂G,
- $|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \omega})^2$,

- $\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \omega^2}$.

$C = C(\ldots)$, $c = c(\ldots)$ denote constants depending only on the quantities appearing in parentheses. In what follows, the same letters C, c will (generally) be used to denote different constants depending on the same set of arguments.

Without loss of generality we can assume that there exists $d > 0$ such that G_0^d is *an angle* with the vertex at $\mathcal{O}$ and the measure $\omega_0 \in (0, 2\pi)$, thus

$$\Gamma_{0\pm}^d = \Big\{ \Big(r, \pm\frac{\omega_0}{2}\Big) \Big| \; x_1 = \pm\cot\frac{\omega_0}{2} \cdot x_2;\; r \in (0, d) \Big\}.$$

By means of the direct calculation we obtain

Lemma 1.2

$$\cos(\vec{n}, x_1)|_{\Gamma_{0\pm}^d} = -\sin\frac{\omega_0}{2}; \quad x_i \cos(\vec{n}, x_i)|_{\Gamma_{0\pm}^d} = 0; \quad x_i\cos(\vec{n}, x_i)|_{\Omega_\varrho} = \varrho.$$

Definition 1.3
A function $u(x)$ is called a *weak* solution of problem (L) provided that $u(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}_0^1(G)$ and satisfies the integral identity

$$\begin{aligned}
&\int_G \{a^{ij}(x)u_{x_j}\eta_{x_i} - b^i(x)u_{x_i}\eta(x) - c(x)u(x)\eta(x)\}\,dx \\
&\quad + \int_{\Gamma_+} \Big(\beta_+\frac{u(x)}{r} + \frac{b}{r}u(\gamma(x))\Big)\eta(x)\,ds + \beta_- \int_{\Gamma_-} \frac{u(x)}{r}\eta(x)\,ds \\
&= \int_{\Gamma_+} g(x)\eta(x)\,ds + \int_{\Gamma_-} h(x)\eta(x)\,ds - \int_G f(x)\eta(x)\,dx
\end{aligned} \tag{II}$$

for all functions $\eta(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}_0^1(G)$.

Lemma 1.4
Let $u(x)$ be a weak solution of (L). For any function $\eta(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}_0^1(G)$ the equality

$$\begin{aligned}
&\int_{G_0^\varrho} \{a^{ij}(x)u_{x_j}\eta_{x_i} + (f(x) - b^i(x)u_{x_i} - c(x)u(x))\eta(x)\}\,dx \\
&= \int_{\Omega_\varrho} a^{ij}(x)u_{x_j}\eta(x)\cos(r, x_i)\,d\Omega_\varrho \\
&\quad + \int_{\Gamma_{0+}^\varrho} \Big(g(x) - \beta_+\frac{u(x)}{r} - \frac{b}{r}u(\gamma(x))\Big)\eta(x)\,ds \\
&\quad + \int_{\Gamma_{0-}^\varrho} \Big(h(x) - \beta_-\frac{u(x)}{r}\Big)\eta(x)\,ds
\end{aligned} \tag{$II)_{loc}$$

holds for almost every $\varrho \in (0, d)$.

Proof. Let $\chi_\varrho(x)$ be the characteristic function of the set G_0^ϱ. We consider the integral identity (II) replacing $\eta(x)$ by $\eta(x)\chi_\varrho(x)$. As the result we obtain

$$\int\limits_{G_0^\varrho}\{a^{ij}(x)u_{x_j}\eta_{x_i}+(f(x)-b^i(x)u_{x_i}-c(x)u(x))\eta(x)\}\,dx$$
$$=-\int\limits_{G_0^\varrho}a^{ij}(x)u_{x_j}\eta(x)\chi_{x_i}\,dx+\int\limits_{\Gamma_{0+}^\varrho}\Big(g(x)-\beta_+\frac{u(x)}{r}-\frac{b}{r}u(\gamma(x))\Big)\eta(x)\,ds$$
$$+\int\limits_{\Gamma_{0-}^\varrho}\Big(h(x)-\beta_-\frac{u(x)}{r}\Big)\eta(x)\,ds.$$

Because of formula (7') of subsection 3 §1 chapter 3 in [2]

$$\chi_{x_i}=-\frac{x_i}{r}\delta(\varrho-r),$$

where $\delta(\varrho-r)$ is the Dirac distribution lumped on the circle $r=\varrho$, we get (see Example 4 of subsection 3 §1 chapter 3 [2])

$$-\int\limits_{G_0^\varrho}a^{ij}(x)u_{x_j}\eta(x)\chi_{x_i}\,dx=\int\limits_{G_0^\varrho}a^{ij}(x)u_{x_j}\eta(x)\frac{x_i}{r}\delta(\varrho-r)\,dx$$
$$=\int\limits_{\Omega_\varrho}a^{ij}u_{x_j}\eta(x)\cos(r,x_i)\,d\Omega_\varrho.$$

Thus the required statement follows.

We will make the following **assumptions**:

(a) *the condition of the uniform ellipticity:*

$$\nu\xi^2\le a^{ij}(x)\xi_i\xi_j\le\mu\xi^2,\qquad\forall x\in\overline{G},\ \forall\xi\in\mathbb{R}^2;$$
$$\nu,\mu=const>0\qquad and\qquad a^{ij}(0)=\delta_i^j,$$

where δ_i^j *is the Kronecker symbol;*

(b) $a^{ij}(x)\in C^0(\overline{G})$, $b^i(x)\in L_p(G)$, $c(x)\in L_{\frac{p}{2}}(G)\cap L_2(G)$; $\forall p>\widetilde{n}$, $\forall\widetilde{n}>2$; *for them the inequalities*

$$\Big(\sum_{i,j=1}^{2}|a^{ij}(x)-a^{ij}(y)|^2\Big)^{\frac12}\le\mathcal{A}(|x-y|);$$
$$|x|\Big(\sum_{i=1}^{2}|b^i(x)|^2\Big)^{\frac12}+|x|^2|c(x)|\le\mathcal{A}(|x|)$$

hold for $x,y\in\overline{G}$, *where* $\mathcal{A}(r)$ *is a monotonically increasing function,* **continuous at 0**, *with* $\mathcal{A}(0)=0$*;*

(c) $c(x) \le 0$ *in* G; $b > 0$, $\beta_\pm \ge \beta_0 > 0$ *and* $\beta_+ > \max\{0; \frac{b^2\omega_0}{4} - b; \frac{b^2\omega_0}{4\nu} - b; \frac{4b^2\omega_0}{\nu} - 2b\}$;

(d) $f(x) \in L_{\frac{p}{2}}(G) \cap L_2(G)$, $g(x) \in L_2(\Gamma_+)$, $h(x) \in L_2(\Gamma_-)$ *and there exist numbers* $f_0 \ge 0$, $g_0 \ge 0$, $h_0 \ge 0$, $s > 2 - \frac{2}{p}$ *such that*

$$|f(x)| \le f_0|x|^{s-2}, \qquad |g(x)| \le g_0|x|^{s-1}, \qquad |h(x)| \le h_0|x|^{s-1};$$

(e) $M_0 = \max_{x\in\overline{G}} |u(x)|$ *is known.*

Our main result is the following theorem. Let

$$\lambda = \sqrt{\vartheta\Big(1 + \frac{b}{4\beta_+}\big(2 + \sqrt{4 + 2\omega_0\beta_+}\,\big)\Big)}, \tag{1.1}$$

where ϑ is the smallest positive eigenvalue of problem (EVP) (see Subsection 2.1).

Theorem 1.5
Let u *be a weak solution of problem* (L), *satisfying the assumptions* $(a) - (e)$ *with* $\mathcal{A}(r)$ *Dini-continuous at zero. Then there are* $d \in (0, \frac{1}{e})$, *where* e *is the Euler number, and a constant* $C > 0$ *depending only on* ν, μ, p, λ, $\|\sum_{i=1}^2 |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}$, ω_0, b, β_+, β_-, M_0, f_0, h_0, g_0, β_0, s, $\operatorname{meas} G$, $\operatorname{meas}\Gamma_+$, $\operatorname{meas}\Gamma_-$ *and on the quantity* $\int_0^{\frac{1}{e}} \frac{\mathcal{A}(r)}{r}\,dr$ *such that for all* $x \in G_0^d$

$$|u(x)| \le C \begin{cases} |x|^{\frac{\lambda k}{\sqrt{q}}}, & \text{if } s > \dfrac{\lambda k}{\sqrt{q}}, \\ |x|^{\frac{\lambda k}{\sqrt{q}}} \ln\left(\dfrac{1}{|x|}\right), & \text{if } s = \dfrac{\lambda k}{\sqrt{q}}, \\ |x|^s, & \text{if } s < \dfrac{\lambda k}{\sqrt{q}}, \end{cases} \tag{1.2}$$

where

$$k = 1 + \frac{b}{2\beta_+} - \frac{b\sqrt{1+\omega_0\beta_+}}{2\beta_+} \quad \text{and} \quad q = 1 + \frac{b}{4\beta_+}\big(2 + \sqrt{4 + 2\omega_0\beta_+}\,\big). \tag{1.3}$$

To prove the main theorem (see Section 6) one ought to derive the following statements:

- the local estimate of the maximum modulus (see Section 3),
- the global estimate of the weighted Dirichlet integral (see Section 4),
- the local estimate of the weighted Dirichlet integral (see Section 5).

2. Preliminaries

2.1. Auxiliary inequalities

In what follows we need some statements and inequalities.

The eigenvalue problem. Let $\Omega = (-\frac{\omega_0}{2}, \frac{\omega_0}{2})$. We consider the following eigenvalue problem:

$$\begin{cases} \psi''(\omega) + \vartheta\psi(\omega) = 0, & \omega \in \Omega, \\ \psi'\Big(\dfrac{\omega_0}{2}\Big) + \beta_+\psi\Big(\dfrac{\omega_0}{2}\Big) = 0, & \\ -\psi'\Big(-\dfrac{\omega_0}{2}\Big) + \beta_-\psi\Big(-\dfrac{\omega_0}{2}\Big) = 0, & \end{cases} \tag{EVP}$$

with $\beta_\pm > 0$, which consist of the determination of all values ϑ (eigenvalues) for which (EVP) has nonzero weak solutions (eigenfunctions).

Definition 2.1
Function ψ is called a *weak solution* of problem (EVP) provided that $\psi \in W^1(\Omega) \cap C^0(\overline{\Omega})$ and satisfies the integral identity

$$\begin{aligned} \int\limits_\Omega (\psi'(\omega)\eta'(\omega) - \vartheta\psi(\omega)\eta(\omega))\,d\omega + \beta_+\psi\Big(\frac{\omega_0}{2}\Big)\eta\Big(\frac{\omega_0}{2}\Big) \\ +\beta_-\psi\Big(-\frac{\omega_0}{2}\Big)\eta\Big(-\frac{\omega_0}{2}\Big) = 0 \end{aligned} \tag{2.1}$$

for all $\eta(\omega) \in W^1(\Omega) \cap C^0(\overline{\Omega})$.

Remark 2.2
We observe that $\vartheta = 0$ is not an eigenvalue of (EVP). In fact, setting in (2.1) $\eta = \psi$ and $\vartheta = 0$ we have

$$\int\limits_\Omega |\psi'(\omega)|^2\,d\omega + \beta_+\Big|\psi\Big(\frac{\omega_0}{2}\Big)\Big|^2 + \beta_-\Big|\psi\Big(-\frac{\omega_0}{2}\Big)\Big|^2 = 0 \implies \psi(\omega) \equiv 0,$$

since $\beta_\pm > 0$.

Now, let us introduce the following functionals on $W^1(\Omega) \cap C^0(\Omega)$

$$\begin{aligned} F[\psi] &= \int\limits_\Omega (\psi'(\omega))^2\,d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big), \\ G[\psi] &= \int\limits_\Omega \psi^2(\omega)\,d\omega, \\ H[\psi] &= \int\limits_\Omega ((\psi'(\omega))^2 - \vartheta\psi^2(\omega))\,d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big). \end{aligned}$$

We introduce also corresponding bilinear forms

$$\mathcal{F}[\psi,\eta] = \int\limits_{\Omega} (\psi'(\omega)\eta'(\omega))\,d\omega + \beta_+\psi\Big(\frac{\omega_0}{2}\Big)\eta\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi\Big(-\frac{\omega_0}{2}\Big)\eta\Big(-\frac{\omega_0}{2}\Big),$$

$$\mathcal{G}[\psi,\eta] = \int\limits_{\Omega} \psi(\omega)\eta(\omega)\,d\omega.$$

We define the set $K = \{\psi \in W^1(\Omega)\cap C^0(\overline{\Omega}) |\ G[\psi] = 1\}$. Since $K \subset W^1(\Omega)\cap C^0(\overline{\Omega})$, $F[\psi]$ is bounded from below for $\psi \in K$. We denote by ϑ the greatest lower bound of $F[\psi]$ for this family:

$$\vartheta := \inf_{\psi\in K} F[\psi].$$

We formulate the following statement:

Theorem 2.3
Let $\Omega \subset S^1$ be an arc. Then there exist $\vartheta > 0$ and a function $\psi \in K$ such that

$$\mathcal{F}[\psi,\eta] - \vartheta\mathcal{G}[\psi,\eta] = 0 \quad \textit{for arbitrary } \eta \in W^1(\Omega)\cap C^0(\overline{\Omega}).$$

In particular $F[\psi] = \vartheta$.

Proof. The proof is similar to Theorem 2.18 [1].

Now from the variational principle we obtain **the Friedrichs–Wirtinger type inequality**:

Theorem 2.4
Let ϑ be the smallest positive eigenvalue of problem (EVP) (it exists according to Theorem 2.3). Let $\Omega \subset S^1$ and assume that $\psi \in W^1(\Omega)\cap C^0(\overline{\Omega})$ satisfies in the weak sense boundary conditions from (EVP). Then

$$\vartheta\int\limits_{\Omega} \psi^2(\omega)\,d\omega \le \int\limits_{\Omega} \Big(\frac{\partial\psi}{\partial\omega}\Big)^2 d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big).$$

Because of (1.1) and the definition of q by (1.3), the Friedrichs–Wirtinger inequality will be written in the following form

$$\int\limits_{\Omega} \psi^2(\omega)\,d\omega \le \frac{q}{\lambda^2}\Bigg\{ \int\limits_{\Omega} \Big(\frac{\partial\psi}{\partial\omega}\Big)^2 d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big)\Bigg\} \tag{2.2}$$

for all $\psi(\omega) \in W^1(\Omega)\cap C^0(\overline{\Omega})$ satisfying boundary conditions from (EVP) in the weak sense.

We formulate the classical Hardy inequality (see Theorem 330 [5]).

Proposition 2.5
Let $v \in C^0[0,d] \cap W^1(0,d)$, $d > 0$ with $v(0) = 0$. Then

$$\int_0^d r^{\alpha-3} v^2(r)\, dr \le \frac{4}{(2-\alpha)^2} \int_0^d r^{\alpha-1} \Big(\frac{\partial v}{\partial r}\Big)^2 dr \tag{2.3}$$

for $\alpha < 2$, provided that the integral on the right hand side is finite.

Proof. It is the corollary of the classical Hardy inequality (see e.g. §2.1 [1]).

Now we use the Hardy inequality and then we get:

Proposition 2.6 (The Hardy–Friedrichs–Wirtinger inequality)
Let $u \in C^0(\overline{G_0^d}) \cap \overset{\circ}{W}{}^1_{\alpha-2}(G_0^d)$, $\alpha \le 2$. Then

$$\begin{aligned} \int_{G_0^d} r^{\alpha-4} u^2(x)\, dx \le \frac{1}{\frac{(2-\alpha)^2}{4} + \frac{\lambda^2}{q}} \cdot \Bigg\{ \int_{G_0^d} r^{\alpha-2} |\nabla u|^2\, dx \\ + \beta_+ \int_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\, ds + \beta_- \int_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\, ds \Bigg\}. \end{aligned} \tag{2.4}$$

Proof. Multiplying inequality (2.2) by $r^{\alpha-3}$ and integrating over $r \in (0,d)$ we obtain

$$\begin{aligned} \int_{G_0^d} r^{\alpha-4} u^2(x)\, dx \le \frac{q}{\lambda^2} \Bigg\{ \int_{G_0^d} r^{\alpha-2} \frac{1}{r^2} \Big(\frac{\partial u}{\partial \omega}\Big)^2 dx \\ + \beta_+ \int_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\, ds + \beta_- \int_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\, ds \Bigg\}. \end{aligned} \tag{2.5}$$

Hence (2.4) follows for $\alpha = 2$. Now, let $\alpha < 2$. We shall show that $u(0) = 0$. In fact, from $u(0) = u(x) - (u(x) - u(0))$ using the Cauchy inequality we have $\frac{1}{2}|u(0)|^2 \le |u(x)|^2 + |u(x) - u(0)|^2$. Multiplying this inequality by $r^{\alpha-4}$, integrating over G_0^d and using $v(x) = u(x) - u(0)$ we obtain

$$\frac{1}{2}|u(0)|^2 \int_{G_0^d} r^{\alpha-4}\, dx \le \int_{G_0^d} r^{\alpha-4} u^2(x)\, dx + \int_{G_0^d} r^{\alpha-4} |v(x)|^2\, dx < \infty \tag{2.6}$$

(the first integral from the right is finite by (2.5) and the second is finite as well in virtue of Proposition 2.5). Since

$$\int_{G_0^d} r^{\alpha-4}\, dx = \operatorname{meas}\Omega \int_0^d r^{\alpha-3}\, dr = \infty$$

because of $\alpha - 2 < 0$, the assumption $u(0) \neq 0$ contradicts (2.6). Then $u(0) = 0$. Now using Hardy inequality (2.3) we obtain

$$\int\limits_{G_0^d} r^{\alpha-4} u^2(x)\, dx \leq \frac{4}{(2-\alpha)^2} \int\limits_{G_0^d} r^{\alpha-2} \Big(\frac{\partial u}{\partial r}\Big)^2 dx. \tag{2.7}$$

Adding inequality (2.5) and (2.7) and using the formula $|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}|\frac{\partial u}{\partial \omega}|^2$, we get the desired (2.4).

Lemma 2.7
Let $u(\varrho, \omega) \in C^0(\overline{\Omega})$ and $\nabla u(\varrho, \omega) \in L_2(\Omega)$ a.e. $\varrho \in (0, d)$. Assume that

$$U(\varrho) = \int\limits_{G_0^\varrho} |\nabla u|^2 dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r} ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r} ds < \infty \tag{2.8}$$

for $\varrho \in (0, d)$. Then

$$\int\limits_\Omega \varrho u \frac{\partial u}{\partial r}\Big|_{r=\varrho} d\omega \leq \frac{\varrho\sqrt{q}}{2\lambda} U'(\varrho),$$

where q is defined by (1.3).

Proof. Writing $U(\varrho)$ in polar coordinates,

$$U(\varrho) = \int\limits_0^\varrho r \int\limits_\Omega \Big(\Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial u}{\partial \omega}\Big|^2\Big) d\omega\, dr + \beta_+ \int\limits_0^\varrho \frac{u^2(r, \frac{\omega_0}{2})}{r} dr + \beta_- \int\limits_0^\varrho \frac{u^2(r, -\frac{\omega_0}{2})}{r} dr$$

and differentiating with respect to ϱ we obtain

$$U'(\varrho) = \int\limits_\Omega \Big(\varrho\Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{\varrho}\Big|\frac{\partial u}{\partial \omega}\Big|^2\Big)\Big|_{r=\varrho} d\omega + \beta_+ \frac{u^2(\varrho, \frac{\omega_0}{2})}{\varrho} + \beta_- \frac{u^2(\varrho, -\frac{\omega_0}{2})}{\varrho}. \tag{2.9}$$

Moreover, by Cauchy's inequality, we have

$$\rho u \frac{\partial u}{\partial r} \leq \frac{\varepsilon}{2} u^2 + \frac{1}{2\varepsilon} \rho^2 \Big(\frac{\partial u}{\partial r}\Big)^2$$

for all $\varepsilon > 0$. Thus, choosing $\varepsilon = \frac{\lambda}{\sqrt{q}}$, by Friedrichs–Wirtinger inequality (2.2), we obtain

$$\begin{aligned}
&\int\limits_\Omega \varrho u \frac{\partial u}{\partial r}\Big|_{r=\varrho} d\omega \\
&\leq \frac{\varepsilon q}{2\lambda^2} \Big\{ \int\limits_\Omega \Big|\frac{\partial u}{\partial \omega}\Big|^2_{r=\varrho} d\omega + \beta_+ u^2\Big(\varrho, \frac{\omega_0}{2}\Big) + \beta_- u^2\Big(\varrho, -\frac{\omega_0}{2}\Big)\Big\} + \frac{\varrho^2}{2\varepsilon} \int\limits_\Omega \Big|\frac{\partial u}{\partial r}\Big|^2_{r=\varrho} d\omega \\
&= \frac{\varrho\sqrt{q}}{2\lambda} \Big\{ \int\limits_\Omega \Big(\frac{1}{\varrho}\Big|\frac{\partial u}{\partial \omega}\Big|^2 + \varrho\Big|\frac{\partial u}{\partial r}\Big|^2\Big)\Big|_{r=\varrho} d\omega + \beta_+ \frac{u^2(\varrho, \frac{\omega_0}{2})}{\varrho} + \beta_- \frac{u^2(\varrho, -\frac{\omega_0}{2})}{\varrho} \Big\} \\
&= \frac{\varrho\sqrt{q}}{2\lambda} U'(\varrho).
\end{aligned}$$

We also need in the sequel well known inequalities (see e.g. (6.23), (6.24) Chapter I [6] or Lemma 6.36 [8])

$$\int\limits_{\Gamma} v\,ds \le C\int\limits_{G}(|v|+|\nabla v|)\,dx, \qquad \forall v(x)\in W^{1,1}(G),\ \forall\Gamma\subseteq\partial G,$$
$$\int\limits_{\partial G} v^2\,ds \le \int\limits_{G}\Big(\delta|\nabla v|^2+\frac{1}{\delta}c_0v^2\Big)\,dx, \qquad \forall v(x)\in W^{1,2}(G),\ \forall\delta>0. \tag{2.10}$$

2.2. The Cauchy problem for differential inequality

THEOREM 2.8
Let $U(\varrho)$ be monotonically increasing, nonnegative differentiable function defined on $[0,d]$ and satisfying the problem

$$\begin{cases} U'(\varrho)-\mathcal{P}(\varrho)U(\varrho)+\mathcal{Q}(\varrho)\ge 0, & 0<\varrho<d,\\ U(d)\le U_0, \end{cases} \tag{CP}$$

where $\mathcal{P}(\varrho),\mathcal{Q}(\varrho)$ are nonnegative continuous functions defined on $[0,d]$, and U_0 is a constant. Then

$$U(\varrho)\le U_0\exp\Bigg(-\int\limits_{\varrho}^{d}\mathcal{P}(\tau)\,d\tau\Bigg)+\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\Bigg(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\Bigg)\,d\tau. \tag{2.11}$$

Proof. For the proof see §1.10 (Theorem 1.57) [1].

3. Local estimate at the boundary

Here we derive the local boundedness (near the boundary corner point) of a weak solution of problem (L).

THEOREM 3.1
Let $u(x)$ be a weak solution of problem (L) and assumptions $(a)-(c)$ be satisfied. Suppose, in addition, that $g(x)\in L_\infty(\Gamma_+)$, $h(x)\in L_\infty(\Gamma_-)$. Then the inequality

$$\sup_{G_0^{\varkappa\varrho}}|u(x)| \le \frac{C}{(1-\varkappa)^{\frac{\widetilde{n}}{2}}}\Big\{\varrho^{-1}||u||_{2,G_0^\varrho}+\varrho^{2(1-\frac{2}{p})}||f||_{\frac{p}{2},G_0^\varrho}+\varrho\big(\|g\|_{\infty,\Gamma_{0+}^\varrho}+\|h\|_{\infty,\Gamma_{0-}^\varrho}\big)\Big\}$$

holds for any $p>\widetilde{n}>2$, $\varkappa\in(0,1)$ and $\varrho\in(0,d)$, where C is a positive constant depending only on μ, ν, p, $\|\sum_{i=1}^{2}|b^i(x)|^2\|_{L_{\frac{p}{2}(G)}}$ and G.

Proof. We apply the Moser iteration method. We consider the integral identity (II) and make the coordinate transformation $x=\varrho x'$. Let G' be the image of

G, Γ'_+ be the image of Γ_+, Γ'_- be the image of Γ_-, then we have $dx = \varrho^2 dx'$, $ds = \varrho ds'$. In addition, we denote

$$\begin{aligned} &v(x') = u(\varrho x'),\ \eta(x') = \eta(\varrho x'),\ \mathcal{F}(x') = \varrho^2 f(\varrho x'), \\ &\mathcal{G}(x') = \varrho g(\varrho x'),\ \mathcal{H}(x') = \varrho h(\varrho x'). \end{aligned} \tag{3.1}$$

Then from (II) we get

$$\begin{aligned} &\int\limits_{G'} \{a^{ij}(\varrho x')v_{x'_j}\eta_{x'_i} - \varrho b^i(\varrho x')v_{x'_i}\eta(x') - \varrho^2 c(\varrho x')v(x')\eta(x')\}\, dx' \\ &\qquad + \int\limits_{\Gamma'_+} \Big(\frac{\beta_+}{|x'|}v(x') + \frac{b}{|x'|}v(\gamma(x'))\Big)\eta(x')\, ds' + \beta_- \int\limits_{\Gamma'_-} \frac{v(x')}{|x'|}\eta(x')\, ds' \\ &\quad = \int\limits_{\Gamma'_+} \mathcal{G}(x')\eta(x')\, ds' + \int\limits_{\Gamma'_-} \mathcal{H}(x')\eta(x')\, ds' - \int\limits_{G'} \mathcal{F}(x')\eta(x')\, dx' \end{aligned} \tag{$(II)'$}$$

for all $\eta(x') \in C^0(\overline{G'}) \cap \overset{\circ}{W}{}^1_0(G')$. We define quantity m by

$$m = m(\varrho) = \frac{1}{\nu}\Big(\|\mathcal{F}\|_{\frac{p}{2},G^1_0} + \|\mathcal{G}\|_{\infty,\Gamma^1_{0+}} + \|\mathcal{H}\|_{\infty,\Gamma^1_{0-}}\Big) \tag{3.2}$$

and we set

$$\overline{v}(x') = |v(x')| + m. \tag{3.3}$$

We observe that

$$\begin{aligned} |\mathcal{F}(x')|\overline{v}(x') &= \frac{1}{m}|\mathcal{F}(x')| \cdot m\overline{v}(x') = \frac{1}{m}|\mathcal{F}(x')|(\overline{v}(x') - |v(x')|) \cdot \overline{v}(x') \\ &= \frac{1}{m}|\mathcal{F}(x')| \cdot \overline{v}^2(x') - \frac{1}{m}|\mathcal{F}(x')| \cdot |v(x')|\overline{v}(x') \\ &\le \frac{1}{m}|\mathcal{F}(x')| \cdot \overline{v}^2(x'); \\ |\mathcal{H}(x')|\overline{v}(x') &\le \frac{1}{m}|\mathcal{H}(x')| \cdot \overline{v}^2(x'); \\ |\mathcal{G}(x')|\overline{v}(x') &\le \frac{1}{m}|\mathcal{G}(x')| \cdot \overline{v}^2(x') \end{aligned} \tag{3.4}$$

in the same way. As the test function in the integral identity $(II)'$ we choose $\eta(x') = \zeta^2(|x'|)v(x')$, where $\zeta(|x'|) \in C_0^\infty([0,1])$ is nonnegative function to be further specified. By the chain and product rules $\eta(x)$ is a valid test function in $(II)'$ and also $\eta_{x'_i} = v_{x'_i}\zeta^2(|x'|) + 2\zeta(|x'|)\zeta_{x'_i}v(x')$, so that by substitution into $(II)'$ with regard to $c(\varrho x') \le 0$ in G' and $v \le |v| \le \overline{v}$, we obtain

$$\begin{aligned} &\int\limits_{G^1_0} a^{ij}(\varrho x')v_{x'_i}v_{x'_j}\zeta^2(|x'|)\, dx' + \beta_+ \int\limits_{\Gamma^1_{0+}} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\, ds' \\ &\qquad + b\int\limits_{\Gamma^1_{0+}} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\, ds' + \beta_- \int\limits_{\Gamma^1_{0-}} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\, ds' \end{aligned}$$

$$\le \varrho \int\limits_{G_0^1} |b^i(\varrho x')v_{x_i}|\overline{v}(x')\zeta^2(|x'|)\,dx' + 2\int\limits_{G_0^1} |a^{ij}(\varrho x')\zeta_{x'_i}v_{x'_j}|\overline{v}(x')\zeta(|x'|)\,dx'$$
$$+ \int\limits_{\Gamma_{0-}^1} \mathcal{H}(x')\overline{v}(x')\zeta^2(|x'|)\,ds' + \int\limits_{\Gamma_{0+}^1} \mathcal{G}(x')\overline{v}(x')\zeta^2(|x'|)\,ds'$$
$$+ \int\limits_{G_0^1} \mathcal{F}(x')\overline{v}(x')\zeta^2(|x'|)\,dx'.$$

By the elliptic conditions and with regard to (3.4), hence it follows

$$\int\limits_{G_0^1} \nu|\nabla' v|^2\zeta^2(|x'|)\,dx' + \beta_+ \int\limits_{\Gamma_{0+}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds'$$
$$+ b\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds' + \beta_- \int\limits_{\Gamma_{0-}^1} \frac{v^2(x')}{|x'|}\zeta^2\,ds'$$
$$\le \int\limits_{G_0^1} \varrho\bigg(\sum_{i=1}^{2}|b^i(\varrho x')|^2\bigg)^{\frac{1}{2}}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \tag{3.5}$$
$$+ 2\mu\int\limits_{G_0^1} |\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}\|\mathcal{G}\|_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1} \overline{v}^2(x')\zeta^2(|x'|)\,ds'$$
$$+ \frac{1}{m}\|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1} \overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1} \big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.$$

We shall estimate the third integral on the left hand side inequality (3.5). Because of $v|_{\Gamma_{0+}^1} = v(r', \frac{\omega_0}{2})$ and, by Remark 1.1, $v(\gamma(x'))|_{\Gamma_{0+}^1} = v(r', 0)$, using the representation $v(r', 0) = v(r', \frac{\omega_0}{2}) - \int_0^{\frac{\omega_0}{2}} \frac{\partial v(r',\omega)}{\partial\omega}\,d\omega$ we obtain:

$$\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds'$$
$$= \int\limits_0^1 \frac{v^2(r', \frac{\omega_0}{2})}{r'}\zeta^2(r')\,dr' - \int\limits_0^1 \frac{v(r', \frac{\omega_0}{2})}{r'}\zeta^2(r')\Bigg(\int\limits_0^{\frac{\omega_0}{2}} \frac{\partial v(r',\omega)}{\partial\omega}\,d\omega\Bigg)\,dr'.$$

Next, by the Cauchy inequality, we have

$$\int\limits_0^1 \frac{v(r', \frac{\omega_0}{2})}{r'}\zeta^2(r')\Bigg(\int\limits_0^{\frac{\omega_0}{2}} \frac{\partial v(r',\omega)}{\partial\omega}\,d\omega\Bigg)\,dr'$$
$$\le \int\limits_{G_0^1} \frac{\zeta^2(r')}{r'^2}\bigg|\frac{\partial v(r',\omega)}{\partial\omega}\bigg|\bigg|v\Big(r', \frac{\omega_0}{2}\Big)\bigg|\,dx'$$

$$\le \int\limits_{G_0^1} \frac{\zeta^2(r')}{r'^2}\Big(\frac{\varepsilon}{2}\Big|\frac{\partial v(r',\omega)}{\partial\omega}\Big|^2 + \frac{1}{2\varepsilon}v^2\Big(r',\frac{\omega_0}{2}\Big)\Big)\,dx' \tag{3.6}$$

$$\le \frac{\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{1}{2\varepsilon}\int\limits_0^1 \frac{\zeta^2(r')}{r'}\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} v^2\Big(r',\frac{\omega_0}{2}\Big)\,d\omega\,dr'$$

$$\le \frac{\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{\omega_0}{2\varepsilon}\int\limits_{\Gamma_{0+}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds', \qquad \forall\varepsilon>0.$$

Choosing in inequality (3.6) $\varepsilon = \frac{\nu}{b}$, from (3.5) we have

$$\begin{aligned}
&\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' \\
&\quad + \Big(\beta_+ + b - \frac{b^2\omega_0}{2\nu}\Big)\int\limits_{\Gamma_{0+}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' + \beta_-\int\limits_{\Gamma_{0-}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' \\
&\le \int\limits_{G_0^1}\varrho\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)^{\frac{1}{2}}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \qquad (3.7)\\
&\quad + 2\mu\int\limits_{G_0^1}|\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}||\mathcal{G}||_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' \\
&\quad + \frac{1}{m}||\mathcal{H}||_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.
\end{aligned}$$

Thus, by the assumption (c) for β_+, from (3.7) it follows that

$$\begin{aligned}
&\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' \\
&\le \int\limits_{G_0^1}\varrho\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)^{\frac{1}{2}}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \\
&\quad + 2\mu\int\limits_{G_0^1}|\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' \qquad (3.8)\\
&\quad + \frac{1}{m}||\mathcal{G}||_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}||\mathcal{H}||_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' \\
&\quad + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.
\end{aligned}$$

We estimate every term by the Cauchy inequality for any $\varepsilon > 0$:

$$2\mu|\nabla' v||\nabla'\zeta|\zeta(|x'|)\overline{v}(x') = 2(|\nabla' v| \cdot \zeta(|x'|))(\mu\overline{v}(x')|\nabla'\zeta|) \\ \leq \varepsilon|\nabla' v|^2\zeta^2(|x'|) + \frac{\mu^2}{\varepsilon}\overline{v}^2(x')|\nabla'\zeta|^2;$$

$$\varrho\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{\frac{1}{2}}|\nabla' v|\overline{v}(x')\zeta^2(|x'|) \\ = \zeta^2(|x'|)\Big(\varrho\overline{v}(x')\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{\frac{1}{2}}\Big) \times |\nabla' v| \\ \leq \frac{\varrho^2}{2\varepsilon}\overline{v}^2(x')\zeta^2(|x'|)\cdot\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big) + \frac{\varepsilon}{2}|\nabla' v|^2\zeta^2(|x'|).$$

For the estimating integrals over the boundaries on the right in (3.8) we apply inequality (2.10). Thus we get

$$\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' \\ \leq \frac{3\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{\mu^2}{\varepsilon}\int\limits_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx' \\ + \frac{\varrho^2}{2\varepsilon}\int\limits_{G_0^1}\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)\overline{v}^2(x')\zeta^2(|x'|)\,dx' \\ + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(|x'|)\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx' \\ + \frac{1}{m}\Big(\|\mathcal{G}\|_{\infty,\Gamma^1_{0-}} + \|\mathcal{H}\|_{\infty,\Gamma^1_{0+}}\Big)\int\limits_{G_0^1}\Big(\delta|\nabla'(\zeta\overline{v})|^2 + \frac{1}{\delta}c_0\overline{v}^2(x')\zeta^2(|x'|)\Big)\,dx', \\ \forall\varepsilon,\delta > 0. \tag{3.9}$$

From relations

$$|\nabla'(\zeta\overline{v})|^2 \leq 2(\zeta^2|\nabla'\overline{v}|^2 + \overline{v}^2(x')|\nabla'\zeta|^2), \qquad |\nabla'\overline{v}|^2 = |\nabla' v|^2 \tag{3.10}$$

it follows the inequality

$$|\nabla'(\zeta\overline{v})|^2 \leq 2|\nabla' v|^2\zeta^2 + 2\overline{v}^2(x')|\nabla'\zeta|^2. \tag{3.11}$$

Now, by (3.9)–(3.11), choosing $\varepsilon = \frac{\nu}{6}$ in (3.9) and, by virtue of (3.2), we find that

$$\frac{\nu}{4}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx'$$

$$\leq \frac{6\mu^2}{\nu}\int\limits_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx' + \frac{3\varrho^2}{\nu}\int\limits_{G_0^1}\bigg(\sum_{i=1}^{2}|b^i(\varrho x')|^2\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx'$$
$$+\,2\delta\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + 2\delta\nu\int\limits_{G_0^1}\overline{v}^2(x')|\nabla'\zeta|^2\,dx'$$
$$+\,\frac{c_0\nu}{\delta}\int\limits_{G_0^1}\overline{v}^2(x')\zeta^2(|x'|)\,dx' + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx', \qquad \forall\delta > 0.$$

Now we choose $\delta = \frac{1}{16}$, then by (3.10), the last inequality means

$$\int\limits_{G_0^1}|\nabla'\overline{v}|^2\zeta^2(|x'|)\,dx' \leq \frac{48\mu^2}{\nu^2}\int\limits_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx'$$
$$+\,\frac{24\varrho^2}{\nu^2}\int\limits_{G_0^1}\bigg(\sum_{i=1}^{2}|b^i(\varrho x')|^2\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx'$$
$$+\int\limits_{G_0^1}\overline{v}^2(x')|\nabla'\zeta|^2\,dx' + 128c_0\int\limits_{G_0^1}\overline{v}^2(x')\zeta^2(|x'|)\,dx'$$
$$+\,\frac{8}{m\nu}\int\limits_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.$$

The above inequality we can rewrite as the following

$$\begin{aligned}&\int\limits_{G_0^1}|\nabla'\overline{v}|^2\zeta^2(|x'|)\,dx'\\ &\quad\leq C_1\int\limits_{G_0^1}(|\nabla'\zeta|^2+\zeta^2(|x'|))\overline{v}^2(x')\,dx'\\ &\qquad+C_2\int\limits_{G_0^1}\bigg(\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2+\frac{|\mathcal{F}(x')|}{m}\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx',\end{aligned}\tag{3.12}$$

where constants C_1, C_2 depend only on c_0, μ, ν. The desired iteration process can now be developed from (3.12). By the Sobolev imbedding theorem (see §2 ch. II [7]) we have

$$\|\zeta\overline{v}\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \leq C^*\int\limits_{G_0^1}((|\nabla'\zeta|^2+\zeta^2)\overline{v}^2(x')+\zeta^2|\nabla'\overline{v}|^2)\,dx', \qquad \widetilde{n} > 2, \tag{3.13}$$

where constant C^* depends only on $\widetilde{n}$ and the domain G. Using the Hölder inequality for integrals

$$\int\limits_{G_0^1} \left(\varrho^2 \sum_{i=1}^{2} |b^i(\varrho x')|^2 + \frac{|\mathcal{F}(x')|}{m} \right) \cdot \overline{v}^2(x')\zeta^2(x')\,dx'$$
$$\leq \left\| \varrho^2 \sum_{i=1}^{2} |b^i(\varrho x')|^2 + \frac{|\mathcal{F}(x')|}{m} \right\|_{\frac{p}{2},G_0^1} \times \|\zeta\overline{v}\|^2_{\frac{2p}{p-2},G_0^1}, \qquad p > 2 \tag{3.14}$$

and from (3.12)–(3.14) we get

$$\|\zeta\overline{v}\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}$$
$$\leq C_3 \int\limits_{G_0^1} (|\nabla'\zeta|^2 + \zeta^2(|x'|))\overline{v}^2(x')\,dx' \tag{3.15}$$
$$+ C_4 \left\| \varrho^2 \sum_{i=1}^{2} |b^i(\varrho x')|^2 + \frac{|\mathcal{F}(x')|}{m} \right\|_{\frac{p}{2},G_0^1} \cdot \|\zeta\overline{v}\|^2_{\frac{2p}{p-2},G_0^1}, \qquad p > \widetilde{n} > 2.$$

By the interpolation inequality for L_p-norms

$$\|\zeta\overline{v}\|_{\frac{2p}{p-2},G_0^1} \leq \varepsilon \|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} + \tilde{c}\varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}} \|\zeta\overline{v}\|_{2,G_0^1}, \qquad p > \widetilde{n} > 2,\ \forall\varepsilon > 0,$$
$$\widetilde{c} = \frac{p-\widetilde{n}}{p} \left(\frac{\widetilde{n}}{p}\right)^{\frac{\widetilde{n}}{p-\widetilde{n}}},$$

and, by virtue of definition (3.2), from (3.16) it follows that

$$\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \leq \sqrt{C_3} \cdot \|(\zeta + |\nabla'\zeta|)\overline{v}\|_{2,G_0^1}$$
$$+ \sqrt{C_4} \left(\left\| \varrho^2 \sum_{i=1}^{2} |b^i(\varrho x')|^2 \right\|_{\frac{p}{2},G_0^1} + \nu \right)^{\frac{1}{2}} \tag{3.16}$$
$$\times \left(\varepsilon ||\zeta\overline{v}||_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} + \widetilde{c}\varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}} \|\zeta\overline{v}\|_{2,G_0^1} \right), \qquad p > \widetilde{n},\ \forall\varepsilon > 0.$$

Choosing

$$\varepsilon = \frac{1}{2\sqrt{C_4}} \left(\left\| \varrho^2 \sum_{i=1}^{2} |b^i(\varrho x')|^2 \right\|_{\frac{p}{2},G_0^1} + \nu \right)^{-\frac{1}{2}}$$

from (3.16) we obtain

$$\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \leq C \|(\zeta + |\nabla'\zeta|)\overline{v}\|_{2,G_0^1}, \qquad 2\widetilde{n} \geq p > \widetilde{n} > 2, \tag{3.17}$$

where C depends only on c_0, μ, ν, p, $\operatorname{diam} G$, $\| \sum_{i=1}^{2} |b^i(x)|^2\|_{\frac{p}{2},G}$. This inequality can now be iterated to yield the desired estimate.

For all $\varkappa \in (0,1)$ we define sets $G'_{(j)} \equiv G_0^{\varkappa+(1-\varkappa)2^{-j}}$, $j = 0, 1, 2 \ldots$. It is easy to verify that $G_0^{\varkappa} \equiv G'_{(\infty)} \subset \ldots \subset G'_{(j+1)} \subset G'_j \subset \ldots \subset G'_{(0)} \equiv G_0^1$. Now we consider the sequence of cut-off function $\zeta_j(x') \in C^{\infty}(G'_{(j)})$ such that

$$0 \leq \zeta_j(x') \leq 1 \text{ in } G'_{(j)} \qquad \text{and} \qquad \zeta_j(x') \equiv 1 \text{ in } G'_{(j+1)},$$

$$\zeta_j(x') \equiv 0 \qquad \text{for } |x'| > \varkappa + 2^{-j}(1-\varkappa),$$
$$|\nabla'\zeta_j| \leq \frac{2^{j+1}}{1-\varkappa} \qquad \text{for } \varkappa + 2^{-j-1}(1-\varkappa) < |x'| < \varkappa + 2^{-j}(1-\varkappa).$$

We also define the number sequence $t_j = 2(\frac{\tilde{n}}{\tilde{n}-2})^j$, $j = 0, 1, 2 \ldots$. Now we rewrite inequality (3.17) replacing $\zeta(|x'|)$ by $\zeta_j(x')$; then, we obtain

$$\|\overline{v}\|_{\frac{2\tilde{n}}{\tilde{n}-2}, G'_{(j+1)}} \leq C\frac{2^{j+2}}{1-\varkappa}\|\overline{v}\|_{2,G'_{(j)}}. \tag{3.18}$$

Putting $w = |\overline{v}|^{(\frac{\tilde{n}}{\tilde{n}-2})^j}$, by (3.18) and the definition on the number sequence t_j, we get

$$\begin{aligned}\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} &= \left(\int\limits_{G_{(j+1)'}} w^{\frac{2\tilde{n}}{\tilde{n}-2}}\, dx'\right)^{\frac{\tilde{n}-2}{2\tilde{n}}\cdot(\frac{\tilde{n}-2}{\tilde{n}})^j} \\ &\leq \left(C\frac{2^{j+2}}{1-\varkappa}\right)^{(\frac{\tilde{n}-2}{\tilde{n}})^j} \|w\|_{2,G'_{(j)}}^{(\frac{\tilde{n}-2}{\tilde{n}})^j} \\ &= \left(\frac{C}{1-\varkappa}\right)^{\frac{2}{t_j}} 4^{\frac{j+2}{t_j}}\|\overline{v}\|_{t_j,G'_{(j)}}.\end{aligned}$$

After iteration, we find that

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \leq \left\{\frac{C}{1-\varkappa}\right\}^{2\sum_{j=0}^{\infty}\frac{1}{t_j}} \cdot 4^{\sum_{j=0}^{\infty}\frac{j+2}{t_j}} \cdot \|\overline{v}\|_{2,G_0^1}. \tag{3.19}$$

Notice that the series $\sum_{j=0}^{\infty}\frac{j+2}{t_j}$ is convergent by the d'Alembert ratio test, and the series $\sum_{j=0}^{\infty}\frac{1}{t_j} = \frac{\tilde{n}}{4}$ as a geometric series. Therefore from (3.19) we get

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \leq \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}}\|\overline{v}\|_{2,G_0^1}.$$

Consequently, letting $j \to \infty$, we have

$$\sup_{x' \in G_0^{\varkappa}} |\overline{v}(x')| \leq \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}}\|\overline{v}\|_{2,G_0^1}.$$

Hence, because of definition of function $\overline{v}(x')$ by (3.3) and definition of number m by (3.2), we get:

$$\sup_{x' \in G_0^{\varkappa}} |v(x')| \leq \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}}\big(\|v\|_{2,G_0^1} + \|\mathcal{F}\|_{\frac{p}{2},G_0^1} + \|\mathcal{G}\|_{\infty,\Gamma^1_{0+}} + \|\mathcal{H}\|_{\infty,\Gamma^1_{0-}}\big).$$

Returning to the variables x and u we obtain the required estimate (3.1).

4. Global integral estimate

In this section we obtain the global estimate for the weighted Dirichlet integral.

Theorem 4.1
Let $u(x)$ be a weak solution of problem (L). Let assumptions $(a)-(c)$, (e) be satisfied. Suppose, in addition, that $g(x) \in L_2(\Gamma_+)$, $h(x) \in L_2(\Gamma_-)$. Then the inequality

$$\int_G |\nabla u|^2\,dx + \int_G \frac{u^2(x)}{r^2}\,dx + \int_{\partial G} \frac{u^2(x)}{r}\,ds \le C\left\{ \int_G f^2(x)\,dx + \int_{\Gamma_+} g^2(x)\,ds + \int_{\Gamma_-} h^2(x)\,ds \right\} \tag{4.1}$$

holds, where constant $C > 0$ depends only on b, β_+, ω_0, β_0, p, ν, M_0, G, $\|\sum_{i=1}^{2}|b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}$.

Proof. Setting in (II) $\eta(x) = u(x)$ and using the classical Hölder inequality, by assumptions (a), (c), we get

$$\nu\int_G |\nabla u|^2\,dx + \int_{\Gamma_+}\Big(\beta_+\frac{u^2(x)}{r} + b\frac{u(x)}{r}u(\gamma(x))\Big)\,ds + \beta_-\int_{\Gamma_-}\frac{u(x)}{r}\,ds \le \int_G \sqrt{\sum_{i=1}^{2}|b^i(x)|^2}|u||\nabla u|\,dx + \int_{\Gamma_+}|u||g(x)|\,ds + \int_{\Gamma_-}|u||h(x)|\,ds + \int_G |u||f(x)|\,dx. \tag{4.2}$$

Now, by assumptions (b), (c), the Cauchy inequality and the Hölder inequality for integrals with $q = \frac{p}{2}$, $q' = \frac{p}{p-2}$, $p > 2$, we have:

$$\begin{aligned}
&\int_G \sqrt{\sum_{i=1}^{2}|b^i(x)|^2}|u||\nabla u|\,dx \\
&= \int_G |\nabla u|\left(\sqrt{\sum_{i=1}^{2}|b^i(x)|^2}|u|\right)dx \\
&\le \frac{\nu}{2}\int_G |\nabla u|^2\,dx + \frac{1}{2\nu}\int_G \sum_{i=1}^{2}|b^i(x)|^2u^2\,dx \\
&\le \frac{\nu}{2}\int_G |\nabla u|^2\,dx + \frac{1}{2\nu}\left(\int_G\left(\sum_{i=1}^{2}|b^i(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{2}{p}} \cdot ||u||^2_{\frac{2p}{p-2}(G)}.
\end{aligned}$$

Next, we apply the inequality

$$\|u\|^2_{L_{\frac{2p}{p-2}}(G)} \le \delta \|\nabla u\|^2_{L_2(G)} + c(\delta, p, G)\|u\|^2_{L_2(G)}, \qquad p > 2,\ \forall \delta > 0$$

(see for example (2.19) §2, chapter II in [7]); hence it follows that

$$\begin{aligned}
&\int\limits_G \sqrt{\sum_{i=1}^2 |b^i(x)|^2} |u||\nabla u|\, dx \\
&\quad \le \frac{\nu}{2} \int\limits_G |\nabla u|^2\, dx + \frac{1}{2\nu} \Big\| \sum_{i=1}^2 |b^i(x)|^2 \Big\|_{L_{\frac{p}{2}}(G)} \\
&\qquad \times \int\limits_G (\delta |\nabla u|^2 + c(\delta, p, G) u^2(x))\, dx, \qquad \forall \varepsilon > 0,\ \forall \delta > 0.
\end{aligned} \tag{4.3}$$

We choose $\delta = \frac{\nu^2}{2\|\sum_{i=1}^2 |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}}$. As a result from (4.2)–(4.3) we obtain

$$\begin{aligned}
&\frac{\nu}{4} \int\limits_G |\nabla u|^2\, dx + \int\limits_{\Gamma_+} \Big(\beta_+ \frac{u^2(x)}{r} + b\frac{u(x)}{r} u(\gamma(x))\Big)\, ds + \beta_- \int\limits_{\Gamma_-} \frac{u(x)}{r}\, ds \\
&\quad \le C \int\limits_G u^2(x)\, dx + \int\limits_{\Gamma_+} |u||g(x)|\, ds + \int\limits_{\Gamma_-} |u||h(x)|\, ds + \int\limits_G |u||f(x)|\, dx,
\end{aligned} \tag{4.4}$$

where $C = \mathrm{const}(p, \nu, \|\sum_{i=1}^2 |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}, G)$. Further, by the Cauchy inequality, in virtue of the assumption (c), we obtain

$$\begin{aligned}
\int\limits_{\Gamma_+} |u||g(x)|\, ds &= \int\limits_{\Gamma_+} \left(\sqrt{\frac{\beta_+}{r}}|u|\right)\left(\sqrt{\frac{r}{\beta_+}}|g(x)|\right) ds \\
&\le \frac{1}{2}\beta_+ \int\limits_{\Gamma_+} \frac{u^2(x)}{r}\, ds + \frac{1}{2\beta_0} \int\limits_{\Gamma_+} r g^2(x)\, ds; \\
\int\limits_{\Gamma_-} |u||h(x)|\, ds &= \int\limits_{\Gamma_-} \left(\sqrt{\frac{\beta_-}{r}}|u|\right)\left(\sqrt{\frac{r}{\beta_-}}|h(x)|\right) ds \\
&\le \frac{1}{2}\beta_- \int\limits_{\Gamma_-} \frac{u^2(x)}{r}\, ds + \frac{1}{2\beta_0} \int\limits_{\Gamma_-} r g^2(x)\, ds; \\
\int\limits_G |u||f(x)|\, dx &\le \frac{1}{2} \int\limits_G |u|^2\, dx + \frac{1}{2} \int\limits_G |f|^2\, dx.
\end{aligned}$$

Hence and from (4.4) we have

$$\begin{aligned}\frac{\nu}{4}\int\limits_G |\nabla u|^2\,dx + \frac{1}{2}\beta_+\int\limits_{\Gamma_+}\frac{u^2(x)}{r}\,ds + b\int\limits_{\Gamma_+}\frac{u(x)}{r}u(\gamma(x))\,ds + \frac{1}{2}\beta_-\int\limits_{\Gamma_-}\frac{u(x)}{r}\,ds\\ \le C\int\limits_G u^2(x)\,dx + \frac{1}{2\beta_0}\int\limits_{\Gamma_+} rg^2(x)\,ds + \frac{1}{2\beta_0}\int\limits_{\Gamma_-} rh^2(x)\,ds + \frac{1}{2}\int\limits_G f^2(x)\,dx.\end{aligned} \tag{4.5}$$

Now we write $\Gamma_+ = \Gamma_{0+}^d \cup \Gamma_{d+}$. At first, we estimate $b\int_{\Gamma_{0+}^d}\frac{u(x)}{r}u(\gamma(x))\,ds$. Because of $u|_{\Gamma_{0+}^d} = u(r,\frac{\omega_0}{2})$ and by Remark 1.1 $u(\gamma(x))|_{\Gamma_{0+}^d} = u(r,0)$, using the representation $u(r,0) = u(r,\frac{\omega_0}{2}) - \int_0^{\frac{\omega_0}{2}}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega$, we obtain:

$$\begin{aligned}b\int\limits_{\Gamma_{0+}^d}\frac{u(x)}{r}u(\gamma(x))\,ds &= b\int\limits_0^d\frac{u(r,\frac{\omega_0}{2})u(r,0)}{r}\,dr\\ &= b\int\limits_0^d\frac{u^2(r,\frac{\omega_0}{2})}{r}\,dr - b\int\limits_0^d\frac{u(r,\frac{\omega_0}{2})}{r}\left(\int\limits_0^{\frac{\omega_0}{2}}\frac{\partial u(r,\omega)}{\partial\omega}d\omega\right)dr.\end{aligned} \tag{4.6}$$

Next, by the Cauchy inequality, we have

$$\begin{aligned}&b\int\limits_0^d\frac{u(r,\frac{\omega_0}{2})}{r}\left(\int\limits_0^{\frac{\omega_0}{2}}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega\right)dr\\ &\le b\int\limits_{G_0^d}\frac{1}{r^2}\Big|u\Big(r,\frac{\omega_0}{2}\Big)\Big|\Big|\frac{\partial u(r,\omega)}{\partial\omega}\Big|\,dx\\ &\le b\int\limits_{G_0^d}\frac{1}{r^2}\left(\frac{\varepsilon}{2}\Big|\frac{\partial u(r,\omega)}{\partial\omega}\Big|^2 + \frac{1}{2\varepsilon}u^2\Big(r,\frac{\omega_0}{2}\Big)\right)dx\\ &\le \frac{b\varepsilon}{2}\int\limits_{G_0^d}|\nabla u|^2\,dx + \frac{b}{2\varepsilon}\int\limits_0^d\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}}\frac{u^2(r,\frac{\omega_0}{2})}{r}\,d\omega\,dr\\ &\le \frac{b\varepsilon}{2}\int\limits_{G_0^d}|\nabla u|^2\,dx + \frac{b\omega_0}{2\varepsilon}\int\limits_{\Gamma_{0+}^d}\frac{u^2(x)}{r}\,ds, \qquad \forall\varepsilon>0.\end{aligned} \tag{4.7}$$

By the assumption (e) the integral over Γ_{d+} we estimate as below:

$$b\int\limits_{\Gamma_{d+}}\frac{u(x)}{r}u(\gamma(x))\,ds \le b\frac{\operatorname{meas}\Gamma_+}{d}M_0^2.$$

Thus, from the assumption (e) and (4.5)–(4.7) we get

$$\Big(\frac{\nu}{4}-\frac{b\varepsilon}{2}\Big)\int\limits_G|\nabla u|^2\,dx + \Big(\frac{1}{2}\beta_+ + b - \frac{b\omega_0}{2\varepsilon}\Big)\int\limits_{\Gamma_+}\frac{u^2(x)}{r}\,ds + \frac{1}{2}\beta_-\int\limits_{\Gamma_-}\frac{u^2(x)}{r}\,ds$$

$$\le C(M_0,\, b,\, d,\, G) + \frac{1}{2\beta_0}\int\limits_{\Gamma_+} rg^2(x)\,ds + \frac{1}{2\beta_0}\int\limits_{\Gamma_-} rh^2(x)\,ds + \frac{1}{2}\int\limits_{G} f^2(x)\,dx, \qquad \forall\varepsilon > 0.$$

If we choose $\varepsilon = \frac{\nu}{4b}$, then, in virtue of assumption (c) for β_+, we obtain

$$\int\limits_G |\nabla u|^2\,dx + \int\limits_{\partial G} \frac{u^2(x)}{r}\,ds \le C\Bigg\{ \int\limits_G f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)\,ds + \int\limits_{\Gamma_-} h^2(x)\,ds \Bigg\}.$$

Finally, by Hardy–Friedrichs–Wirtinger inequality (2.4) with $\alpha = 2$, we get the desired estimate (4.1).

5. Local integral weighted estimates

Theorem 5.1

Let $u(x)$ be a weak solution of problem (L) and λ be as in (1.1). Let assumptions (a) − (e) be satisfied with $\mathcal{A}(r)$ being Dini-continuous at zero. Then there are $d \in (0, \frac{1}{e})$ and a constant $C > 0$ depending only on s, λ, ν, b, β_+, d, G, M_0 and on $\int_0^{\frac{1}{e}} \frac{\mathcal{A}(r)}{r}\,dr$ such that $\forall\varrho \in (0, d)$

$$\int\limits_{G_0^\varrho} \Big(|\nabla u|^2 + \frac{u^2(x)}{r^2}\Big)\,dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds$$
$$\le C\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2) + \|f\|^2_{2,G} + \|g\|^2_{2,\Gamma_+} + \|h\|^2_{2,\Gamma_-}\Big)$$
$$\cdot \begin{cases} \varrho^{\frac{2\lambda k}{\sqrt{q}}}, & \text{if } s > \dfrac{\lambda k}{\sqrt{q}}, \\ \varrho^{\frac{2\lambda k}{\sqrt{q}}} \ln^2\Big(\dfrac{1}{\varrho}\Big), & \text{if } s = \dfrac{\lambda k}{\sqrt{q}}, \\ \varrho^{2s}, & \text{if } 1 < s < \dfrac{\lambda k}{\sqrt{q}}, \end{cases} \tag{5.1}$$

where k and q are defined by (1.3).

Proof. Setting $\eta(x) = u(x)$ in $(II)_{loc}$, we obtain

$$\int\limits_{G_0^\varrho} |\nabla u|^2\,dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds$$
$$= \varrho \int\limits_{\Omega} u(x)\frac{\partial u}{\partial r}\bigg|_{r=\varrho}\,d\omega + \int\limits_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0))u(x)u_{x_j}\cos(r, x_i)\,d\Omega_\varrho$$
$$+ \int\limits_{\Gamma_{0+}^\varrho} u(x)g(x)\,ds - b\int\limits_{\Gamma_{0+}^\varrho} \frac{u(x)}{r}u(\gamma(x))\,ds + \int\limits_{\Gamma_{0-}^\varrho} u(x)h(x)\,ds$$

$$+\int_{G_0^\varrho}\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)-u(x)f(x)\}\,dx.$$

To estimate the integral $b\cdot\int_{\Gamma_{0+}^\varrho}\frac{u(x)}{r}u(\gamma(x))\,ds$ we behave similarly to (4.6)–(4.7). Then we get:

$$\begin{aligned}
&\Big(1-\frac{b\varepsilon}{2}\Big)\int_{G_0^\varrho}|\nabla u|^2\,dx+\beta_+\Big(1+\frac{b}{\beta_+}-\frac{b\omega_0}{2\beta_+\varepsilon}\Big)\int_{\Gamma_{0+}^\varrho}\frac{u^2(x)}{r}\,ds+\beta_-\int_{\Gamma_{0-}^\varrho}\frac{u^2(x)}{r}\,ds\\
&\quad\le\varrho\int_{\Omega}u(x)\frac{\partial u}{\partial r}\Big|_{r=\varrho}\,d\omega+\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho \qquad (5.2)\\
&\quad+\int_{\Gamma_{0+}^\varrho}u(x)g(x)\,ds+\int_{\Gamma_{0-}^\varrho}u(x)h(x)\,ds\\
&\quad+\int_{G_0^\varrho}\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)-u(x)f(x)\}\,dx.
\end{aligned}$$

By assumption (c) $\beta_+>\frac{b^2\omega_0}{4}-b$. Therefore we can choose in (5.3) $\varepsilon=\frac{\sqrt{1+\omega_0\beta_+}-1}{\beta_+}$. Hence it follows that

$$0<1-\frac{b\varepsilon}{2}=1+\frac{b}{\beta_+}-\frac{b\omega_0}{2\varepsilon\beta_+}=1+\frac{b}{2\beta_+}-\frac{b\sqrt{1+\omega_0\beta_+}}{2\beta_+}=k \qquad (5.3)$$

(see (1.3)) and recalling (2.8) we obtain

$$\begin{aligned}
&kU(\varrho)\\
&\quad\le\varrho\int_{\Omega}u(x)\frac{\partial u}{\partial r}\Big|_{r=\varrho}\,d\Omega+\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho\\
&\quad+\int_{\Gamma_{0+}^\varrho}u(x)g(x)\,ds+\int_{\Gamma_{0-}^\varrho}u(x)h(x)\,ds \qquad (5.4)\\
&\quad+\int_{G_0^\varrho}\big\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)-u(x)f(x)\big\}\,dx.
\end{aligned}$$

Now, we shall derive an upper bound for the each integral from the right hand side of (5.4). The first integral we estimate by Lemma 2.7; next, in virtue of assumption (b) and the Cauchy inequality,

$$\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho\le\varrho\mathcal{A}(\varrho)\int_{\Omega}|u(x)||\nabla u|\,d\omega,$$

$$\int_{G_0^\varrho}\{(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u_{x_i}u(x)+c(x)u^2(x)\}\,dx \qquad (5.5)$$

$$\leq \mathcal{A}(\varrho)\int\limits_{G_0^\varrho}\left\{|\nabla u|^2+\frac{u^2(x)}{r^2}\right\}dx.$$

Thus, from (5.4)–(5.5) it follows that

$$\begin{aligned} kU(\varrho) \leq{}& \frac{\varrho\sqrt{q}}{2\lambda}U'(\varrho)+\varrho\mathcal{A}(\varrho)\int\limits_{\Omega}|u(x)||\nabla u|\,d\omega \\ &+\int\limits_{\Gamma_{0+}^\varrho}|u(x)||g(x)|\,ds+\int\limits_{\Gamma_{0-}^\varrho}|u(x)||h(x)|\,ds \\ &+\mathcal{A}(\varrho)\int\limits_{G_0^\varrho}\Big(|\nabla u|^2+\frac{u^2(x)}{r^2}\Big)\,dx+\int\limits_{G_0^\varrho}|u(x)||f(x)|\,dx. \end{aligned} \tag{5.6}$$

Further, we derive an upper bound for each integral on the right hand side of (5.6). At first, applying the Cauchy and Friedrichs–Wirtinger inequalities (see (2.2)) with regard to (2.9), we have

$$\begin{aligned} &\mathcal{A}(\varrho)\int\limits_{\Omega}\varrho|u(x)||\nabla u|\,d\omega \\ &\leq \frac{1}{2}\mathcal{A}(\varrho)\int\limits_{\Omega}(\varrho^2|\nabla u|^2+|u(x)|^2)\,d\omega \\ &\leq \frac{1}{2}\mathcal{A}(\varrho)\int\limits_{\Omega}\varrho^2\Big[\Big(\frac{\partial u}{\partial r}\Big)^2+\frac{1}{\varrho^2}\Big(\frac{\partial u}{\partial \omega}\Big)^2\Big]\Big|_{r=\varrho}\,d\omega \\ &\quad+\frac{1}{2}\mathcal{A}(\varrho)\frac{q}{\lambda^2}\Bigg\{\int\limits_{\Omega}\Big(\frac{\partial u}{\partial\omega}\Big)^2\,d\omega+\beta_+u^2\Big(\varrho,\frac{\omega_0}{2}\Big)+\beta_-u^2\Big(\varrho,-\frac{\omega_0}{2}\Big)\Bigg\} \\ &\leq \frac{1}{2}\varrho\mathcal{A}(\varrho)\Big(1+\frac{q}{\lambda^2}\Big)\Bigg\{\int\limits_{\Omega}\Big[\varrho\Big(\frac{\partial u}{\partial r}\Big)^2+\frac{1}{\varrho}\Big(\frac{\partial u}{\partial\omega}\Big)^2\Big]\Big|_{r=\varrho}\,d\omega \\ &\quad+\beta_+\frac{u^2(\varrho,\frac{\omega_0}{2})}{\varrho}+\beta_-\frac{u^2(\varrho,-\frac{\omega_0}{2})}{\varrho}\Bigg\} \\ &\leq c_1(b,\beta_+,\omega_0,\lambda)\varrho\mathcal{A}(\varrho)U'(\varrho). \end{aligned} \tag{5.7}$$

Next, using the Cauchy and Hardy–Friedrichs–Wirtinger (see (2.4) for $\alpha = 2$) inequalities, by (2.8), we obtain

$$\begin{aligned} &\mathcal{A}(\varrho)\int\limits_{G_0^\varrho}\Big(|\nabla u|^2+\frac{|u|^2}{r^2}\Big)\,dx \\ &\leq c(b,\beta_+,\omega_0\lambda)\mathcal{A}(\varrho)\Bigg\{\int\limits_{G_0^\varrho}|\nabla u|^2\,dx+\beta_+\int\limits_{\Gamma_{0+}^\varrho}\frac{u^2(x)}{r}\,ds+\beta_-\int\limits_{\Gamma_{0-}^\varrho}\frac{u^2(x)}{r}\,ds\Bigg\} \\ &\leq c_2(b,\beta_+,\omega_0,\lambda)\mathcal{A}(\varrho)U(\varrho), \end{aligned} \tag{5.8}$$

and for all $\delta > 0$

$$\begin{aligned}
\int\limits_{\Gamma_{0+}^{\varrho}} |u(x)||g(x)|\,ds &= \int\limits_{\Gamma_{0+}^{\varrho}} \left(\sqrt{\frac{\beta_+}{r}}|u(x)|\right)\left(\sqrt{\frac{r}{\beta_+}}|g(x)|\right) ds \\
&\le \frac{\delta\beta_+}{2}\int\limits_{\Gamma_{0+}^{\varrho}} \frac{u^2(x)}{r}\,ds + \frac{1}{2\delta\beta_0}\int\limits_{\Gamma_{0+}^{\varrho}} rg^2(x)\,ds; \\
\int\limits_{\Gamma_{0-}^{\varrho}} |u(x)||h(x)|\,ds &= \int\limits_{\Gamma_{0-}^{\varrho}} \left(\sqrt{\frac{\beta_-}{r}}|u(x)|\right)\left(\sqrt{\frac{r}{\beta_-}}|h(x)|\right) ds \\
&\le \frac{\delta\beta_-}{2}\int\limits_{\Gamma_{0-}^{\varrho}} \frac{u^2(x)}{r}\,ds + \frac{1}{2\delta\beta_0}\int\limits_{\Gamma_{0-}^{\varrho}} rh^2(x)\,ds; \\
\int\limits_{G_0^{\varrho}} |u(x)||f(x)|dx &\le \frac{\delta}{2}\int\limits_{G_0^{\varrho}} \frac{u^2(x)}{r^2}\,dx + \frac{1}{2\delta}\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx \\
&\le \frac{\delta}{2}c_3(b,\beta_+,\omega_0,\lambda)U(\varrho) + \frac{1}{2\delta}\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx
\end{aligned} \tag{5.9}$$

in virtue of inequality (2.4). From (5.6)–(5.9) it follows

$$\begin{aligned}
&\langle k - c_4(\delta + \mathcal{A}(\varrho))\rangle\, U(\varrho) \\
&\quad \le \frac{\varrho\sqrt{q}}{2\lambda}(1 + c_5\mathcal{A}(\varrho))U'(\varrho) \\
&\quad + \frac{1}{2\delta}\left\{\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx + \frac{1}{\beta_0}\int\limits_{\Gamma_{0+}^{\varrho}} rg^2(x)\,ds + \frac{1}{\beta_0}\int\limits_{\Gamma_{0-}^{\varrho}} rh^2(x)\,ds\right\}, \qquad \forall\delta > 0.
\end{aligned} \tag{5.10}$$

But, by condition (d),

$$\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx + \frac{1}{\beta_0}\int\limits_{\Gamma_{0+}^{\varrho}} rg^2(x)\,ds + \frac{1}{\beta_0}\int\limits_{\Gamma_{0-}^{\varrho}} rh^2(x)\,ds \le \frac{1}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\cdot \varrho^{2s}.$$

Now we take into account that, by (5.3), $0 < k < 1$ and therefore

$$\begin{aligned}
\frac{k - c_4(\delta + \mathcal{A}(\varrho))}{1 + c_5\mathcal{A}(\varrho)} &= 1 - \frac{1 - k + c_4(\delta + \mathcal{A}(\varrho)) + c_5\mathcal{A}(\varrho)}{1 + c_5\mathcal{A}(\varrho)} \\
&\ge k[1 - c_6\delta - c_7\mathcal{A}(\varrho)], \qquad \forall\delta > 0.
\end{aligned}$$

Thus, from (5.10) we have differential inequality (CP) of Subsection 2.2 with

$$\begin{aligned}
\mathcal{P}(\varrho) &= \frac{2\lambda k}{\varrho\sqrt{q}}\cdot[1 - c_6\delta - c_7\mathcal{A}(\varrho)], \qquad \forall\delta > 0; \\
\mathcal{Q}(\varrho) &= \frac{\lambda}{2s\sqrt{q}}\left(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\right)\cdot\delta^{-1}\varrho^{2s-1}, \qquad \forall\delta > 0;
\end{aligned} \tag{5.11}$$

and, by (2.8) and Theorem 4.1,

$$U_0 = C(1+\beta_+ + \beta_-)\left\{ \int\limits_G f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x) ds + \int\limits_{\Gamma_-} h^2(x)\,ds \right\}.$$

We shall consider three cases:

1) $\boldsymbol{s > \frac{\lambda k}{\sqrt{q}}}$.

Choosing $\delta = \varrho^{\varepsilon}$, $\forall \varepsilon > 0$,

$$\mathcal{P}(\varrho) = \frac{2\lambda k}{\varrho\sqrt{q}} \cdot [1 - c_6 \varrho^{\varepsilon} - c_7 \mathcal{A}(\varrho)];$$
$$\mathcal{Q}(\varrho) = \frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \cdot \varrho^{2s-1-\varepsilon}.$$

Since $\mathcal{P}(\varrho) = \frac{2\lambda k}{\varrho\sqrt{q}} - \frac{\mathcal{K}(\varrho)}{\varrho}$, where $\mathcal{K}(\varrho)$ satisfies the Dini condition at zero, we have

$$-\int\limits_{\varrho}^{\tau} \mathcal{P}(s)\,ds = -\frac{2\lambda k}{\sqrt{q}} \ln\Big(\frac{\tau}{\varrho}\Big) + \int\limits_{\varrho}^{\tau} \frac{\mathcal{K}(s)}{s}\,ds \le \ln\Big(\frac{\varrho}{\tau}\Big)^{\frac{2\lambda k}{\sqrt{q}}} + \int\limits_0^d \frac{\mathcal{K}(r)}{r}\,dr$$
$$\Longrightarrow \exp\Bigg(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)\,d\sigma\Bigg) \le \Big(\frac{\varrho}{\tau}\Big)^{\frac{2\lambda k}{\sqrt{q}}} \exp\Bigg(\int\limits_0^d \frac{\mathcal{K}(\tau)}{\tau} d\tau\Bigg) = K_0 \Big(\frac{\varrho}{\tau}\Big)^{\frac{2\lambda k}{\sqrt{q}}};$$
$$\exp\Bigg(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)\,d\tau\Bigg) \le \Big(\frac{\varrho}{d}\Big)^{\frac{2\lambda k}{\sqrt{q}}} \exp\Bigg(\int\limits_0^d \frac{\mathcal{K}(\tau)}{\tau}\,d\tau\Bigg) = K_0 \Big(\frac{\varrho}{d}\Big)^{\frac{2\lambda k}{\sqrt{q}}}.$$

As well we have:

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau) \exp\Bigg(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)\,d\sigma\Bigg) d\tau$$
$$\le \frac{\lambda K_0}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \varrho^{\frac{2\lambda k}{\sqrt{q}}} \int\limits_{\varrho}^{d} \tau^{2s - \frac{2\lambda k}{\sqrt{q}} - \varepsilon - 1}\,d\tau$$
$$\le \frac{\lambda K_0}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}\big(g_0^2 + h_0^2\big)\Big) \cdot \frac{d^{s - \frac{\lambda k}{\sqrt{q}}}}{s - \frac{\lambda k}{\sqrt{q}}} \varrho^{2\frac{\lambda k}{\sqrt{q}}},$$

since $s > \frac{\lambda k}{\sqrt{q}}$ and we can choose $\varepsilon = s - \frac{\lambda k}{\sqrt{q}}$.

Now we apply Theorem 2.8: from (2.11), by virtue of the deduced inequalities and with regard to (2.4) for $\alpha = 2$, we obtain the required statement for $s > \frac{\lambda k}{\sqrt{q}}$.

2) $\boldsymbol{s = \frac{\lambda k}{\sqrt{q}}}$.

Taking in (5.11) any function $\delta(\varrho) > 0$ instead of $\delta > 0$, we obtain problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda k(1 - c_6\delta(\varrho))}{\varrho\sqrt{q}} - c_8\frac{\mathcal{A}(\varrho)}{\varrho};$$
$$\mathcal{Q}(\varrho) = \frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \cdot \delta^{-1}(\varrho)\varrho^{2\frac{\lambda k}{\sqrt{q}}-1}.$$

We choose $\delta(\varrho) = \frac{\sqrt{q}}{2c_6\lambda k\ln(\frac{ed}{\varrho})}$, $0 < \varrho < d$, where e is the Euler number. Then we obtain

$$-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma \le -\frac{2\lambda k}{\sqrt{q}}\ln\frac{\tau}{\varrho} + \int_\varrho^\tau \frac{d\sigma}{\sigma\ln(\frac{ed}{\sigma})} + c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma$$
$$= \ln\Big(\frac{\varrho}{\tau}\Big)^{2\frac{\lambda k}{\sqrt{q}}} + \ln\left(\frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\tau}}\right) + c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma$$
$$\Longrightarrow\ \exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\right) \le \Big(\frac{\varrho}{\tau}\Big)^{2\frac{\lambda k}{\sqrt{q}}} \cdot \frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\tau}} \cdot \exp\left(c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}d\sigma\right),$$
$$\exp\left(-\int_\varrho^d \mathcal{P}(\tau)\,d\tau\right) \le \Big(\frac{\varrho}{d}\Big)^{2\frac{\lambda k}{\sqrt{q}}} \cdot \ln\frac{ed}{\varrho} \cdot \exp\left(c_8\int_0^d \frac{\mathcal{A}(\tau)}{\tau}\,d\tau\right).$$

In this case we also have

$$\int_\varrho^d \mathcal{Q}(\tau)\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\right)d\tau$$
$$\le \frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \cdot \varrho^{2\frac{\lambda k}{\sqrt{q}}}\exp\left(c_8\int_0^d \frac{\mathcal{A}(\tau)}{\tau}\,d\tau\right)\ln\frac{ed}{\varrho}$$
$$\times\int_\varrho^d \delta^{-1}(\tau)\tau^{-1}\frac{1}{\ln(\frac{ed}{\tau})}\,d\tau$$
$$\le c_9\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \cdot \varrho^{2\frac{\lambda k}{\sqrt{q}}}\ln^2\Big(\frac{ed}{\varrho}\Big).$$

Now we apply Theorem 2.8, and from (2.11), by virtue of the deduced inequalities, we obtain

$$U(\varrho) \le c_{10}\Big(U_0 + \omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big)\varrho^{2\frac{\lambda k}{\sqrt{q}}}\ln^2\frac{1}{\varrho}, \qquad 0 < \varrho < d < \frac{1}{e}.$$

Thus, we proved the required statement for $s = \frac{\lambda k}{\sqrt{q}}$.

3) $\mathbf{0 < s < \frac{\lambda k}{\sqrt{q}}}$.

Now, similar to case 1) with regard to (5.11) we have

$$\exp\left(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\right)\leq\left(\frac{\varrho}{\tau}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}\exp\left(\int\limits_{0}^{d}\frac{\mathcal{A}(\sigma)}{\sigma}\,d\tau\right)=c_{11}\left(\frac{\varrho}{\tau}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}},$$

and

$$\exp\left(-\int\limits_{\varrho}^{d}\mathcal{P}(\tau)\,d\tau\right)\leq\left(\frac{\varrho}{d}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}\exp\left(\int\limits_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}\,d\tau\right)=c_{11}\left(\frac{\varrho}{d}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}.$$

In this case we also have

$$\begin{aligned}&\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\right)d\tau\\&\quad\leq\frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2+\frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\delta^{-1}\varrho^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}\times\int\limits_{\varrho}^{d}\tau^{2s-\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}-1}d\tau\\&\quad\leq c_{12}\Big(\omega_0 f_0^2+\frac{1}{\nu_0}g_0^2+\frac{1}{\nu_0}h_0^2\Big)\cdot\varrho^{2s},\end{aligned}$$

if we choose $\delta\in(0,\frac{1}{c_6}(1-\frac{s\sqrt{q}}{\lambda k}))$.

We again apply Theorem 2.8 and from (2.11), by virtue of the deduced inequalities, we obtain

$$\begin{aligned}U(\varrho)&\leq c_{13}\Big\{U_0\varrho^{\frac{2\lambda k(1-c_5\delta)}{\sqrt{q}}}+\Big(\omega_0 f_0^2+\frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\varrho^{2s}\Big\}\\&\leq c_{14}\Big(U_0+f_0^2+\frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\varrho^{2s}.\end{aligned}$$

Thus, we proved the required statement of Theorem 5.1 for $0<s<\frac{\lambda k}{\sqrt{q}}$.

6. The power modulus of continuity at the conical point for weak solutions

Proof of Theorem 1.5. We define the function

$$\psi(\varrho)=\begin{cases}\varrho^{\frac{\lambda k}{\sqrt{q}}}, & \text{if } s>\dfrac{\lambda k}{\sqrt{q}},\\ \varrho^{\frac{\lambda k}{\sqrt{q}}}\ln\left(\dfrac{1}{\varrho}\right), & \text{if } s=\dfrac{\lambda k}{\sqrt{q}},\\ \varrho^{s}, & \text{if } 1<s<\dfrac{\lambda k}{\sqrt{q}}\end{cases}$$

for $0<\varrho<d$.

For the proof we apply theorem 3.1 about the local bound of the weak solution modulus

$$\sup_{G_0^{\varkappa\varrho}} |u(x)| \leq \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}} \Big\{ \varrho^{-1}||u||_{2,G_0^\varrho} + \varrho^{2(1-\frac{2}{p})}||f||_{\frac{p}{2},G_0^\varrho} + \varrho\Big(||g||_{\infty,\Gamma_{0+}^\varrho} + ||h||_{\infty,\Gamma_{0-}^\varrho}\Big)\Big\}.$$

Then, by Theorem 5.1, we obtain

$$\begin{aligned}\varrho^{-1}\|u\|_{2,G_0^\varrho} &\leq \Bigg(\int_{G_0^\varrho} \frac{u^2(x)}{r^2}\,dx\Bigg)^{\frac{1}{2}} \\ &\leq C(\|f\|_{2,G} + \|g\|_{2,\Gamma_+} + \|h\|_{2,\Gamma_-} + \sqrt{\omega_0}f_0 + \frac{1}{\sqrt{\beta_0}}(g_0+h_0))\psi(\varrho).\end{aligned} \tag{6.1}$$

Further, by the assumption (d), we get

$$\begin{aligned}&\varrho^{2(1-\frac{2}{p})}\|f\|_{\frac{p}{2},G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_{0+}^\varrho} + \|h\|_{\infty,\Gamma_{0-}^\varrho}) \\ &\quad\leq c\Big(f_0 + \frac{1}{\sqrt{\beta_0}}(g_0+h_0)\Big)\psi(\varrho),\end{aligned} \tag{6.2}$$

for $\widetilde{n} < p < 2\widetilde{n}$, $\forall \widetilde{n} > 2$. From (3.1), (6.1)–(6.2) it follows that

$$\sup_{G_{\varrho/4}^{\varrho/2}} |u(x)| \leq C\Big(\|f\|_{2,G} + \|g\|_{2,\Gamma_+} + \|h\|_{2,\Gamma_-} + f_0 + \frac{1}{\sqrt{\beta_0}}(g_0+h_0)\Big)\psi(\varrho).$$

Putting $|x| = \frac{1}{3}\varrho$ we obtain finally the desired estimate (1.2).

Acknowledgement

This is part of the authors Ph.D. thesis, written under the supervision of prof. dr hab. Mikhail Borsuk at the University of Warmia and Mazury in Olsztyn.

References

[1] M. Borsuk, V. Kondratiev, *Elliptic boundary value problems of second order in piecewise smooth domains.* North-Holland Mathematical Library, **69**, Elsevier, 2006.

[2] I.M. Gel'fand, G.E. Shilov, *Generalized functions*, vol. 1, Academic Press, New York-London, 1964.

[3] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin/Heidelberg/New York, 1977. Revised Third Printing, 1998.

[4] P.L. Gurevich, *Asymptotics of solutions for nonlocal elliptic problems in plane angles*, J. Math. Sci. **120** (2004), 1295–1312.

[5] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, 1934; University Press, Cambridge, 1952.

[6] O.A. Ladyzhenskaya, *Boundary value problems of mathematical physics*, Nauka , Moscow, 1973 (in Russian).

[7] O.ALadyzhenskaya, N.N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[8] Gary M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing, Singapore-New Jersey-London-Hong Kong, 1996.

[9] M.H.V. Murthy, G. Stampacchia, *Boundary value problem for some degenerate elliptic operators*, Ann. Math. Pura Appl. **80** (1968), 1–122.

[10] K. Rektorys, *Variational methods in mathematics, sience and engineering*, SNTL, Prague, 1980.

[11] A.L. Skubachevskii, *Nonclassical boundary-value problems I, II*, I: J. Math. Sci. **155** (2008), 199–334; II: Contemporary Mathematics. Fundamental Directions **33** (2009), 3–179 (in Russian).

[12] S.L. Sobolev, *Some applications of functional analysis in mathematical physics*, Trans. of Math. Monographs, **90** AMS, Providence, R. I. 1991.

Department of Mathematics and Informatics
University of Warmia and Mazury in Olsztyn
10-957 Olsztyn-Kortowo,
Poland
E-mail: krzysztof.zyjewski@uwm.edu.pl

Received: 25 September 2009; final version: 23 October 2010;
available online: 11 March 2011.