FOLIA 70

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica VIII (2009)

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A note on some iterative roots

Abstract. In this paper some orientation-preserving iterative roots of an orientation-preserving homeomorphism $F\colon S^1 \to S^1$ which possess periodic points of order n are considered. Namely, iterative roots with periodic points of order n. All orders of such roots are determined and their general construction is given.

Let X be a nonempty set. A function $g\colon X \to X$ is called an *iterative root* of a given function $f\colon X \to X$ if $g^m(x) = f(x)$ for $x \in X$. The number $m \geq 2$ is called the *order of the iterative root* and g^m denotes m-th iterate of g. Moreover, we say that $x \in X$ is a *periodic point* of f of order $n \in \mathbb{N}$, $n > 1$ if

$$f^n(x) = x \quad \text{and} \quad f^k(x) \neq x \text{ for } k \in \{1, \dots, n-1\}.$$

If $f(x) = x$, then x is said to be a *fixed point of* f. The set of all periodic (fixed) points of f will be denoted by $\operatorname{Per} f$ ($\operatorname{Fix} f$).

In [9] M.C. Zdun proved that every orientation-preserving homeomorphism $F\colon S^1 \to S^1$ possessing periodic points of order n is a composition of two orientation-preserving homeomorphisms $T, G\colon S^1 \to S^1$. Function G has no periodic points except fixed points and T is such that $T^n = \mathrm{id}_{S^1}$. Using this result he determined all continuous iterative roots with periodic points for homeomorphisms having fixed points.

In the present paper we apply Zdun's theorem to the problem of finding some continuous iterative roots for an orientation-preserving homeomorphism $F\colon S^1 \to S^1$ with periodic points of order n. Namely, we shall give conditions under which continuous iterative roots with periodic points of order n exist and give the construction of these roots.

Now, we recall some useful notations and definitions related to the mappings of the circle. Let $u, w \in S^1$ and $u \neq w$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2 < t_1 + 1$ and $u = e^{2\pi i t_1}$ and $w = e^{2\pi i t_2}$. Put

$$\overrightarrow{(u,w)} := \left\{ e^{2\pi i t} : \ t \in (t_1, t_2) \right\}, \quad \overrightarrow{[u,w]} := \overrightarrow{(u,w)} \cup \{u, w\}, \quad \overrightarrow{[u,w)} := \overrightarrow{(u,w)} \cup \{u\}.$$

AMS (2000) Subject Classification: 39B12, 26A18.

Volumes I-VII appeared as *Annales Academiae Paedagogicae Cracoviensis Studia Mathematica.*

These sets are called arcs.

For every homeomorphism $F\colon S^1 \to S^1$ there exists a unique (up to translation by an integer) homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$F\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} f(x)}$$

and

$$f(x+1) = f(x) + k$$

for all $x \in \mathbb{R}$, where $k \in \{-1, 1\}$. We call F *orientation-preserving* if $k = 1$, which is equivalent to the fact that f is increasing.

Moreover, for every continuous function $G\colon I \to J$, where $I = \{e^{2\pi \mathrm{i} t} : t \in [a,b]\}$ and $J = \{e^{2\pi \mathrm{i} t} : t \in [c,d]\}$ there exists a unique continuous function $g\colon [a,b] \to [c,d]$ such that

$$G\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} g(x)}, \qquad x \in [a,b].$$

In this case we also call g the lift of G and we say that G preserves orientation if g is strictly increasing.

For any orientation-preserving homeomorphism $F\colon S^1 \to S^1$, the limit

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1, \qquad x \in \mathbb{R}$$

always exists and does not depend on the choice of x and f. This number is called the *rotation number* of F (see [3]). It is known that $\alpha(F)$ is a rational and positive number if and only if F has a periodic point (see for example [3]). If $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where q, n are positive integers with $q < n$ and $\gcd(q,n) = 1$, then $\operatorname{Per} F$ contains only periodic points of order n (see [7], [5]). Moreover, there exists a unique number $p \in \{1, \ldots, n-1\}$, called the *characteristic number of* F, satisfying $pq = 1 \pmod n$. From now on put $n_F := n$ and $\operatorname{char} F := p$. The following result comes from [8].

Lemma 1

If $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism with $\operatorname{Per} F \neq \emptyset$, then for every $z \in \operatorname{Per} F$,

$$\operatorname{Arg} \frac{F^{k \operatorname{char} F}(z)}{z} < \operatorname{Arg} \frac{F^{(k+1) \operatorname{char} F}(z)}{z}, \qquad k = 0, \ldots, n_F - 2.$$

For fixed $z \in \operatorname{Per} F$ we define the partition of S^1 onto the following arcs

$$I_k = I_k(z) := \overrightarrow{\left[F^{k \operatorname{char} F}(z), F^{(k+1) \operatorname{char} F}(z)\right)}, \qquad k \in \{0, \ldots, n_F - 1\}. \tag{1}$$

Let us note that

$$\begin{aligned} F[I_k] &= \overrightarrow{\left[F^{k \operatorname{char} F + 1}(z), F^{(k+1) \operatorname{char} F + 1}(z)\right)} \\ &= \overrightarrow{\left[F^{k \operatorname{char} F + q \operatorname{char} F}(z), F^{(k+1) \operatorname{char} F + q \operatorname{char} F}(z)\right)} \\ &= I_{(k+q) \pmod{n_F}}, \qquad\qquad k \in \{0, \ldots, n_F - 1\}, \end{aligned}$$

where $q = n_F \alpha(F)$.

We shall use the following property (see [9]).

Remark 1
Let $n \in \mathbb{N}$ and $p, q \in \{0, \ldots, n-1\}$ satisfy $pq = 1 \pmod n$ and $\gcd(q, n) = 1$. The mapping $\{0, \ldots, n-1\} \ni d \mapsto i_d := -dp \pmod n \in \{0, \ldots, n-1\}$ is a bijection. Moreover, $d + i_d q = 0 \pmod n$.

The next theorem also comes from [9] and it is a modification of the factorization theorem (see [9], Theorems 5 and 9).

Theorem 1
Let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism, $z \in \operatorname{Per} F$ and let $\{I_d\}_{d \in \{0,\ldots,n_F-1\}}$ be the family defined in (1). Then there exists a unique orientation-preserving homeomorphism $T\colon S^1 \to S^1$ having periodic points of order n_F and such that $\operatorname{Per} T = S^1$ and

$$F^{k+jn_F}_{|I_d} = T^{\alpha(F)n_F k} \circ \begin{cases} T^d \circ (F^{n_F})^{j+1} \circ T^{-d}_{|I_d}, & \text{if } i_d \le k-1, \\ T^d \circ (F^{n_F})^{j} \circ T^{-d}_{|I_d}, & \text{if } i_d > k-1 \end{cases}$$

for $d, k \in \{0, 1, \ldots, n_F - 1\}$, $j \in \mathbb{N}$.

Let us stress that T is unique up to a periodic point of F. Moreover, $F^{n_F}[I_d] = I_d$, $T[I_d] = I_{(d+1) \pmod n}$ for $d \in \{0, \ldots, n_F - 1\}$ and $T^{n_F} = \mathrm{id}_{S^1}$. Such a function T will be called a *Babbage function of* F (see [9]).

In view of the above theorem (see also [9], Corollary 6) for every orientation-preserving homeomorphism $F\colon S^1 \to S^1$ with $\emptyset \ne \operatorname{Per} F$ and for every $z_0 \in \operatorname{Per} F$ we have

$$F(z) := \begin{cases} T^q\left(F^{n_F}(z)\right), & z \in I_0(z_0), \\ T^q(z), & z \in S^1 \setminus I_0(z_0), \end{cases} \tag{2}$$

where $q = \alpha(F)n_F$ and T is a Babbage function of F.

We start with the following

Remark 2
Let $n, m \ge 2$ be integers and let $q, q' \in \{1, \ldots, n-1\}$ be such that $\gcd(q, n) = 1$ and $mq' = q \pmod n$, then $\gcd(m, n) = 1$.

Proof. To obtain a contradiction suppose that $m = ka$ and $n = kb$ for some integers $k > 1$ and $a, b \ge 1$. This and the fact that $mq' = q \pmod n$ give $kaq' = q + jkb$ for some $j \in \mathbb{Z}$. Therefore $k(aq' - jb) = q$, which contradicts the fact that $\gcd(q, n) = 1$.

Remark 3
Let $n, m \ge 2$ be relatively prime integers and let $q \in \{1, \ldots, n-1\}$ be such that $\gcd(q, n) = 1$. There is a unique $q' \in \{1, \ldots, n-1\}$ such that $\gcd(q', n) = 1$ and $mq' = q \pmod n$.

Proof. The fact that $\gcd(m, n) = 1$ implies that the equation $mx + ny = q$ has integer solutions x, y. In particular, there is exactly one pair (q', j), where $q' \in \{0, \ldots, n-1\}$ and $j \in \mathbb{Z}$ such that $mq' + jn = q$. Thus $mq' = q \pmod n$. Moreover, $q' \ne 0$ as $\gcd(q, n) = 1$. In the same manner as in the proof of Remark 2 we can see that $\gcd(q', n) = 1$.

From Remark 2 we can conclude that

Corollary 1
Let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with $\emptyset \neq \operatorname{Per} F$ and let $m \geq 2$ be an integer. If equation

$$G^m(z) = F(z), \qquad z \in S^1 \tag{3}$$

has continuous and orientation-preserving solution such that $n_G = n_F$, then

$$\gcd(m, n_F) = 1.$$

It appears that $\gcd(m, n_F) = 1$ is also a sufficient condition for the existence of continuous and orientation-preserving solutions of (3) with $n_G = n_F$. The proof of this property and the description of the solution of (3) in the case $\operatorname{Per} F = S^1$ can be found in [6]. Therefore, from now on assume that $\operatorname{Per} F \neq S^1$. Before we present some results let as recall that if (3) holds, then $\operatorname{Per} F = \operatorname{Per} G$.

Lemma 2
Let $F, G\colon S^1 \to S^1$ be orientation-preserving homeomorphisms possessing periodic points of order $n_F = n_G = n$ and satisfying equation (3) for an $m \geq 2$. Let moreover $z_0 \in \operatorname{Per} F = \operatorname{Per} G$ and $J_k := \overrightarrow{[G^{k \operatorname{char} G}(z_0), G^{(k+1) \operatorname{char} G}(z_0))}$, $k \in \{0, \ldots, n-1\}$. Then

(i) *$J_k = I_k(z_0)$ for $k \in \{0, \ldots, n-1\}$, where the arcs $I_k(z_0)$ are defined by (1);*

(ii) *$(G^{n_G})^m = F^{n_F}$;*

(iii) *if T and V are Babbage functions of F and G, respectively, $[x]$ stands for an integer part of $x \in \mathbb{R}$ and $i'_d := -d \operatorname{char} G \pmod n$ for $d \in \{0, \ldots, n-1\}$, then*

$$V^q_{|I_d} = T^{\alpha(F)n} \circ T^d \circ G^{n\beta_d} \circ T^{-d}_{|I_d}, \tag{4}$$

where

$$\beta_d := \begin{cases} m - \left[\frac{m}{n}\right] - 1, & d = 0, \\ -\left[\frac{m}{n}\right] - 1, & d \in \{1, \ldots, n-1\},\ i'_d \leq m - \left[\frac{m}{n}\right] n - 1, \\ -\left[\frac{m}{n}\right], & d \in \{1, \ldots, n-1\},\ i'_d > m - \left[\frac{m}{n}\right] n - 1. \end{cases} \tag{5}$$

Proof. Fix $z_0 \in \operatorname{Per} F$ and assume that (3) holds, $n_F = n_G = n$. Put $q := \alpha(F)n$, $q' := \alpha(G)n$ and $b := \left[\frac{m}{n}\right]$. From the fact that $\gcd(m, n) = 1$ (see Corollary 1), we get

$$m = k + bn \qquad \text{for some } k \in \{1, \ldots, n-1\}. \tag{6}$$

To prove (i) it suffices to show that $G^{\operatorname{char} G}(z_0) = F^{\operatorname{char} F}(z_0)$. Equation (3) yields $m\alpha(G) = \alpha(F) \pmod 1$ (see [2]). Thus $mq' = q \pmod n$, hence

$$mq' \operatorname{char} F \operatorname{char} G = q \operatorname{char} F \operatorname{char} G \pmod n$$

and finally, in view of the definition of $\operatorname{char} F$,

$$m \operatorname{char} F = \operatorname{char} G \pmod{n}. \tag{7}$$

From (7), (3) and since z_0 is a periodic point of G of order n we obtain

$$G^{\operatorname{char} G}(z_0) = G^{m \operatorname{char} F}(z_0) = F^{\operatorname{char} F}(z_0).$$

Note that (ii) is an immediate consequence of equation (3) and equality $n_F = n_G$.

Now we prove (iii). From Theorem 1, (6) and (i) we get

$$G^m_{|I_d} = G^{k+bn}_{|I_d} = V^{q'k} \circ \begin{cases} V^d \circ (G^n)^{b+1} \circ V^{-d}_{|I_d}, & \text{if } i'_d \le k-1, \\ V^d \circ (G^n)^b \circ V^{-d}_{|I_d}, & \text{if } i'_d > k-1 \end{cases}$$

for $d \in \{0, 1, \ldots, n-1\}$. Furthermore, observe that condition $mq' = q \pmod{n}$ and (6) give $kq' = q \pmod{n}$, which, in view of the fact that V is a Babbage function of G of order n, implies $V^{q'k} = V^q$. Therefore,

$$G^m_{|I_d} = V^q \circ \begin{cases} V^d \circ (G^n)^{b+1} \circ V^{-d}_{|I_d}, & \text{if } i'_d \le k-1, \\ V^d \circ (G^n)^b \circ V^{-d}_{|I_d}, & \text{if } i'_d > k-1 \end{cases} \tag{8}$$

for $d \in \{0, 1, \ldots, n-1\}$.

On the other hand, we may write (5), as follows

$$\beta_d = \begin{cases} m-b-1, & d = 0, \\ -b-1, & d \in \{1, \ldots, n-1\},\ i'_d \le k-1, \\ -b, & d \in \{1, \ldots, n-1\},\ i'_d > k-1. \end{cases}$$

Let $d = 0$, then $i'_0 = 0 \le k-1$ and $b = m - \beta_0 - 1$. Combining these with (8) we obtain

$$G^m_{|I_0} = V^q \circ (G^n_{|I_0})^{b+1} = V^q \circ (G^n_{|I_0})^{m-\beta_0}.$$

Let $d \in \{1, \ldots, n-1\}$. Replacing b by $-\beta_d - 1$ if $i'_d \le k-1$ (resp. by $-\beta_d$ if $i'_d > k-1$) in (8) yields

$$G^m_{|I_d} = V^q \circ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}.$$

Finally,

$$G^m_{|I_d} = V^q \circ \begin{cases} V^d \circ G^{-n\beta_d + mn} \circ V^{-d}_{|I_d}, & d = 0, \\ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}, & d \in \{1, \ldots, n-1\}. \end{cases} \tag{9}$$

Equating (9) with (2) yields for $d \in \{1, \ldots, n-1\}$,

$$T^q_{|I_d} = V^q \circ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}. \tag{10}$$

While, for $d = 0$, we get

$$T^q \circ F^n_{|I_0} = V^q \circ G^{-n\beta_0 + nm}_{|I_0}.$$

which, in view of (ii), gives

$$T^q_{|I_0} = V^q \circ G^{-n\beta_0}_{|I_0}. \tag{11}$$

From (10) and (11) we have

$$T^q_{|I_d} = V^q \circ V^d \circ G^{-n\beta_d} \circ V^{-d}_{|I_d}, \qquad d \in \{0, \dots, n-1\}. \tag{12}$$

Hence

$$T^q_{|I_{p \,(\mathrm{mod}\, n)}} = V^q \circ V^{p \,(\mathrm{mod}\, n)} \circ G^{-n\beta_{p \,(\mathrm{mod}\, n)}} \circ V^{-p \,(\mathrm{mod}\, n)}_{|I_{p \,(\mathrm{mod}\, n)}}, \qquad p \in \mathbb{N}.$$

As $V^p = V^{p \,(\mathrm{mod}\, n)}$ for $p \in \mathbb{N}$ we obtain

$$T^q_{|I_{p \,(\mathrm{mod}\, n)}} = V^q \circ V^p \circ G^{-n\beta_{p \,(\mathrm{mod}\, n)}} \circ V^{-p}_{|I_{p \,(\mathrm{mod}\, n)}}, \qquad p \in \mathbb{N}. \tag{13}$$

Now let us recall that $T^q[I_d] = I_{(d+q) \,(\mathrm{mod}\, n)}$ for $d \in \{0, \dots, n-1\}$. This, (11) and (13) imply

$$\begin{aligned} T^{lq}_{|I_0} = (T^q)^l_{|I_0} &= \Big(V^q \circ V^{(l-1)q} \circ G^{-n\beta_{q(l-1) \,(\mathrm{mod}\, n)}} \circ V^{-(l-1)q}\Big) \\ &\quad \circ \Big(V^q \circ V^{(l-2)q} G^{-n\beta_{q(l-2) \,(\mathrm{mod}\, n)}} \circ V^{-(l-2)q}\Big) \\ &\quad \circ \dots \circ \big(V^q \circ V^q \circ G^{-n\beta_q} \circ V^{-q}\big) \circ \Big(V^q \circ G^{-n\beta_0}_{|I_0}\Big) \\ &= V^{lq} \circ G^{-n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)}_{|I_0}, \end{aligned}$$

which gives

$$V^{lq}_{|I_0} = T^{lq} \circ G^{n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)}_{|I_0} \tag{14}$$

for $l \in \{1, \dots, n\}$. Now fix $d \in \{1, \dots, n-1\}$. Since $\gcd(q, n) = 1$ there is a unique $l \in \{1, \dots, n\}$ such that $lq = d \,(\mathrm{mod}\, n)$. Hence by (13) we have

$$T^q_{|I_d} = T^q_{|I_{lq \,(\mathrm{mod}\, n)}} = V^q \circ V^{lq} \circ G^{-n\beta_{lq \,(\mathrm{mod}\, n)}} \circ V^{-lq}_{|I_{lq \,(\mathrm{mod}\, n)}}.$$

By substituting (14) twice to the above equation we obtain

$$\begin{aligned} T^q_{|I_d} &= V^q \circ \Big(T^{lq} \circ G^{n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)}\Big) \circ G^{-n\beta_{lq \,(\mathrm{mod}\, n)}} \\ &\quad \circ \Big(G^{-n\left(\beta_{q(l-1) \,(\mathrm{mod}\, n)} + \beta_{q(l-2) \,(\mathrm{mod}\, n)} + \dots + \beta_q + \beta_0\right)} \circ T^{-lq}_{|I_{lq \,(\mathrm{mod}\, n)}}\Big) \\ &= V^q \circ T^{lq} \circ G^{-n\beta_{lq \,(\mathrm{mod}\, n)}} \circ T^{-lq}_{|I_{lq \,(\mathrm{mod}\, n)}}. \end{aligned}$$

This and the fact that T is a Babbage homeomorphism of F of order n, i.e., $T^{lq} = T^{lq \,(\mathrm{mod}\, n)} = T^d$, yield

$$V^q_{|I_d} = T^q \circ T^d \circ G^{n\beta_d} \circ T^{-d}_{|I_d},$$

which in view of (11) completes the proof of (4).

Lemma 3
Let $u, w \in S^1$, $u \neq w$ and $I := \overrightarrow{[u, w]}$. For every integer $m \geq 2$ and every orientation-preserving homeomorphism $F\colon I \to I$ with $\operatorname{Fix} F \neq \emptyset$ there exist infinitely many orientation-preserving homeomorphisms $G\colon I \to I$ satisfying (3) *and such that* $\operatorname{Fix} G \neq \emptyset$.

Proof. Let $a, b \in \mathbb{R}$ be such that $a < b < a + 1$ and $u = e^{2\pi \mathrm{i} a}$ and $w = e^{2\pi \mathrm{i} b}$. Then

$$F\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} f(x)}, \qquad x \in [a, b]$$

for a unique increasing homeomorphism $f\colon [a, b] \to [a, b]$. Clearly, f possesses fixed points. By Theorem 11.2.2 (see [4] ch. 11), there exist infinitely many strictly increasing continuous solutions of

$$g^m(x) = f(x), \qquad x \in [a, b],$$

with $\operatorname{Fix} g \neq \emptyset$. For every such function $g\colon [a, b] \to [a, b]$ define $G\colon I \to I$ by

$$G\left(e^{2\pi \mathrm{i} x}\right) := e^{2\pi \mathrm{i} g(x)}, \qquad x \in [a, b].$$

Then $\operatorname{Fix} G \neq \emptyset$ and

$$G^m\left(e^{2\pi \mathrm{i} x}\right) = e^{2\pi \mathrm{i} g^m(x)} = e^{2\pi \mathrm{i} f(x)} = F(e^{2\pi \mathrm{i} x}), \qquad x \in [a, b].$$

In the proof of the next theorem we will use the following result (see for example [7]).

Lemma 4
Suppose that $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism, $z \in \operatorname{Per} F$, $\{z, F(z), \ldots, F^{n_F - 1}(z)\} = \{z_0, z_1, \ldots, z_{n_F - 1}\}$, where $z_0 = z$,

$$\operatorname{Arg}\frac{z_d}{z_0} < \operatorname{Arg}\frac{z_{d+1}}{z_0} < 2\pi, \qquad d \in \{0, \ldots, n_F - 2\}$$

and $F(z_0) = z_q$. Then $\alpha(F) = \frac{q}{n_F}$.

Theorem 2
Let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and let $m \geq 2$ be an integer such that $\gcd(m, n_F) = 1$. There exists an orientation-preserving homeomorphism $G\colon S^1 \to S^1$ satisfying (3) *and such that $n_G = n_F$.*
For every such an m and every $z_0 \in \operatorname{Per} F$, providing that $I_d = I_d(z_0)$ for $d \in \{0, \ldots, n_F - 1\}$ are defined by (1)*, the solution of* (3) *is of the form:*

$$G(z) := \begin{cases} \left(\Psi^{\operatorname{char} F}\right)^{q'}(H(z)), & z \in I_0, \\ \left(\Psi^{\operatorname{char} F}\right)^{q'}(z), & z \in S^1 \setminus I_0, \end{cases} \tag{15}$$

where $q' \in \{1, \ldots, n_F - 1\}$ fulfils $mq' = q \pmod{n_F}$, $q := n_F \alpha(F)$, $H\colon I_0 \to I_0$ is an orientation-preserving homeomorphism such that $\operatorname{Fix} H \neq \emptyset$, $H^m = F^{n_F}_{|I_0}$ and

$$\Psi(z) := T^q \circ T^d \circ H^{\beta_d} \circ T^{-d}(z), \qquad z \in I_d,\ d \in \{0, \ldots, n_F - 1\}, \tag{16}$$

where $T\colon S^1 \to S^1$ *is a Babbage function of* F *and* β_d *for* $d \in \{0,\dots,n_F-1\}$ *are defined by* (5) *with* $n = n_F$ *and* i'_d *uniquely determined by* $(d + i'_d q')\,(\mathrm{mod}\, n_F) = 0$ *for* $d \in \{0,\dots,n_F-1\}$.

Proof. Fix $z_0 \in \operatorname{Per} F$ and a mapping $H\colon I_0 \to I_0$ such that $\operatorname{Fix} H \neq \emptyset$ and $H^m = F^{n_F}_{|I_0}$ (by Lemma 3 there are infinitely many such mappings). Observe that

$$\Psi[I_d] = T^q \circ T^d \circ H^{\beta_d}[I_0] = T^q[I_d] = I_{(d+q)\,(\mathrm{mod}\, n)}, \qquad d \in \{0,\dots,n_F-1\}. \tag{17}$$

Moreover, as a composition of orientation-preserving homeomorphisms, $\Psi_{|I_d}$ is an orientation-preserving homeomorphism. Hence $\Psi\colon S^1 \to S^1$ and G are orientation-preserving homeomorphisms.

Now we show that $n_G = n_F$. Put $z_d := F^{d\,\mathrm{char}\,F}(z_0)$ for $d \in \{1,\dots,n_F-1\}$. Thus by (1), (17) and since Ψ preserves the orientation we get

$$\Psi(z_d) = z_{(d+q)\,(\mathrm{mod}\, n_F)}, \qquad d \in \{0,\dots,n_F-1\}.$$

This, Lemma 1 and Lemma 4 yield $\alpha(\Psi) = \frac{q}{n_F} = \alpha(F)$ and, in consequence, $n_\Psi = n_F$ and $\mathrm{char}\,\Psi = \mathrm{char}\,F$. Next note that $H(z_0) = z_0$ and $H(z_1) = z_1$. Therefore, by (15) and the definition of $\mathrm{char}\,F$,

$$G(z_d) = \Psi^{q'\,\mathrm{char}\,F}(z_d) = z_{(d+qq'\,\mathrm{char}\,F)\,(\mathrm{mod}\, n_F)} = z_{(d+q')\,(\mathrm{mod}\, n_F)}, \tag{18}$$

for $d \in \{0,\dots,n_F-1\}$. As $\gcd(q', n_F) = 1$ (see Remark 3) we get $n_F = n_G$.

Our next goal is to prove that $\Psi^{n_F} = \mathrm{id}_{S^1}$. From (17) and (16), in view of the fact that $T^p = T^{p\,(\mathrm{mod}\, n_F)}$ for $p \in \mathbb{N}$, we obtain

$$\begin{aligned}
\Psi^{n_F}_{|I_d} &= \Big(T^q \circ T^{d+(n_F-1)q} \circ H^{\beta_{(d+(n_F-1)q)\,(\mathrm{mod}\, n_F)}} \circ T^{-d+(n_F-1)q}\Big) \\
&\quad \circ \ldots \circ \Big(T^q \circ T^{d+q} \circ H^{\beta_{(d+q)\,(\mathrm{mod}\, n_F)}} \circ T^{-d+q}\Big) \circ \Big(T^q \circ T^d \circ H^{\beta_d} \circ T^{-d}_{|I_d}\Big) \\
&= T^q \circ T^{d+(n_F-1)q} \circ H^{\beta_{(d+(n_F-1)q)\,(\mathrm{mod}\, n_F)}+\ldots+\beta_d} \circ T^{-d}_{|I_d}
\end{aligned}$$

for $d \in \{0,\dots,n_F-1\}$. Moreover, since $\gcd(q, n_F) = 1$ we get

$$\{d, (d+q)\,(\mathrm{mod}\, n_F), \dots, (d+(n_F-1)q)\,(\mathrm{mod}\, n_F)\} = \{0,1,\dots,n_F-1\}.$$

We thus get

$$\Psi^{n_F}_{|I_d} = T^q \circ T^{d+(n_F-1)q} \circ H^{\beta_{n_F-1}+\ldots+\beta_0} \circ T^{-d}_{|I_d}, \qquad d \in \{0,\dots,n_F-1\}. \tag{19}$$

Putting $b := \left[\frac{m}{n_F}\right]$ we have (6) with $n = n_F$. By Remark 3 and Remark 1 it follows that the mapping $\{0,\dots,n_F-1\} \ni d \mapsto i'_d \in \{0,\dots,n_F-1\}$ is a bijection. Therefore, $i'_d \le m - bn_F - 1 = k-1$ holds true for exactly k arguments and one of them is 0, as $i'_0 = 0 \le k-1$. Hence in view of (5),

$$\beta_{n_F-1} + \ldots + \beta_0 = (n_F - k)(-b) + (k-1)(-b-1) + m - b - 1 = 0.$$

This and (19) give $\Psi^{n_F} = \mathrm{id}_{S^1}$.

What is left is to show that (3) holds. Put $\Psi^{\operatorname{char} F} = V$. By Theorem 1 homeomorphism V is a Babbage function of G. Since $q \operatorname{char} F = 1 \pmod{n_F}$ and $\Psi^{n_F} = \mathrm{id}_{S^1}$ we have $\Psi = \Psi^{q \operatorname{char} F} = V^q$. Hence by (16),

$$V^q_{|I_d} = T^q \circ T^d \circ H^{\beta_d} \circ T^{-d}_{|I_d}, \qquad d \in \{0, \dots, n_F - 1\}. \tag{20}$$

Applying the similar reasoning as in the proof of (iii) of Lemma 2 we obtain

$$T^q_{|I_d} = V^q \circ V^d \circ H^{-\beta_d} \circ V^{-d}_{|I_d}, \qquad d \in \{0, \dots, n_F - 1\}. \tag{21}$$

Indeed, as $T^p = T^{p \pmod{n_F}}$ for $p \in \mathbb{N}$ from (20) we get

$$V^q_{|I_{p \pmod{n_F}}} = T^q \circ T^p \circ H^{\beta_{p \pmod{n_F}}} \circ T^{-p}_{|I_{p \pmod{n_F}}}, \qquad p \in \mathbb{N}. \tag{22}$$

Thus

$$V^{lq}_{|I_0} = T^{lq} \circ H^{\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \dots + \beta_q + \beta_0\right)}_{|I_0},$$

which gives

$$T^{lq}_{|I_0} = V^{lq} \circ H^{-\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \dots + \beta_q + \beta_0\right)}_{|I_0} \tag{23}$$

for $l \in \{1, \dots, n_F\}$. Now fix $d \in \{1, \dots, n_F - 1\}$. Since $\gcd(q, n_F) = 1$ there is a unique $l \in \{1, \dots, n_F\}$ such that $lq = d \pmod{n_F}$. Hence by (22) we have

$$V^q_{|I_d} = V^q_{|I_{lq \pmod{n_F}}} = T^q \circ T^{lq} \circ H^{\beta_{lq \pmod{n_F}}} \circ T^{-lq}_{|I_{lq \pmod{n_F}}}.$$

By substituting (23) twice to the above equation we obtain

$$\begin{aligned} V^q_{|I_d} &= T^q \circ \left(V^{lq} \circ H^{-\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \dots + \beta_q + \beta_0\right)}\right) \circ H^{\beta_{lq \pmod{n_F}}} \\ &\quad \circ \left(H^{\left(\beta_{q(l-1) \pmod{n_F}} + \beta_{q(l-2) \pmod{n_F}} + \dots + \beta_q + \beta_0\right)} \circ V^{-lq}_{|I_{lq \pmod{n_F}}}\right) \\ &= T^q \circ V^{lq} \circ G^{\beta_{lq \pmod{n_F}}} \circ V^{-lq}_{|I_{lq \pmod{n_F}}}. \end{aligned}$$

This and the fact that V is a Babbage homeomorphism of G of order n_F, i.e., $V^{lq} = V^{lq \pmod{n_F}} = T^d$, yield (21).

Now observe that from (2) and (21), since $H^m = F^{n_F}_{|I_0}$ and $kq' = q \pmod{n_F}$, we get

$$F_{|I_d} = V^{kq'} \circ \begin{cases} V^d \circ H^{-\beta_d + m} \circ V^{-d}_{|I_d}, & d = 0, \\ V^d \circ H^{-\beta_d} \circ V^{-d}_{|I_d}, & d \in \{1, \dots, n_F - 1\}, \end{cases}$$

which in view of (15), (6) and Theorem 1 gives $F = G^m$.

We finish with the following observations

Remark 4

If the assumptions of Theorem 2 are fulfilled, then

(i) from (18), Lemma 4, Lemma 1 it follows that $\alpha(G) = \frac{q'}{n_F}$,

(ii) by Lemma 3 there are infinitely many solutions of (3) with $n_G = n_F$,

(iii) Lemma 2 and Theorem 2 imply that every orientation-preserving continuous solution of (3) with $n_G = n_F$ is given by (15) and (16).

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Received: 28 February 2009; final version: 24 July 2009;
available online: 12 October 2009.