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A combinatorial proof of non-speciality of systems with at most 9 imposed base points

Abstract. It is known that the Segre–Gimigliano–Harbourne–Hirschowitz Conjecture holds for linear systems of curves with at most 9 imposed base fat points. We give a nice proof based on a combinatorial method of showing non-speciality of such systems. We will also prove, by the same method, that systems $\mathcal{L}(km; m^{\times k^2})$ and $\mathcal{L}(km+1; m^{\times k^2})$ are non-special.

1. Introduction

Let $p_1, \dots, p_r \in \mathbb{P}^2 = \mathbb{P}^2(\mathbb{K})$ be distinct points, where $\mathbb{K}$ is a field of characteristic 0. The points $p_1, \dots, p_r$ will be called *imposed base points*. Let $m_1, \dots, m_r$ be nonnegative integers. By $\mathcal{L}(d; m_1p_1, \dots, m_rp_r)$ we denote the linear system of plane curves of degree d with multiplicity at least m_j at p_j, $j = 1, \dots, r$. The dimension of $\mathcal{L}(d; m_1p_1, \dots, m_rp_r)$ is upper semicontinuous in the position of imposed base points and reaches minimum for points in general position. This minimum will be denoted by

$$\dim \mathcal{L}(d; m_1, \dots, m_r).$$

We will also write $\mathcal{L}(d; m_1, \dots, m_r)$ for a system with imposed base points in general position, and $\mathcal{L}(d; m_1^{\times s_1}, \dots, m_r^{\times s_r})$ for repeated multiplicities. Define the *virtual dimension of* $\mathcal{L}(d; m_1, \dots, m_r)$

$$\operatorname{vdim} \mathcal{L}(d; m_1, \dots, m_r) = \frac{d(d+3)}{2} - \sum_{j=1}^{r} \binom{m_j+1}{2}$$

and the *expected dimension of* $\mathcal{L}(d; m_1, \dots, m_r)$

$$\operatorname{edim} \mathcal{L}(d; m_1, \dots, m_r) = \max\{\operatorname{vdim} \mathcal{L}(d; m_1, \dots, m_r), -1\}.$$

By linear algebra one has

$$\dim \mathcal{L}(d; m_1, \dots, m_r) \geq \operatorname{edim} \mathcal{L}(d; m_1, \dots, m_r)$$

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and $\mathcal{L}(d; m_1, \ldots, m_r)$ is said to be *special* if strict inequality holds for points in general position, *non-special* otherwise.

For systems $L = \mathcal{L}(d; m_1, \ldots, m_r)$, $L' = \mathcal{L}(d'; m'_1, \ldots, m'_r)$ we have the *intersection number* denoted by $L \cdot L'$,

$$L \cdot L' = dd' - \sum_{j=1}^{r} m_j m'_j.$$

Definition 1

The system $L = \mathcal{L}(d; m_1, \ldots, m_r)$ satisfying

- $\dim L = \operatorname{edim} L = 0$,
- *self-intersection* $L^2 = L \cdot L = -1$,
- the only curve in L is irreducible,

will be called a -1-*system*.

A curve $C \subset \mathbb{P}^2$ is said to be *in the base locus of* $\mathcal{L}(d; m_1, \ldots, m_r)$ if C is the component of each curve in $\mathcal{L}(d; m_1, \ldots, m_r)$. Observe that, by Bézout Theorem, if L is nonempty and $L \cdot L' = -t < 0$ for -1-system L', then the curve $C \in L'$ is in the base locus of L at least t times, i.e., the equation of each curve in L is divisible by f^t, where f is the equation of C. Such C is said to be a *multiple* -1-*curve in the base locus*, and it forces the system to be special:

$$\dim L \overset{\text{(by Lemma 2)}}{=} \dim(L - tL') \geq \operatorname{vdim}(L - tL') \overset{\text{(by Lemma 2)}}{>} \operatorname{vdim} L,$$

thus, by nonemptiness of L, we have also

$$\dim L > \operatorname{edim} L.$$

Lemma 2

Let $L = \mathcal{L}(d; m_1, \ldots, m_r)$, let $L' = \mathcal{L}(d'; m'_1, \ldots, m'_r)$ be a -1-system, let $L - tL' = \mathcal{L}(d - d'; m_1 - m'_1, \ldots, m_r - m'_r)$. If $L \cdot L' = -t < 0$, then

$$\dim(L - tL') = \dim L,$$
$$\operatorname{vdim}(L - tL') = \operatorname{vdim} L + \frac{t^2 - t}{2}.$$

The proof of the Lemma is postponed to the next section. The system with multiple -1-curve in the base locus will be called -1-*special*. We have seen that every -1-special system is special. The following conjecture due to Harbourne [13], Gimigliano [10] and Hirschowitz [15] states the following.

Conjecture 3

A system $\mathcal{L}(d; m_1, \ldots, m_r)$ with imposed base points in general position is special if and only if it is -1-special.

In [5] it is shown that the above Conjecture is equivalent to the conjecture posed by Segre [18].

Conjecture 4
If a system $L = \mathcal{L}(d; m_1, \dots, m_r)$ with imposed base points in general position is special, then every curve in L is non-reduced.

We will refer to either one of the above conjectures as to Segre–Harbourne–Gimigliano–Hirschowitz (SHGH for short) Conjecture. From now on we will assume that imposed base points are always in general position.

The SHGH Conjecture can be reformulated using standard systems. A system $\mathcal{L}(d; m_1, \dots, m_r)$ is called *standard* if $m_1 \geq m_2 \geq \dots \geq m_r$ and

$$d \geq m_1 + m_2 + m_3.$$

Theorem 5
In order to show that the SHGH Conjecture holds for at most r points it suffices to show that each standard system for at most r points is non-special.

For completeness, we will give a proof of this well-known Theorem in the next section.

The fact that the SHGH Conjecture holds for $r \leq 9$ points has been shown by various methods in [16], [10] and [12], but the first results appeared already in [2]. A nice idea is to use the following well-known fact.

Proposition 6
Let d, m_1, m_2, m_3 be nonnegative integers. If $d \geq m_1 + m_2 + m_3$, $m_1 \geq m_2 \geq m_3$ and the system $\mathcal{L}(d; m_1, m_2^{\times 3}, m_3^{\times 5})$ is non-special, then any standard system $\mathcal{L}(d; m_1, m_2, m_3, m_4, \dots, m_9)$ is non-special.

For completeness, we will give a proof of this proposition in the next section.

In the paper we will prove that SHGH holds for $r \leq 9$ points using only elementary facts based on linear algebra. In fact we must prove the following.

Theorem 7
Let d, m_1, m_2, m_3 be nonnegative integers. If $d \geq m_1 + m_2 + m_3$ and $m_1 \geq m_2 \geq m_3$, then the system $\mathcal{L}(d; m_1, m_2^{\times 3}, m_3^{\times 5})$ is non-special.

One of the main ingredients is the *cutting diagram algorithm* from [7]. Briefly, it is proved that in order to show non-speciality of a given system it suffices to find an appropriate finite set of points in $\mathbb{N}^2$ enjoying some combinatorial properties. To be precise, we must first define, for any finite $D \subset \mathbb{N}^2$, the system

$$\mathcal{L}(D; m_1, \dots, m_r)$$

of polynomials with support in D and with multiplicity at least m_j at p_j, $j = 1, \dots, r$. Formally, we identify $\mathbb{N}^2$ with monomials in $\mathbb{K}[X, Y]$

$$\mathbb{N}^2 \ni (x, y) \mapsto X^x Y^y \in \mathbb{K}[X, Y]$$

and put

$$\mathcal{L}(d; m_1, \dots, m_r) = \{f \in \mathbb{K}[X, Y] : \ \mathrm{supp}(f) \in D, \ \mathrm{mult}_{p_j}(f) \geq m_j, \ j = 1, \dots, k\}.$$

The set $\mathcal{L}(D; m_1, \dots, m_r)$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}[X, Y]$. We say that *conditions in $\mathcal{L}(D; m_1, \dots, m_r)$ are independent* if

$$\dim_{\mathbb{K}} \mathcal{L}(D; m_1, \dots, m_r) = \#D - \sum_{j=1}^{r} \binom{m_j + 1}{2}.$$

The system $\mathcal{L}(D; m_1, \dots, m_r)$ is called *empty* if

$$\dim_{\mathbb{K}} \mathcal{L}(D; m_1, \dots, m_r) = 0.$$

Observe that, by dehomogenizing and generality assumption, if conditions in $\mathcal{L}(D; m_1, \dots, m_r)$ are independent for $D = \{(x, y) : x + y \leq d\}$, then $\mathcal{L}(d; m_1, \dots, m_r)$ is non-special, similarly $\mathcal{L}(D; m_1, \dots, m_r)$ is empty if and only if $\mathcal{L}(d; m_1, \dots, m_r)$ is empty.

The cutting diagram algorithm is based on the following two theorems.

Theorem 8 ([7], Theorem 14)
Let $D, D' \subset \mathbb{N}^2$ be finite, let $m_1, \dots, m_r, m'_1, \dots, m'_s$ be nonnegative integers. If

- *$D \cap D' = \varnothing$,*
- *conditions in $\mathcal{L}(D; m_1, \dots, m_r)$ are independent (resp. $\mathcal{L}(D; m_1, \dots, m_r)$ is empty),*
- *conditions in $\mathcal{L}(D'; m'_1, \dots, m'_s)$ are independent (resp. $\mathcal{L}(D'; m'_1, \dots, m'_s)$ is empty),*
- *there exists an affine function $\mathbb{N}^2 \colon f \ni (a, b) \mapsto q_1 a + q_2 b + q_3 \in \mathbb{Q}$, $q_1, q_2, q_3 \in \mathbb{Q}$ such that f has strictly negative values on D and nonnegative values on D',*

then conditions in

$$L = \mathcal{L}(D \cup D'; m_1, \dots, m_r, m'_1, \dots, m'_s)$$

are independent (resp. L is empty).

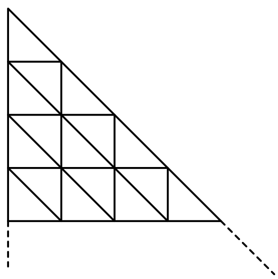
Theorem 9 ([7], Proposition 13)
Let $D \subset \mathbb{N}^2$ be finite, let m_1 be a nonnegative integer. Then conditions in $\mathcal{L}(D; m_1)$ are independent if and only if D, considered as a set of points in $\mathbb{Q}^2$, does not lie on a curve of degree $m_1 - 1$. If $\#D = \binom{m_j+1}{2}$ and conditions in $\mathcal{L}(D; m_1)$ are independent, then $\mathcal{L}(D; m_1)$ is empty.

The proofs are technical but use only simple linear algebra.

Theorem 10
Let k, m be nonnegative integers. Then systems $\mathcal{L}(km; m^{\times k^2})$ and $\mathcal{L}(km{+}1; m^{\times k^2})$ are non-special.

It is known that the above theorem holds. More generally, homogeneous systems with the square number of imposed base points are always non-special, see [8]. Such systems, i.e., homogeneous with the number of imposed base points satisfying some property have been widely studied. For example, systems of the form $\mathcal{L}(d; m^{\times 4^h})$ have been considered in [9]; this consideration has been extended to systems of the form $\mathcal{L}(d; m^{\times 4^h 9^k})$ in [1]; systems with the number of imposed base points being nearly a square have been considered in [4]; systems of the form $\mathcal{L}(d; m_1^{\times 9}, m_2, \dots, m_r)$ for $m_1 \geq m_2 \geq \dots \geq m_r$ (so called *quasiuniform*) in [14], and systems of the form $\mathcal{L}(d; m^{\times r})$ for $r \geq 4m^2$ in [17].

The proof of Theorem 10 using toric degenerations can be found in [3]. We will give a simple combinatorial proof in a sequence of lemmas. Both proofs exploit the natural dissection of a two-dimensional simplex into k^2 simplexes:



but the idea behind is slightly different. In the degeneration approach one controls the behaviour of the system "along" the intersection of two meeting regions (given always by weak inequalities). In our approach it is better to completely separate regions by defining them with strict inequalities.

Lemma 11
Conditions in the system $\mathcal{L}(D; m^{\times 16})$ are independent for

$$D = \{(x, y) \in \mathbb{N}^2 : \ x + y \leq 4m + 1\};$$

conditions in the system $\mathcal{L}(D; m^{\times 25})$ are independent for

$$D = \{(x, y) \in \mathbb{N}^2 : \ x + y \leq 5m + 1\};$$

thus systems $\mathcal{L}(4m + 1; m^{\times 16})$ and $\mathcal{L}(5m + 1; m^{\times 25})$ are non-special.

Lemma 12
Systems $\mathcal{L}(4m; m^{\times 16})$, $\mathcal{L}(5m; m^{\times 25})$, $\mathcal{L}(6m; m^{\times 36})$ and $\mathcal{L}(6m+1; m^{\times 36})$ are empty.

Lemma 13
Systems $\mathcal{L}(km; m^{\times k^2})$ and $\mathcal{L}(km + 1; m^{\times k^2})$ are empty for $k \geq 7$.

Proofs of lemmas are postponed to the next section.

2. Proofs

Proof of Lemma 2. To prove that $\dim(L - tL') = \dim L$ observe that multiplication by the equation of $C \in L'$ in tth power induces an isomorphism between $L - tL'$ and L. By a straightforward calculation one shows that

$$\operatorname{vdim}(L - tL') = \operatorname{vdim} L - tL \cdot L' + \frac{t^2 L'^2}{2} + \frac{t(-3d' + \sum_{j=1}^{r} m'_j)}{2}.$$

Moreover,

$$L'^2 - 2\operatorname{vdim} L' = -3d' + \sum_{j=1}^{r} m'_j,$$

which completes the proof.

Proof of Theorem 5. Let $L = \mathcal{L}(d; m_1, \ldots, m_r)$. Consider the following procedure:

Step 1. Sort multiplicities in non-increasing order.

Step 2. If $k = d - m_1 - m_2 < 0$, then take $d \longleftarrow d+k$, $m_1 \longleftarrow m_1+k$, $m_2 \longleftarrow m_2+k$ and go back to Step 1.

Step 3. If $k = d - m_1 - m_2 - m_3 < 0$, then take $d \longleftarrow d + k$, $m_j \longleftarrow m_j + k$ for $j = 1, 2, 3$ and go back to Step 1.

We finish with a system L'. We will show that in each step the dimension does not change. Indeed, if $k = d - m_1 - m_2$ is negative, then each curve in $\mathcal{L}(d; m_1, m_2, m_3, \ldots)$ is reducible and contains the line passing through p_1, p_2 at least $-k$ times. In other words, we have the isomorphism

$$\varphi\colon \mathcal{L}(d - k; m_1 - k, m_2 - k, m_3, \ldots) \to \mathcal{L}(d; m_1, m_2, m_3, \ldots)$$

given by multiplication by the equation of the line in kth power. In Step 3 the result follows from applying the Cremona transformation based on p_1, p_2, p_3 to our system (see eg. [11, Section 3]). This transformation induces the isomorphism

$$\varphi\colon \mathcal{L}(d - k; m_1 - k, m_2 - k, m_3 - k, m_4, \ldots) \to \mathcal{L}(d; m_1, m_2, m_3, m_4, \ldots)$$

(the proof of this fact using only linear algebra can be found in [6, proof of Theorem 3]; we use the fact that the system passed Step 2, so $d - m_1 - m_2 \geq 0$). By an easy computation one can show that the virtual dimension does not change in Step 3, while in Step 2 it increases by $\frac{k^2+k}{2}$. Thus for $k \leq -2$ we obtain L to be either empty or special. In the second case, we know that after some Cremona transformations there exists a multiple line in the base locus. Again, by easy computations we can show that Cremona transformation preserves the intersection number, hence the multiple line from the base locus will be mapped, by the reversed process, into a multiple -1-curve in the base locus of L. Therefore L is either -1-special or enjoys the same properties (dimension, virtual dimension, emptiness, speciality…) as L', which is standard.

Proof of Proposition 6. Assume, by hypothesis, that $L_2 = \mathcal{L}(d; m_1, \dots, m_9)$ is special. We will show that $L_1 = \mathcal{L}(d; m_1, m_2^{\times 3}, m_3^{\times 5})$ is special. Let c be the difference between the number of conditions in L_1 and the number of conditions in L_2,

$$c = \binom{m_1+1}{2} + 3\binom{m_2+1}{2} + 5\binom{m_3+1}{2} - \sum_{j=1}^{9}\binom{m_j+1}{2}.$$

Since each condition can lower the dimension by at most one, we have

$$\dim L_1 \geq \dim L_2 - c > \operatorname{edim} L_2 - c \geq \operatorname{vdim} L_2 - c = \operatorname{vdim} L_1.$$

Since for $d \geq m_1 + m_2 + m_3$, the virtual dimension

$$\begin{aligned} \operatorname{vdim} L_1 \geq\ & \frac{(m_1+m_2+m_3)(m_1+m_2+m_3+3)}{2} \\ & - \frac{m_1(m_1+1) + 3m_2(m_2+1) + 5m_3(m_3+1)}{2} \\ =\ & (m_1 - m_3) + m_2(m_1 - m_2) + m_3(m_1 + m_2 - 2m_3) \\ \geq\ & 0, \end{aligned}$$

we have $\operatorname{vdim} L_1 = \operatorname{edim} L_1$ and consequently

$$\dim L_1 > \operatorname{edim} L_1.$$

Before proving Theorem 7 we must prepare some helpful systems with independent conditions.

Definition 14
Let m be a positive integer. Define an *m-rectangle* to be the set

$$\left\{(x,y) \in \mathbb{N}^2 :\ a - \frac{1}{2} < x < a + m + \frac{1}{2},\ b - \frac{1}{2} < y < b + m - \frac{1}{2}\right\}$$

or the set

$$\left\{(x,y) \in \mathbb{N}^2 :\ a - \frac{1}{2} < x < a + m - \frac{1}{2},\ b - \frac{1}{2} < y < b + m + \frac{1}{2}\right\}$$

for some nonnegative integers a, b. Define an *m-triangle* to be the set

$$\left\{(x,y) \in \mathbb{N}^2 :\ x > a - \frac{1}{2},\ y > a - \frac{1}{2},\ x + y < 2a + m - \frac{1}{2}\right\}$$

for some nonnegative integer a. The examples are shown on Figure 1.

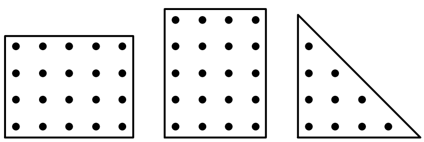


Figure 1. Example of 4-rectangles and 4-triangle

Lemma 15
Let T be an m-triangle, let R be an m-rectangle. Then conditions in the systems $\mathcal{L}(T; m)$ and $\mathcal{L}(R; m^{\times 2})$ are independent and these systems are empty.

Proof. Observe that there exists parallel lines $\ell_1, \dots, \ell_m$ such that $\#(T \cap \ell_j) = j$. The proof for $\mathcal{L}(T; m)$ is completed by Theorem 9 and Bézout Theorem.

To deal with $\mathcal{L}(R; m^{\times 2})$ observe that R can be divided into two pieces R_1, R_2, such that R_1 is an m-triangle, while R_2 is a rotated m-triangle. By Theorem 8 the proof is completed.

Proof of Theorem 7. Let $D = \{(x, y) \in \mathbb{N}^2 : x+y \le d\}$. We want to show that conditions in $\mathcal{L}(D; m_1, m_2^{\times 3}, m_3^{\times 5})$ are independent. Take the following cutting of D into three pieces:

$$D_1 = \Big\{(x, y) \in D : \ y > m_2 + m_3 + \frac{1}{2}\Big\},$$
$$D_2 = \Big\{(x, y) \in D : \ y < m_2 + m_3 + \frac{1}{2} \ \text{ and } \ (m_3 + 2)y + x > m_3^2 + 3m_3 - \frac{1}{2}\Big\},$$
$$D_3 = \Big\{(x, y) \in D : \ (m_3 + 2)y + x < m_3^2 + 3m_3 - \frac{1}{2}\Big\}.$$

By Theorem 8 it is enough to show that conditions in systems $\mathcal{L}(D_1; m_1)$, $\mathcal{L}(D_2; m_2^{\times 3})$, $\mathcal{L}(D_3; m_3^{\times 5})$ are independent. Observe that, by easy computations, an m_1-triangle with vertices $(0, m_2 + m_3 + 1)$, $(m_1 - 1, m_2 + m_3 + 1)$ and $(0, m_1 + m_2 + m_3)$ is contained in D_1. Similarly, observe that an m_2-rectangle with vertices $(0, m_3 + 1)$, $(m_2, m_3 + 1)$, $(m_2, m_3 + m_2)$, $(0, m_3 + m_2)$ and an m_2-triangle with vertices $(m_2 + 1, m_3)$, $(2m_2, m_3)$, $(m_2 + 1, m_3 + m_2 - 1)$ are contained in D_2. Moreover, these two shapes can be separated from each other by an affine line. For D_3, we take three shapes — an m_3-rectangle with vertices $(0, 0)$, $(m_3 - 1, 0)$, $(m_3 - 1, m_3)$, $(0, m_3)$, another m_3-rectangle with vertices $(m_3, 0)$, $(2m_3, 0)$, $(2m_3, m_3 - 1)$, $(m_3, m_3 - 1)$ and finally an m_3-triangle with vertices $(2m_3 + 1, 0)$, $(3m_3, 0)$, $(2m_3 + 1, m_3 - 1)$. By Theorem 8 and Lemma 15 the proof is completed.

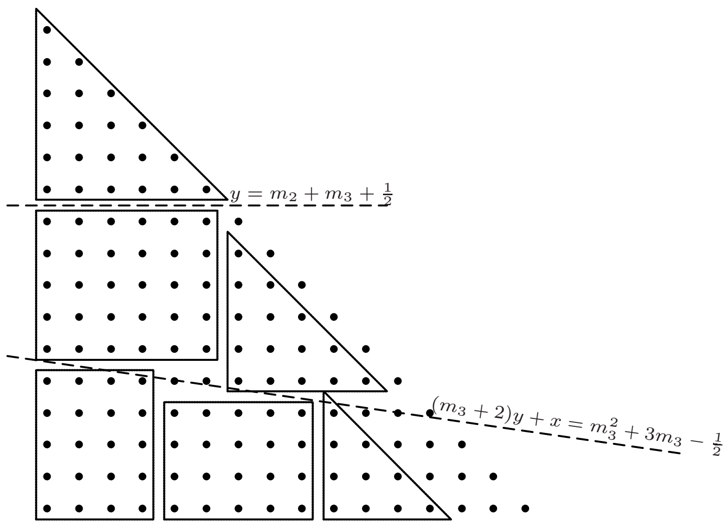


Figure 2. Example of divisions for $m_1 = 6$, $m_2 = 5$, $m_3 = 4$

Proof of Lemma 11. The proofs can be easily read off from Figures 3 and 4. The pictures are drawn for $m = 3$, but can be easily rescaled. Less obvious cuttings are presented, the details are left to the reader. By ε we denote a sufficiently small positive rational number.

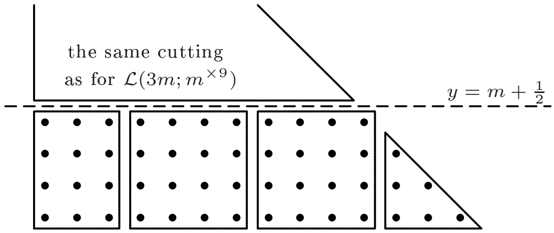


Figure 3. Divisions for $\mathcal{L}(4m + 1; m^{\times 16})$

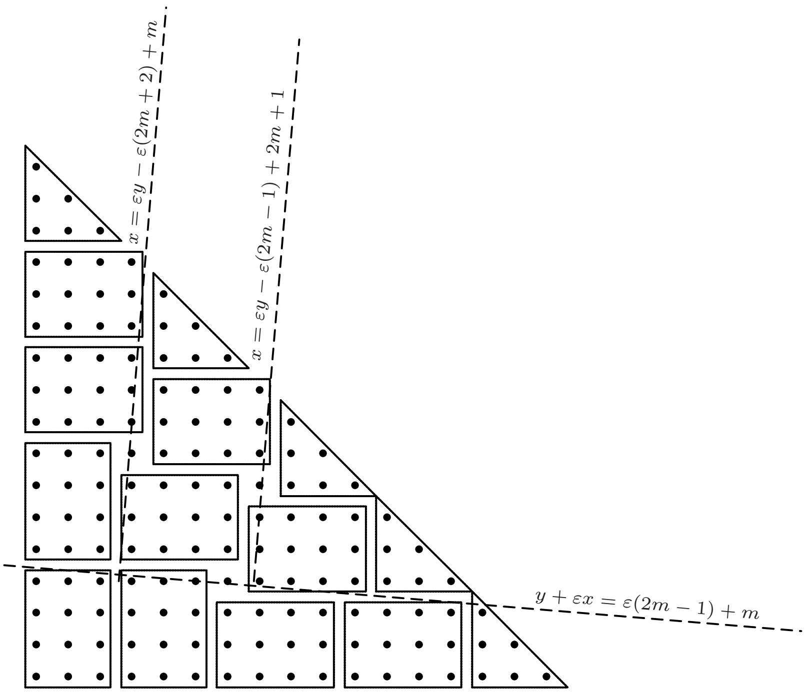


Figure 4. Divisions for $\mathcal{L}(5m + 1; m^{\times 25})$

Proof of Lemma 12. Emptiness of $\mathcal{L}(6m; m^{\times 36})$ would follow from emptiness of $\mathcal{L}(6m+1; m^{\times 36})$. Again, the proofs can be easily read off from Figures 5, 6 and 7. Observe that if $R \subset \mathbb{N}^2$ is contained in some m-rectangle, then $\mathcal{L}(R; m^{\times 2})$ is empty.

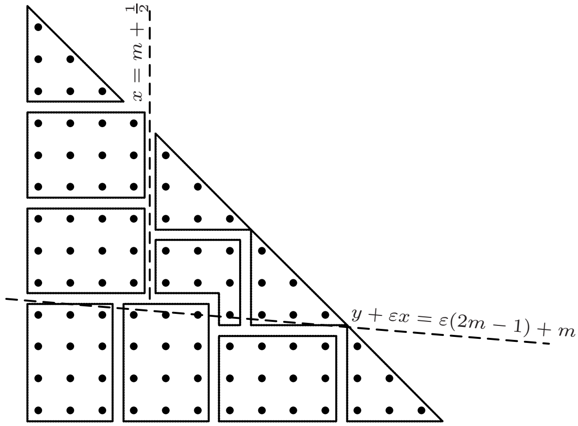


Figure 5. Divisions for $\mathcal{L}(4m; m^{\times 16})$

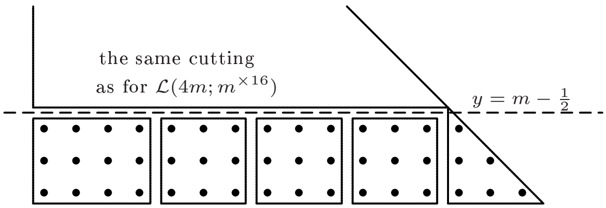


Figure 6. Divisions for $\mathcal{L}(5m; m^{\times 25})$

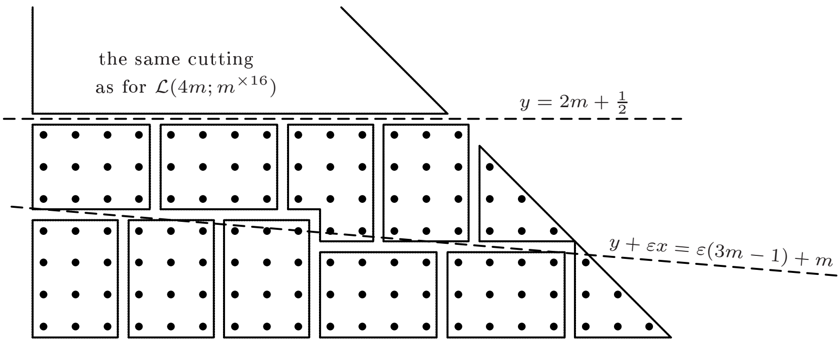


Figure 7. Divisions for $\mathcal{L}(6m + 1; m^{\times 36})$

Proof of Lemma 13. Emptiness of $\mathcal{L}(km; m^{\times k^2})$ would follow from emptiness of $\mathcal{L}(km + 1; m^{\times k^2})$. The first cutting, into upper and bottom part, is given by the line $y = m - \frac{1}{2}$. Since $k - 1 \geq 6$, we use induction to the upper part, cutting it exactly as $\mathcal{L}((k-1)m + 1; m^{\times (k-1)^2})$. The bottom part

$$B = \{(x, y) \in \mathbb{N}^2 : \; x + y \leq km + 1, \; y \leq m\}$$

gives the system $\mathcal{L}(B; m^{\times(2k-1)})$. We will cover B from right to left with one m-triangle and $(k-1)$ m-rectangles of hight m. This allows to cover $(k-1)(m+1)+m = km+k-1$ lattice points $(x,0) \in B$, while $\#\{(x,0) \in B\} = km+2$. Thus we can entirely cover B and the proof is completed.

Remark 16
There is no theoretical obstruction to make similar proofs for systems of the form $\mathcal{L}(km+k_0; m^{\times k^2})$ for fixed k_0. In fact, for k satisfying $k \geq k_0+2$ the induction step (emptiness of $\mathcal{L}(km+k_0; m^{\times k^2})$ implies emptiness of $\mathcal{L}((k+1)m+k_0; m^{\times (k+1)^2})$) will work. One can even hope that for k's satisfying $k \leq K+1$,

$$K = \max\{k : \ \operatorname{vdim} \mathcal{L}(km+k_0; m^{\times k^2}) \geq 0 \text{ for some } m\},$$

it is always possible to prove non-speciality by the presented method.

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