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Report of Meeting

12th International Conference on Functional Equations and Inequalities, Będlewo, September 7-14, 2008

Contents

The *Twelfth International Conference on Functional Equations and Inequalities* was held from September 7 to 14, 2008 in Będlewo, Poland. The series of ICFEI meetings has been organized by the *Institute of Mathematics of the Pedagogical University of Cracow* since 1984. For the third time, the conference was organized jointly with the *Stefan Banach International Mathematical Center* and hosted by the *Mathematical Research and Conference Center* in Będlewo. As usual, the conference was devoted mainly to various aspects of functional equations and inequalities. A special emphasis was given to applications of functional equations. A Workshop on the latter theme, chaired by Prof. Vladimir Mityushev, followed the regular ICFEI meeting.

The Scientific Committee consisted of Professors Nicole Brillouët-Belluot, Dobiesław Brydak (Honorary Chairman), Janusz Brzdęk (Chairman), Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Marek Cezary Zdun and Jacek Chmieliński (Secretary). The Organizing Committee consisted of Janusz Brzdęk (Chairman), Vladimir Mityushev, Paweł Solarz, Janina Wiercioch and Władysław Wilk.

The 57 participants came from 11 countries: China, Germany, Greece, Hungary, Israel, Japan, Romania, Slovenia, USA, Russia and from Poland.

The Conference was opened on Monday morning, September 8 by Professor Janusz Brzdęk – Chairman of the Scientific and Organizing Committees. This ceremony was followed by the first scientific session chaired by Professor Bogdan Choczewski and the first lecture was given by Professor Roman Ger. Altogether, during 20 scientific sessions 4 lectures and 47 short talks were delivered. They focused on functional equations in a single variable and in several variables, functional inequalities, stability theory, convexity, multifunctions, theory of iteration, means, differential and difference equations, functional equations in functional analysis, functional equations in physics and other topics. Several contributions have been made during special *Problems and Remarks* sessions.

On Tuesday, September 9, a picnic was organized in the park surrounding the Center. On the next day afternoon participants visited Poznań with its old city, baroque parish church and National Museum. In the evening the piano recital was performed by Marek Czerni and Hans-Heinrich Kairies. On Thursday, September 21, a banquet was held in the Palace in Będlewo. It was an occasion to honour Prof. Zoltán Daróczy on the occasion of his 70th birthday, celebrated this year. On the following day a *Flamenco evening* was hosted by Małgorzata Drzał (dance & vocal) and Grzegorz Guzik (guitar).

The ICFEI conference was closed on Saturday, September 13 by Professor Bogdan Choczewski. In the closing address, he gave some concluding information about the meeting and conveyed best regards for the participants from the Honorary Chairman of the ICFEI, Professor Dobiesław Brydak. It was announced that Professor Zsolt Páles joined the ICFEI Scientific Committee and that the 13th ICFEI will be organized in 2009.

On Saturday afternoon, September 13 and Sunday morning, September 14 the Workshop devoted to applications of functional equations was held. 4 sessions were organized with 3 lectures, 2 talks and discussion.

The following part of the report contains abstracts of talks (in alphabetical order of the authors' names), problems and remarks (in chronological order of presentation) and the list of participants (with addresses). It has been compiled by Jacek Chmieliński.

Abstracts of Talks

Marcin Adam *On the double quadratic difference property*

Let X be a real normed space and Y a real Banach space. Denote by $C^n(X,Y)$ the class of n-times continuously differentiable functions $f\colon X \to Y$. We prove that the class C^n has the double quadratic difference property, that is if

$$Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y) \in C^n(X \times X, Y),$$

then there exists exactly one quadratic function $K\colon X \to Y$ such that $f - K \in C^n(X,Y)$.

Mirosław Adamek *On two variable functional inequality and related functional equation*

We present the result stating that the lower semicontinuous solutions of a large class of functional inequalities can be obtained from particular solutions of the related functional equations. Our main theorem reads as follows.

THEOREM
Let $\lambda\colon I^2 \longrightarrow (0,1)$ be a function and $n, m\colon I^2 \longrightarrow I$ be continuous strict means. If there exists a non-constant and continuous solution $\phi\colon I \longrightarrow \mathbb{R}$ of the equation

$$T^{\lambda}_{(n(x,y),m(x,y))}\phi = T^{\lambda}_{(x,y)}\phi, \qquad x, y \in I,$$

then ϕ is one-to-one, and a lower semicontinuous function $f\colon I \longrightarrow \mathbb{R}$ satisfies the inequality

$$T^{\lambda}_{(n(x,y),m(x,y))}f \leq T^{\lambda}_{(x,y)}f, \qquad x, y \in I,$$

if and only if $f \circ \phi^{-1}$ is convex on $\phi(I)$.

This result improves results presented in [1] and [2].

[1] J. Matkowski, M. Wróbel, *A generalized a-Wright convexity and related function equation*, Ann. Math. Silesianae **10** (1996), 7-12.

[2] Zs. Páles, *On two variable functional inequality*, C. R. Math. Rep. Acad. Sci. Canada, **10** (1988), 25-28.

Anna Bahyrycz *A system of functional equations related to plurality functions*

We consider the system of functional equations related to plurality functions:

$$f(x) \cdot f(y) \neq 0_m \implies f(x+y) = f(x) \cdot f(y),$$
$$f(rx) = f(x),$$

where $f\colon \mathbb{R}(n) := [0, +\infty)^n \setminus \{0_n\} \longrightarrow \mathbb{R}(m)$, $n, m \in \mathbb{N}$, $r \in \mathbb{R}(1)$ and

$$x+y := (x_1+y_1, ..., x_k+y_k), \quad x \cdot y := (x_1 \cdot y_1, ..., x_k \cdot y_k), \quad rx := (rx_1, ..., rx_k)$$

for $x = (x_1, ..., x_k) \in \mathbb{R}(k)$, $y = (y_1, ..., y_k) \in \mathbb{R}(k)$.

We investigate systems of cones over $\mathbb{R}$, which are the parameter determining the solutions of this system.

Karol Baron *Random-valued functions and iterative equations*

As emphasized in [1; 0.3], iteration is the fundamental technique for solving functional equations of the form

$$F(x, \varphi(x), \varphi \circ f(x, \cdot)) = 0,$$

and iterates usually appear in the formulae for solutions. Moreover, many results may be interpreted in both ways: either as theorems about the behaviour of iterates, or as theorems about solutions of functional equations. In this survey we are interested in formulae of the form

$$\varphi(x) = \text{probability that the sequence } (f^n(x,\cdot))_{n\in\mathbb{N}} \text{ converges and its limit belongs to } B,$$

where the iterates f^n, $n \in \mathbb{N}$, are defined as in [1; 1.4] and B is a Borel set. Such formulae defining solutions of

$$\varphi(x) = \int_\Omega \varphi(f(x,\omega))\mathrm{Prob}(d\omega)$$

are rather new in the theory of iterative functional equations, but as in more classical cases also results involving them may be read in two ways above described.

[1] Marek Kuczma, Bogdan Choczewski, Roman Ger, *Iterative Functional Equations*, Encyclopedia of Mathematics and its Applications, Vol. **32**, Cambridge University Press, Cambridge, 1990.

Nicole Brillouët-Belluot *Some aspects of functional equations in physics* (presented by **Joachim Domsta**)

Functional equations represent a way of modelling problems in physics. The physical problem is often directly stated in terms of one or several functional equations. However, a problem in physics may also be firstly described by a partial differential equation from which we derive a functional equation whose solutions solve the problem.

In this talk, I will present several examples of functional equations modelling physical problems in various fields of physics. In each example, I will mainly explain how the functional equation appears in the physical problem.

Janusz Brzdęk *Fixed point results and stability of functional equations in single variable*

Joint work with Roman Badora.

We show that stability of numerous functional equations in single variable is an immediate consequence of very simple fixed point results. We consider a generalization of the classical Hyers–Ulam stability (as suggested by T. Aoki, D.G. Bourgin and Th.M. Rassias), a modification of it, quotient stability (in the sense of R. Ger), and iterative stability.

Liviu Cădariu *Fixed points method for the generalized stability of monomial functional equations*

Joint work with Viorel Radu.

D.H. Hyers in 1941 gave an affirmative answer to a question of S.M. Ulam, concerning the stability of group homomorphisms in Banach spaces: *Let* E_1 *and* E_2 *be Banach spaces and* $f\colon E_1 \longrightarrow E_2$ *be such a mapping that*

$$\|f(x+y)-f(x)-f(y)\| \le \delta \tag{1}$$

for all $x, y \in E_1$ *and a* $\delta > 0$, *that is* f *is* $\delta-$additive. *Then there exists a unique* additive $T\colon E_1 \longrightarrow E_2$, *given by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \qquad x \in E_1\,, \tag{2}$$

which satisfies $\|f(x)-T(x)\| \le \delta$, $x \in E_1$.

T. Aoki, D. Bourgin and Th.M. Rassias studied the stability problem with unbounded Cauchy differences. Generally, the constant δ in (1) is replaced by a control function, $||\mathcal{D}_f(x,y)|| \le \delta(x,y)$, where, for example, $\mathcal{D}_f(x,y) = f(x+y)-f(x)-f(y)$ for Cauchy equation. The *stability estimations* are of the form $||f(x)-S(x)|| \le \varepsilon(x)$, where S *verifies the functional equation* $\mathcal{D}_S(x,y) = 0$, and for $\varepsilon(x)$ explicit formulae are given, which depend on the control δ as well as on the equation.

We use a fixed point method, initiated in [3] and developed, e.g., in [1], to give a generalized Ulam–Hyers stability result for functional equations in single variable and functions defined on groups, with values in sequentially complete locally convex spaces. This result is then used to obtain the generalized stability for some abstract monomial functional equations.

[1] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure and Appl. Math. **4(1)** (2003), Art. 4 (http://jipam.vu.edu.au).

[2] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications vol. **34**, Birkhäuser, Boston – Basel – Berlin, 1998.

[3] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, Cluj-Napoca **IV(1)** (2003), 91-96.

Bogdan Choczewski *Special solutions of an iterative functional inequality of second order*

This a report on a joint work by Dobiesław Brydak, Marek Czerni and the speaker [1].

The inequality reads:

$$\psi[f^2(x)] \le (p(x)+q(f(x))\psi[f(x)] - p(x)q(x)\psi(x), \tag{1}$$

where ψ is the unknown function. We aim at investigating these continuous solutions of (1) that behave at the fixed point of f like a prescribed "test" function T, in particular, like one from among the functions p, q or f.

Inequality (1) has been first studied by Maria Stopa [2].

[1] D. Brydak, B. Choczewski, M. Czerni, *Asymptotic properties of solutions of some iterative functional inequalities*, Opuscula Math., Volume dedicated to the memory of Professor Andrzej Lasota, in print.

[2] M. Stopa, *On the form of solutions of some iterative functional inequality*, Publ. Math. Debrecen **45** (1994), 371-377.

Jacek Chudziak *On some property of the Gołąb–Schinzel equation*

Let X be a linear space over a field K of real or complex numbers. Given nonempty subset A of X, we say that $a \in A$ is an algebraically interior point to A provided, for every $x \in X \setminus \{0\}$, there is an $r_x > 0$ such that

$$\{a + bx : \; |b| < r_x\} \subset A.$$

By $\mathrm{int}_a A$ we denote the set of all algebraically interior points to A.

We show that, rather surprisingly, in a class of functions $f \colon X \longrightarrow K$ such that $F_f := \{x \in \; X : \; f(x) = 0\} \neq \emptyset$ and $\mathrm{int}_a(X \setminus F_f) \neq \emptyset$, the following two conditions are equivalent:

(i) $f(x + f(x)y) = 0$ if and only if $f(x)f(y) = 0$ for $x, y \in X$;

(ii) $f(x + f(x)y) = f(x)f(y)$ for $x, y \in X$.

Some consequences of this fact are also presented.

Marek Czerni *Representation theorems for solutions of a system of linear inequalities*

In the talk we present representation theorems for continuous solutions of a system of functional inequalities

$$\begin{cases} \psi[f(x)] \leq g(x)\psi(x), \\ (-1)^p\psi[f^2(x)] \leq (-1)^p g[f(x)]g(x)\psi(x) \end{cases} \tag{1}$$

where ψ is an unknown function, f, g are given functions, f^2 denotes the second iterate of f and $p \in \{0, 1\}$.

We assume the following hypotheses about the given functions f and g:

(H_1) The function $f \colon I \longrightarrow I$ is continuous and strictly increasing in an interval $I = [0, a|$ ($a > 0$ may belong to I or not). Moreover $0 < f(x) < x$ for $x \in I^\star = I \setminus \{0\}$, $f(I) = I$.

(H_2) The function $g \colon I \longrightarrow \mathbb{R}$ is continuous in I and $g(x) < 0$ for $x \in I$.

We shall be concerned with such solutions of (1) that for some fixed solution φ of a linear homogeneous functional equation

$$\varphi[f(x)] = g(x)\varphi(x)$$

or

$$\varphi[f(x)] = -g(x)\varphi(x)$$

the finite limit

$$\lim_{x\to 0^+} \frac{\psi(x)}{\varphi(x)}$$

exists.

Stefan Czerwik *Effective formulas for the Stirling numbers*

It is known that Stirling numbers play important role in many areas of mathematics and applications. We shall present some results about the Stirling numbers. We introduce new definition of the Stirling numbers of second kind. Moreover, we shall present some effective formulas for the Stirling numbers of the first kind.

Zoltán Daróczy *Nonconvexity and its application*

Joint work with Zsolt Páles.

Let $I \subset \mathbb{R}$ be a nonempty open interval. The following characterization of a continuous nonconvex function $f : I \longrightarrow \mathbb{R}$ is applicable for a number of questions in the theory of mean values.

THEOREM
Let $f\colon I \longrightarrow \mathbb{R}$ be a nonconvex continuous function on I. Then there exist $a \neq b$ in I such that

$$f(ta + (1-t)b) > tf(a) + (1-t)f(b)$$

holds for all $0 < t < 1$.

Judita Dascăl *On a functional equation with a symmetric component*

Let $I \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1,\ q \in (0,1)$, such that $r \neq q$, $r \neq \frac{1}{2}$ and $q \neq \frac{1}{2}$. In this presentation we give all the functions $f, g\colon I \longrightarrow \mathbb{R}_+$ such that

$$f\left(\frac{x+y}{2}\right)[r(1-q)g(y) - (1-r)qg(x)] = \frac{r-q}{1-2q}[(1-q)f(x)g(y) - qf(y)g(x)]$$

for all $x, y \in I$. Our main result is the following.

If the functions $f, g\colon I \longrightarrow \mathbb{R}_+$ are solutions of the above functional equation, then the following cases are possible:

(1) If $r \neq \frac{q^2}{q^2+(1-q)^2}$ and $r \neq \frac{q}{2q-1}$ then there exist constants $a, b \in \mathbb{R}_+$ such that

$$f(x) = a \quad \text{and} \quad g(x) = b \qquad \text{for all } x \in I;$$

(2) If $r = \frac{q^2}{q^2+(1-q)^2}$ then there exists an additive function $A\colon \mathbb{R} \longrightarrow \mathbb{R}$ and real numbers $c_1, c_2 > 0$ such that

$$g(x) = c_1 e^{A(x)} \quad \text{and} \quad f(x) = c_2 e^{2A(x)} \qquad \text{for all } x \in I;$$

(3) If $r = \frac{q}{2q-1}$ then there exist real numbers d_1, d_2, d_3 such that

$$g(x) = \frac{1}{d_1 x + d_2} > 0 \quad \text{and} \quad f(x) = d_3 \frac{1}{d_1 x + d_2} > 0 \qquad \text{for all } x, y \in I.$$

Conversely, the functions given in the above cases are solutions of the previous equation.

Joachim Domsta *An example of a group of commuting boosts*

In this talk a construction of a particular group of invariant linear maps for $\mathbb{R}^{1+3}$ is given. The set of mappings is the same as the one of the Lorentz group, but the group action is not simply the composition of maps. It is chosen in such a way, that the group of rotations is the same, as in the Lorentz group. But all boosts form a subgroup, which does not hold in the Lorentz group. Additionally, this subgroup is abelian. An interesting fact is, that the one dimensional subgroups of the boosts are simultaneously (one dimensional) subgroups of the Lorentz group.

Piotr Drygaś *Functional equations and effective conductivity in composite material with non perfect contact.*

We consider a conjugation problem for harmonic functions in multiply connected circular domains. This problem is rewritten in the form of the $\mathbb{R}$-linear boundary value problem by using equivalent functional-differential equations in a class of analytic functions. It is proven that the operator corresponding to the functional-differential equations is compact in the Hardy-type space. Moreover, these equations can be solved by the method of successive approximations under some natural conditions. This problem has applications in mechanics of composites when the contact between different materials is imperfect. It is given information about effective conductivity tensor with fixed accuracy for macroscopic isotropy composite material.

Włodzimierz Fechner *On some functional-differential inequalities related to the exponential mapping*

We examine some functional-differential inequalities which are related to the exponential function. In particular, we show that its solutions can be written as a product of the exponential function and a convex mapping. Our results are closely connected with the Hyers–Ulam stability of functional-differential equations and, in particular, with some of the results obtained in 1998 by Claudi Alsina and Roman Ger in [1].

[1] C. Alsina, R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373-380.

Roman Ger *On functional equations related to functional analysis – selected topics*

The talk focuses on the occurrence of various functional analysis aspects in the theory of functional equations and vice versa. Among others, the topics discussed concern representation theorems, generalizations of the Hahn–Banach type theorems and their geometric counterparts (separation results), characterizations of various kinds of Banach spaces, functional equations in Banach algebras, convex analysis, algebraic analysis, generalized polynomials, abstract orthogonalities and related equations, global isometries and their perturbations, stability and approximation theory and geometry of Banach spaces.

Dorota Głazowska *An invariance of the geometric mean in the class of Cauchy means*

We determine all the Cauchy conditionally homogeneous mean-type mappings for which the geometric mean is invariant, assuming that one of the generators of Cauchy mean is a power function.

Grzegorz Guzik *Derivations in some model of quantum gravity*

A sketch of a role of derivations, i.e., linear operators satisfying a Leibniz's rule in a new model of quantum gravity proposed by polish astrophysicist M. Heller and co-workers is presented. This promising model is an alternative to popular modern superstrings theories and it gives a hope to unification of relativity and quanta.

Konrad J. Heuvers *Some partial Cauchy difference equations for dimension two*

Let G be an abelian group and X a vector space over the rationals. For $\Phi\colon G \longrightarrow X$ its 1-st Cauchy difference is the function $K_2\Phi\colon G^2 \longrightarrow X$ defined by

$$K_2\Phi(x_1, x_2) := \Phi(x_1 + x_2) - \Phi(x_1) - \Phi(x_2)$$

and in general, for $n = 2, 3, \dots$, the $(n-1)$-th Cauchy difference of Φ is the function $K_n\Phi\colon G^n \longrightarrow X$ defined by

$$K_n\Phi(x_1, \dots, x_n) := \sum_{r=1}^{n}(-1)^{n-r} \sum_{|J|=r} \Phi(x_J)$$

where $\emptyset \neq J \subset I_n = \{1,\dots,n\}$ and $x_J = \sum_{j\in J} x_j$. If $\Psi\colon G^n \longrightarrow X$, then its i-th partial difference of order r ($r = 2,3,\dots$), $K_r^{(i)}\Psi\colon G^{n+r-1} \longrightarrow X$, is its Cauchy difference of order r with respect to its i-th variable with all the others held fixed. For $n = 2$ and $i = 1,2$ we have

$$K_2^{(1)}\Psi(x_1,x_2;x_3) = \Psi(x_1+x_2,x_3) - \Psi(x_1,x_3) - \Psi(x_2,x_3)$$

and

$$K_2^{(2)}\Psi(x_1;x_2,x_3) = \Psi(x_1,x_2+x_3) - \Psi(x_1,x_2) - \Psi(x_1,x_3).$$

In this talk the solutions of the following equations are given.

1. $K_2^{(1)}f_2 = K_2^{(2)}f_1$, where $f = \langle f_1,f_2\rangle\colon G^2 \longrightarrow X^2$ (a 2-dim “curl”$=0$).
2. $K_2^{(1)}f_1 + K_2^{(2)}f_2 = 0$, where $f = \langle f_1,f_2\rangle\colon G^2 \longrightarrow X^2$ (a 2-dim “div”$=0$).
3. $K_2^{(1)}f = K_2^{(2)}f$, where $f\colon G^2 \longrightarrow X$.
4. $K_2^{(1)}f = \lambda K_2^{(2)}f$, where $\lambda \neq 0,1$ and $f\colon G^2 \longrightarrow X$. (Here the special case $\lambda = -i$ corresponds to a “Cauchy–Riemann equation”.)

Eliza Jabłońska *On Christensen measurability and a generalized Gołąb–Schinzel equation*

Let X be a real linear space. We consider solutions $f\colon X \longrightarrow \mathbb{R}$ and $M\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the functional equation

$$f(x+M(f(x))y) = f(x)f(y) \qquad \text{for } x,y \in X, \tag{1}$$

where f is bounded on a Christensen measurable nonzero set as well as f is Christensen measurable. Our results refer to some results of C.G. Popa and J. Brzdęk.

Justyna Jarczyk *On an equation involving weighted quasi-arithmetic means*

We report on a progress made recently in studying solutions (φ,ψ) of the equation

$$\begin{aligned}\kappa x + (1-\kappa)y = {}& \lambda\varphi^{-1}\left(\mu\varphi(x) + (1-\mu)\varphi(y)\right)\\ &+ (1-\lambda)\psi^{-1}\left(\nu\psi(x) + (1-\nu)\psi(y)\right),\end{aligned} \tag{1}$$

where $\kappa,\lambda \in \mathbb{R}\setminus\{0,1\}$ and $\mu,\nu \in (0,1)$. When $\kappa = \mu = \nu = \frac{1}{2}$ all twice continuously differentiable solutions of (1) were found by D. Głazowska, W. Jarczyk, and J. Matkowski. Later Z. Daróczy and Zs. Páles determined all continuously differentiable solutions of (1) in the case $\kappa = \mu = \nu$.

Witold Jarczyk *Iterability in a class of mean-type mappings*

Joint work with Janusz Matkowski.

Embeddability of a given pair of means in a continuous iteration semigroup of pairs of homogeneous symmetric strict means is considered.

Hans-Heinrich Kairies *On Artin type characterizations of the Gamma function*

E. Artin's monograph on the Gamma function contains two characterizations using the functional equation

$$f(x+1) = xf(x), \qquad x \in \mathbb{R}_+ \tag{F}$$

and the multiplication formula

$$f\Big(\frac{x}{p}\Big)f\Big(\frac{x+1}{p}\Big)\dots f\Big(\frac{x+p-1}{p}\Big) = (2\pi)^{\frac{1}{2}(p-1)}p^{\frac{1}{2}-x}f(x), \qquad x \in \mathbb{R}_+\,. \tag{M_p}$$

They read as follows.

Theorem A
Assume that $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is continuously differentiable and satisfies* (F) *and* (M_p) *for some* $p \in \{2,3,4,\dots\}$*. Then* $f = \Gamma$.

Theorem B
Assume that $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is continuous and satisfies* (F) *and* (M_p) *for every* $p \in \{2,3,4,\dots\}$*. Then* $f = \Gamma$.

We discuss both theorems with respect to their optimality.

Zygfryd Kominek *Stability of a quadratic functional equation on semigroups*

The stability problem of the functional equation of the form

$$f(x+2y) + f(x) = 2f(x+y) + 2f(y),$$

is investigated. We prove that if the norm of the difference between the left-hand side and the right-hand side of the equation is majorized by a function ω of two variables having some standard properties, then there exists a unique solution F of our equation and the norm of the difference between F and the given function f is controlled by a function depending on ω.

Krzysztof Król *Application of the least squares method and the decomposition method to solving functional equations*

In the talk we consider the approximate solution of the linear functional equation

$$y[f(x)] = g(x)y(x) + F(x), \tag{1}$$

in the class of continuous functions.

We use the least squares method for finding the approximate solution of the equation (1). In this method the accurate solution of the equation (1) may be approximated by the function

$$y_n(x) = \sum_{j=1}^{n} p_j \Phi_j(x),$$

where Φ_j , $j = 1, \dots, n$, are given, continuous, linear independent functions, and coefficients p_j, $j = 1, \dots, n$, are solutions of the system of equations

$$\sum_{j=1}^{n} p_j \int_a^b \Psi_i(x)\Psi_j(x)\,dx = \int_a^b \Psi_i(x)F(x)\,dx,$$

where $\Psi_i(x) = \Phi_i[f(x)] - g(x)\Phi_i(x)$ and $i = 1, \dots, n$. We apply the least squares method to solving the exemplary equation.

Next we use the decomposition method for finding the approximate solution of the equation (1). At certain assumptions we show that the accurate solution of the equation (1) may be uniformly approximated by the function

$$y(x) = \sum_{n=0}^{\infty} \varphi_n(x),$$

where

$$\varphi_0(x) = -\frac{F(x)}{g(x)}, \quad \varphi_n(x) = \frac{\varphi_{n-1}(x)}{g(x)}, \qquad n = 1, 2, \dots .$$

We prove that if there exists $0 \le \alpha < 1$ such that

$$\|\varphi_{n+1}\| \le \alpha \|\varphi_n\|, \qquad n = 0, 1, \dots$$

then the series $\sum_{n=0}^{\infty} \varphi_n(x)$ is uniformly convergent to the accurate solution of the equation (1). Finally, we apply the decomposition method to solving the exemplary equation.

Arkadiusz Lisak *Some remarks on solutions of functional equations stemming from trapezoidal rule*

Joint work with Maciej Sablik.

The following functional equation (stemming from trapezoidal rule)

$$f_1(y) - g_1(x) = (y - x)[f_2(x) + f_3(sx + ty) + f_4(tx + sy) + f_5(y)]$$

with six unknown functions $g_1, f_i : \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, \dots, 5$, where s and t are two fixed real parameters, has been solved by Prasanna K. Sahoo (University of Louisville, Louisville, USA). However, the solutions have been determined in particular for $s^2 \ne t^2$ (with $st \ne 0$) under high regularity assumptions on unknown functions (twice and four times differentiability). We solve this equation without any regularity assumptions on unknown functions for rational parameters s and t and with lesser regularity assumptions on unknown functions for real parameters s and t.

Fruzsina Mészáros *Functional equations stemming from probability theory*

Joint work with Károly Lajkó.

Special cases of the almost everywhere satisfied functional equation

$$g_1\left(\frac{x}{c(y)}\right)\frac{1}{c(y)}f_Y(y) = g_2\left(\frac{y}{d(x)}\right)\frac{1}{d(x)}f_X(x)$$

are investigated for the given positive functions c, d and unknown functions g_1, g_2, f_X and f_Y. This functional equation has important role in the characterization of distributions, whose conditionals belong to given scale families and have specified regressions.

Vladimir Mityushev *Application of functional equations to composites and to porous media*

Boundary value problems for multiply connected domains describe various physical phenomena in composites and porous media. One of the important constant of such problems constructed as a functional is the effective conductivity. Estimation of the effective conductivity can help to predict and to optimize properties of new created materials. It is shown that discussed boundary value problems can be effectively solved by reduction to iterative functional equations. New exact and approximate analytical formulae for the effective conductivity have been deduced. Further possible applications are discussed.

Takeshi Miura *A note on stability of Volterra type integral equation*

Let $\mathbb{R}$ be the real number field and let X be a complex Banach space. Suppose that p is a continuous function from $\mathbb{R}$ to the complex number field. The purpose of this talk is to give a sufficient condition in order that the equation

$$f(t) - f(0) = \int_0^t p(s)f(s)\,ds \qquad (\forall\, t \in \mathbb{R}) \tag{$*$}$$

has the stability in the sense of Hyers–Ulam: for every $\varepsilon \geq 0$ and continuous map $f\colon \mathbb{R} \longrightarrow X$ satisfying

$$\left\| f(t) - f(0) - \int_0^t p(s)f(s)\,ds \right\| \leq \varepsilon \qquad (\forall\, t \in \mathbb{R}),$$

there exists a solution $g\colon \mathbb{R} \longrightarrow X$ of the equation $(*)$ such that

$$\|f(t) - g(t)\| \le K\varepsilon \qquad (\forall\, t \in \mathbb{R}),$$

where K is a non-negative constant, depending only on the function p.

Janusz Morawiec *On the set of probability distribution solutions of a linear equation of infinite order*

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tau\colon \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ be a function which is strictly increasing and continuous with respect to the first variable, measurable with respect to the second variable. We are interested in the following problem: How much can we say about the class of all probability distribution solutions of the equation

$$F(x) = \int_\Omega F(\tau(x,\omega))\, dP(\omega)\,?$$

Jacek Mrowiec *On nonsymmetric t-convex functions*

Let $t \in (0,1)$ be a fixed number. It is known that if a function f defined on a convex domain D is t-convex, i.e., satisfies the condition

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \qquad x, y \in D, \tag{$*$}$$

then it is a midconvex (Jensen-convex) function, i.e., it satisfies the inequality

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x) + f(y)}{2}$$

for all $x, y \in D$ (see [1] or [2], Lemma 1). Some years ago Zs. Páles has posed the following problem: Suppose that a function f satisfies the condition $(*)$ but only for $x < y$. Does this imply midconvexity of f? The partial answer to this question is given.

[1] N. Kuhn, *A note on t-convex functions*, General Inequalities **4**, Birkhäuser Verlag, 1984, 269-276.

[2] Z. Daróczy, Zs. Páles, *Convexity with given infinite weight sequences*, Stochastica **XI-1** (1987), 5-12.

Anna Mureńko *A generalization of the Gołąb-Schinzel functional equation*

We consider solutions $M, f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\circ : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y) = f(x) \circ f(y),$$

under the following additional assumptions:

(a) f is continuous at a point;

(b) $M^{-1}(\{0\}) = \{0\}$;

(c) $\circ$ is commutative and associative.

Adam Najdecki *On stability of some functional equation*

Let $\mathcal{A}$ be a complex Banach algebra, S and T nonempty sets, and $h\colon T \longrightarrow \mathcal{A}$. Moreover, let $a_j \in \mathbb{C}$ and $g_j\colon S \times T \longrightarrow S$ for $j \in \mathbb{N}$. We are going to discuss the stability of the functional equation

$$\sum_{j=1}^{\infty} a_j f(g_j(s,t)) = h(t)f(s), \qquad s \in S,\ t \in T,$$

in the class of functions $f : S \longrightarrow \mathcal{A}$.

Andrzej Olbryś *On some functional inequality connected with t-Wright convexity and Jensen-convexity*

Let $t \in (0,1)$ be a fixed number, $L(t)$ – the smallest field containing the set $\{t\}$, and let X be a linear space over the field K, where $L(t) \subset K \subset R$. Let, moreover, $D \subset X$ be a $L(t)$-convex set, i.e., such set that $\alpha D + (1-\alpha)D \subset D$ for all $\alpha \in L(t) \cap (0,1)$.

In the talk we study connections between functions $f\colon D \longrightarrow \mathbb{R}$ satisfying the inequality

$$\frac{f(tx+(1-t)y) + f((1-t)x+ty)}{2} + f\Big(\frac{x+y}{2}\Big) \le f(x) + f(y), \qquad x, y \in D$$

and Jensen convex functions.

Boris Paneah *On the solvability of the identifying problem for general functional operators with linear arguments*

We start with a new problem for a general linear functional operator

$$(\mathcal{P}F)(x) = \sum_{j=1}^{N} c_j(x) F\big(a_j(x)\big), \qquad x \in D \subset \mathbb{R}^n,$$

where $F \in C(I,B)$, $I = [-1,1]$, B a Banach space, a_j and c_j given functions. This problem is intimately connected in some sense with approximation theory and can be described shortly as follows: find a finite-dimensional subspace $\mathcal{K} \subset C(I,B)$, a one-dimensional manifold $\Gamma \subset D$ and a subspace $C_{\langle\tau\rangle} = C_{\langle\tau\rangle}(I,B) \subset C(I,B)$ such that for an arbitrary $\varepsilon > 0$ the relation $|\mathcal{P}F|_{\langle\tau\rangle} < \varepsilon$ implies the inequality

$$\inf_{\varphi \in \mathcal{K}} |F - \varphi|_{\langle\tau\rangle} < c\varepsilon$$

with c a positive constant not depending on ε nor on F. If such a triple $(\mathcal{K},\ \Gamma,\ C_{\langle\tau\rangle})$ is found, we say that the identifying problem for the operator $\mathcal{P}$ is $(\Gamma,\ \mathcal{K})$ - solvable in the space $C_{\langle\tau\rangle}$. In particular, the well-known Hyers–Ulam result related to the functional Cauchy operator $\mathfrak{C}F = F(x,y) - F(x) - F(y)$ with $(x,y) \in \mathbb{R}^2$ can be reformulated as follows: the identifying problem for the operator $\mathfrak{C}$ is $(\mathbb{R}^2,\ \ker\mathfrak{C})$ – solvable in the space $C_{\langle\tau\rangle}$.

In the second part of the talk we give a solution of the identifying problem for a wide class of operators $\mathcal{P}$ with real c_j and linear functions a_j.

Boris Paneah *On the theory of the general linear functional operators with applications in analysis*

In the talk we discuss the recent results related to the solvability and qualitative properties of solutions of the general linear functional equations

$$\sum_{j=1}^{N} c_j(x)F\big(a_j(x)\big) = H(x),$$

where F are compact supported Banach-valued functions of a single variable and x are the points in a bounded domain $D \subset \mathbb{R}^n$, $n \geq 2$. When obtaining these results the new approach has been used. This is, first of all, the functional-analytic point of view which makes it possible to use the results and the methods of the classical functional analysis. Another novelty consists in systematical applying dynamical methods, based on the theory of the new dynamical systems introduced by the speaker (especially in connection with the problems in question). These results and methods will be considered at the first part of the talk. The second one is devoted to the (completely unexpected) connection of the above results with such divers fields of analysis as integral geometry, partial differential equations, and approximate solvability of the linear functional equations. The corresponding problems from these fields will be formulated (only some basic analysis is required for understanding) and their solutions will be given together with a list of unsolved problems (both in the theory of functional operators and in the applications).

Zsolt Páles *Comparison theorems in various classes of generalized quasi-arithmetic means*

Given a strictly increasing continuous function $f\colon I \longrightarrow \mathbb{R}$, the A_f quasi-arithmetic mean of the numbers $x_1, \dots, x_n \in I$ is defined by

$$A_f(x_1, \dots, x_n) = f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right).$$

The following classical result has attracted the attention of many researchers during the last decades.

Theorem

Let $f, g\colon I \longrightarrow \mathbb{R}$ be continuous strictly increasing functions. Then the following conditions are equivalent:

— *For all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I$,*

$$A_f(x_1, \ldots, x_n) \leq A_g(x_1, \ldots, x_n);$$

— *for all $p \in I$ there exists $\delta > 0$ such that, for all $x, y \in]p-\delta, p+\delta[$,*

$$A_f(x, y) \leq A_g(x, y);$$

— *$g \circ f^{-1}$ is convex;*

— *there exists a function $h : I \longrightarrow \mathbb{R}$ such that, for all $x, y \in I$,*

$$f(x) - f(y) \leq h(y)\big(g(x) - g(y)\big);$$

— *if f, g are twice differentiable with $f'g' \neq 0$ then, for all $x \in I$,*

$$\frac{f''(x)}{f'(x)} \leq \frac{g''(x)}{g'(x)}.$$

Our aim is to survey several extensions and of the above theorem related to various generalizations of quasi-arithmetic means.

Magdalena Piszczek *On a multivalued second order differential problem with Jensen multifunctions*

Let K be a closed convex cone with a nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F_t : t \geq 0\}$ is a regular cosine family of continuous Jensen set-valued functions $F_t\colon K \longrightarrow cc(K)$, $x \in F_t(x)$ for $t \geq 0$, $x \in K$ and $F_t \circ F_s = F_s \circ F_t$ for $s, t \geq 0$, then such family is twice differentiable and

$$DF_t(x)|_{t=0} = \{0\}, \qquad D^2 F_t(x) = A_t(A(x) + D)$$

for $x \in K$ and $t \geq 0$, where $DF_t(x)$ denotes the Hukuhara derivative of $F_t(x)$ with respect to t, $\{A_t : t \geq 0\}$ is a regular cosine family of continuous additive multifunctions, $D \in cc(K)$ and $A(x) = D^2 A_t(x)|_{t=0}$.

This result is a motivation for studying the existence and uniqueness of a solution

$$\Phi\colon [0, +\infty) \times K \longrightarrow cc(K),$$

which is Jensen with respect to the second variable, of the following differentiable problem

$$\begin{aligned} \Phi(0,x) &= \Psi(x), \\ D\Phi(t,x)|_{t=0} &= \{0\}, \\ D^2\Phi(t,x) &= A_\Phi(t,H(x)), \end{aligned}$$

where $H,\Psi\colon K \longrightarrow cc(K)$ are given continuous Jensen multifunctions, $D\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t and A_Φ is the additive, with respect to the second variable, part of Φ.

Vladimir Protasov *Self-similarity equations in L_p spaces*

We consider functional difference equations with linear contractions of the argument (self-similarity equations). Let $L_p[0,1]$ be the space of vector-functions from the segment $[0,1]$ to $\mathbb{R}^d$ with the norm $\|v\|_p = \left(\int_0^1 |v(t)|^p\,dt\right)^{\frac{1}{p}}$. Suppose we have an arbitrary family of affine operators $\{\tilde{A}_1,\ldots,\tilde{A}_m\}$ in $\mathbb{R}^d$. We always assume this family to be irreducible (there is no common invariant affine subspace, different from the whole $\mathbb{R}^d$). Let us also have a partition of the segment $[0,1]$ with nodes $0=b_0<\ldots<b_m=1$. We denote $\Delta_k=[b_{k-1},b_k]$, $r_k=b_k-b_{k-1}$. The affine function $g_k(t)=tb_k+(1-t)b_{k-1}$ maps $[0,1]$ to the segment Δ_k. The *self-similarity operator* $\tilde{\mathbf{A}}$:

$$\big[\,\tilde{\mathbf{A}}\,v\,\big](t) \;=\; \tilde{A}_k\,v\big(g_k^{-1}(t)\big), \qquad t\in\Delta_k,\; k=1,\ldots,m,$$

is defined on $L_1[0,1]$. The equation $\tilde{\mathbf{A}}v=v$ is called *self-similarity equation.* Special cases of such equations are applied in the ergodic theory, wavelets theory, approximation theory, probability, etc. Most of the classical fractal curves (such as Cantor singular function, Koch and de Rham curve, etc.) are solution of suitable self-similarity equations. Refinement equations from wavelets theory and approximation subdivision algorithms are also actually self-similarity equations.

We consider the following problem: what are the conditions on the operators $\{\tilde{A}_k\}$ and on the partition of the segment $[0,1]$ necessary and sufficient for the self-similarity equation to possess an L_p-solution? What can be said about the uniqueness and regularity of the solutions?

We derive a sharp criterion of solvability for these equations in the spaces L_p and C, compute the exponents of regularity and estimate the moduli of continuity. We show that the solution is always unique, whenever exists. The answers are given in terms of the so-called p-radius of the family of operators $\{\tilde{A}_k\}$. This, in particular, gives a geometric interpretation of the p-radius in terms of spectral radii of certain operators in the space $L_p[0,1]$.

Viorel Radu *The fixed point method to generalized stability of functional equations in normed and random normed spaces*

D.H. Hyers in 1941 gave an affirmative answer to a question of S.M. Ulam, concerning the stability of group homomorphisms, for Banach spaces: *Let E_1*

and E_2 *be Banach spaces and* $f\colon E_1 \longrightarrow E_2$ *be such a mapping that*

$$\|f(x+y) - f(x) - f(y)\| \le \delta, \tag{1}$$

for all $x, y \in E_1$ *and a* $\delta > 0$*, that is* f *is* $\delta-$additive. *Then there exists a unique* additive $T\colon E_1 \longrightarrow E_2$, *which satisfies* $\|f(x) - T(x)\| \le \delta$, $x \in E_1$. *In fact,*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \qquad x \in E_1, \tag{Hyers}$$

T. Aoki, D. Bourgin and Th.M. Rassias studied the stability problem with unbounded Cauchy differences: it is supposed that $||\mathcal{D}f(x,y)|| \le \delta(x,y)$ and the *stability estimations* are of the form $||f(x) - S(x)|| \le \varepsilon(x)$, where S is a solution, that is, it *verifies the functional equation* $\mathcal{D}S(x,y) = 0$, and for $\varepsilon(x)$ explicit formulae are given, which depend on the control δ as well as on the equation.

We discuss the generalized Ulam–Hyers stability for functional equations in abstract spaces and show how the stability results can be obtained by a fixed point method, initiated in (Radu [4], 2003) and developed in (Cădariu & Radu [2], 2004) as well as in subsequent papers.

[1] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure and Appl. Math. **4(1)** (2003), Art.4 (http://jipam.vu.edu.au).

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed points approach*, Grazer Math. Ber. **346** (2004), 323-350.

[3] L. Cădariu, V. Radu, *Fixed points method for the stability of some functional equations*, Carpathian J. Math. **23**, No. 1-2, (2007), 63-72.

[4] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, Cluj-Napoca **IV(1)** (2003), 91-96.

Ewa Rak *Distributivity between uninorms and nullnorms*

The problem of distributivity has been posed many years ago (cf. Aczel [1], pp. 318-319). A new direction of investigations is mainly concerned of distributivity between triangular norms and triangular conorms ([5] p. 17). Recently, many authors have dealt with solution of distributivity equation for aggregation functions ([3]), fuzzy implications ([2], [10]), uninorms and nullnorms ([6], [7], [8], [9]), which are generalization of triangular norms and conorms.

Our consideration was motivated by intention of determining algebraic structures which have weaker assumptions than uninorms and nullnorms. In particular, the assumption of associativity is not necessary in consideration of distributivity equation. Moreover, if we omit commutativity assumption, consideration of the left and right distributivity conditions is reasonable. A characterization of such binary operations is interesting not only from a theoretical point of view, but also for their applications, since they have proved to be useful in several fields like fuzzy logic framework, expert system, neural networks or fuzzy quantifiers (cf. [4]).

Previous results about distributivity between uninorms and nullnorms can be obtained as simple corollaries.

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[2] M. Baczyński, *On a class of distributive fuzzy implications*, Internat. J. Uncertainty, Fuzzines Knowledge-Based Syst. **9** (2001), 229-238.

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[4] J. Drewniak, P. Drygaś, E. Rak, *Distributivity equations for uninorms and nullnorms*, Fuzzy Sets and Systems **159** (2008), 1646-1657.

[5] J. Fodor, M. Roubens, *Fuzzy Preference Modeling and Multicriteria Decision Support*, Kluwer Acad. Publ., New York, 1994.

[6] M. Mas, G. Mayor, J. Torrens, *The distributivity condition for uninorms and t-operators*, Fuzzy Sets and Systems **128** (2002), 209-225.

[7] E. Rak, *Distributivity equation for nullnorms*, J. Electrical Engin. **56**, 12/s (2005), 53-55.

[8] E. Rak, P. Drygaś, *Distributivity between uninorms*, J. Electrical Engin. **57**, 7/s (2006), 35-38.

[9] D. Ruiz, J. Torrens, *Distributive idempotent uninorms*, Internat. J. Uncertainty, Fuzzines Knowledge-Based Syst. **11** (2003), 413-428.

[10] D. Ruiz, J. Torrens, *Distributivity of residual implications over conjunctive and disjunctive uninorms*, Fuzzy Sets and Systems **158** (2007), 23-37.

Themistocles M. Rassias *On some major trends in mathematics*

In this talk I shall attempt to present some ideas regarding the present state and the near future of mathematics. Since assessments and any predictions in this field of science are necessarily subjective, I shall communicate to you the opinions of renowned contemporary mathematicians with some of whom I have recently come into contact. I will include of course the significant contribution of Polish mathematicians.

Themistocles M. Rassias *New and old problems in mathematical analysis*

We present some new and old problems that are inspired by D. Hilbert problems [Göttinger Nachrichten (1900), 253-297, and the Bull. Amer. Math. Soc. **8** (1902), 437-479] and S. Smale problems [Mathematics: Frontiers and Perspectives, Mathematical Problems for the Next Century, International Mathematical Union, Amer. Math. Soc., 2000].

In particular emphasis is given to problems related to the representation of functions in several variables by means of functions of a smaller number of variables (J. d'Alembert, V. Arnold, N. Kolmogorov), A.D. Aleksandrov problem for isometric mappings and S.M. Ulam problem for approximate homomorphisms.

The interaction between analysis and geometry is discussed through old and new results, examples and further questions for future work.

[1] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications vol. **34**, Birkhäuser, Boston – Basel – Berlin, 1998.

[2] Th.M. Rassias, J. Šimša, *Finite Sums Decompositions in Mathematical Analysis*, John Wiley & Sons, Wiley-Interscience Series in Pure and Applied Mathematics, Chichester, 1995.

[3] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[4] G. Isac, Th.M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. & Math. Sci. **19**(2) (1996), 219-228.

[5] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Applicandae Mathematicae **62**(1) (2000), 23-130.

Maciej Sablik *Generalized homogeneity of some means*

We deal with means $M\colon I \times I \longrightarrow I$ which are $\circ$-homogenous, i.e., satisfying the equation

$$M(s \circ x, s \circ y) = s \circ M(x, y)$$

for all $s, x, y \in I$, where $\circ$ is a binary operation defined on $I \times I$. In particular, given a quasiarithmetic mean, we determine all continuous, associative and commutative operations $\circ$ with respect to which the mean is homogeneous. Also, we characterize given quasiarithmetic means as homogeneous with respect to a couple of suitable operations. This is a generalization of the well known result on characterization of the arithemtic mean as the only one which is homogeneous both with respect to ordinary multiplication and addition (see eg. J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989).

The results have been partially obtained in collaboration with Małgorzata Pałys.

Ekaterina Shulman *Some extensions of the Levi–Civita equation*

Let T be a representation of a topological group G on a Banach space X. A vector x is called *finite* if there is a finite dimensional subspace $M \subset X$ such that $T_g x \in M$ for each $g \in G$. A finite dimensional subspace $L \subset X$ is called *special* if there is a finite dimensional subspace $M \subset X$ such that $T_g L \bigcap M \neq 0$, for each $g \in G$. We prove that a subspace is special if and only if it contains a finite vector. Using this result we describe continuous solutions $f_j(x)$ of the functional equation

$$\sum_{j=1}^{m} a_j(x) f_j(x+y) = \sum_{i=1}^{n} u_i(x) v_i(y)$$

which extends the well known Levi–Civita equation

$$f(x+y) = \sum_{i=1}^{n} u_i(x) v_i(y).$$

Justyna Sikorska *On a conditional exponential functional equation and its stability*

Joint work with Janusz Brzdęk.

We study a conditional functional equation of the form

$$\gamma(x+y) = \gamma(x-y) \implies f(x+y) = f(x)f(y) \tag{$*$}$$

for a given function γ. Condition $(*)$ with $\gamma = \|\cdot\|$ is the so called isosceles orthogonally exponential functional equation. We show the form of the solutions and investigate the stability of the presented equation. Moreover, we study the pexiderized version of $(*)$.

Barbara Sobek *Pexider equation on a restricted domain*

Let $(X,+)$ be a uniquely 2-divisible Abelian topological group which has a base $\mathcal{B}$ of open neighbourhoods of 0 satisfying the following conditions:

(a) if $B \in \mathcal{B}$ and $x \in B$, then $\frac{x}{2} \in B$,

(b) if $B \in \mathcal{B}$ and $x \in X$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $\frac{x}{2^n} \in B$.

Assume that U is a nonempty, open and connected subset of $X \times X$. Let

$$U_1 := \{x : \ (x,y) \in U \text{ for some } y \in X\},$$
$$U_2 := \{y : \ (x,y) \in U \text{ for some } x \in X\}$$

and

$$U_+ := \{x+y : \ (x,y) \in U\}.$$

We consider the Pexider functional equation

$$f(x+y) = g(x) + h(y) \qquad \text{for } (x,y) \in U,$$

where $f\colon U_+ \longrightarrow K$, $g\colon U_1 \longrightarrow K$ and $h\colon U_2 \longrightarrow K$ are unknown functions and $(K,+)$ is an Abelian group. In particular, we improve Theorem 1 in [F. Radó, J.A. Baker, *Pexider's equation and aggregation of allocations*, Aequationes Math. **32** (1987), 227-239].

Paweł Solarz *Some iterative roots for homeomorphisms with periodic points*

Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism such that $\operatorname{Per} F$, the set of all periodic points of F, is nonempty. It is known that there is an integer $n > 1$ such that

$$\operatorname{Per} F = \{z \in S^1 :\ F^n(z) = z \text{ and } \forall_{0<k<n}\ F^k(z) \neq z\}.$$

If $\operatorname{Per} F \neq S^1$, the equation

$$G^m(z) = F(z), \qquad z \in S^1,$$

where $m \geq 2$, may not have continuous and orientation-preserving solutions. However, if $\gcd(m, n) = 1$, then there are infinitely many such solutions having periodic points of period n. These solutions depend on an arbitrary function. We give the general construction of these solutions.

Tomasz Szostok *On an equation connected to Lobatto quadrature rule*

Joint work with Barbara Koclęga-Kulpa.

Quadrature rules are used in numerical analysis for estimating integrals by the following formula

$$\int_x^y f(t)\,dt \approx (y-x)\sum_{i=1}^n \alpha_i f(a_i x + (1-a_i)y)$$

where the error term depends on the derivative of f. Further for the polynomials of certain degree (depending on the length and form of the quadrature considered) the above formula is exact. This means that polynomials satisfy equations of the type

$$F(y) - F(x) = (y-x)\sum_{i=1}^n \alpha_i f(a_i x + (1-a_i)y)$$

where F is the primitive function of f. In the current talk we solve an equation of this type with the right-hand side containing two endpoints and two other points from the interval $[x, y]$ which are symmetric with respect to the midpoint of this interval. Thus we deal with the equation

$$F(y) - F(x) = (y-x)[\alpha f(x) + \beta f(ax + (1-a)y) + \beta f((1-a)x + ay) + \alpha f(y)]$$

where functions $f, F\colon \mathbb{R} \longrightarrow \mathbb{R}$ and constants $\alpha, \beta, a \in \mathbb{R}$ are unknown.

Jacek Tabor *Extensions of conditionally convex functions*

Joint work with Józef Tabor.

Let $V \subset \mathbb{R}^N$ be a closed bounded convex set and let $f\colon \partial V \longrightarrow \mathbb{R}$ be a continuous conditionally convex function, that is

$$f(\alpha x+(1-\alpha)y)\le \alpha f(x)+(1-\alpha)f(y) \quad \text{for } \alpha\in[0,1],\ x,y\in V:[x,y]\subset V,$$

where $[x,y]=\{\alpha x+(1-\alpha)y:\ \alpha\in[0,1]\}$. Then there exists a continuous convex function $F\colon V\longrightarrow\mathbb{R}$ such that $F|_{\partial V}=f$.

We also show that the assumption that V is bounded is essential.

Józef Tabor *Generalized approximate midconvexity*

Joint work with Jacek Tabor.

Let X be a normed space and let $V\subset X$ be an open convex set. Let $\alpha\colon[0,\infty)\longrightarrow\mathbb{R}$ be a given nondecreasing function. A function $f\colon V\longrightarrow\mathbb{R}$ is $\alpha(\cdot)$-midconvex if

$$f\Big(\frac{x+y}{2}\Big)\le\frac{f(x)+f(y)}{2}+\alpha(\|x-y\|) \qquad \text{for all } x,y\in V.$$

We prove that if f is $\alpha(\cdot)$-midconvex and locally bounded at a point then

$$f(rx+(1-r)y)\le rf(x)+(1-r)f(y)+P_\alpha(r,\|x-y\|)$$

for $x,y\in V$, $r\in[0,1]$, where $P_\alpha\colon[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is a function depending on α. Three different estimations of P_α are considered.

Aleksej Turnšek *Mappings approximately preserving orthogonality*

We present some results on orthogonality preserving and approximately orthogonality preserving mappings in the setting of inner product C^*-modules (Hilbert spaces). Some open questions are also considered.

Jian Wang *The relation between isometric and affine operators on F^*-spaces*

In this talk, we study the relation between isometries and affine operators on F^*-spaces, showing that for AL_β-spaces $(0<\beta\le1)$ E and F if E possesses a normalized complete disjoint atoms system $\{e_\gamma\}_{\gamma\in\Gamma}$, then an isometric embedding $T\colon E\longrightarrow F$ with $T\emptyset=\emptyset$ is linear if and only if, for any $\gamma\in\Gamma$,

(i) $P_\gamma(TE)\subseteq\operatorname{span}(Te_\gamma)$ when $0<\beta<1$, and

(ii) $P_\gamma(TE)\subseteq\operatorname{span}(Te_\gamma)$ and $T(-e_\gamma)\in\operatorname{span}(Te_\gamma)$ when $\beta=1$,

where P_γ is a principal band projection from F onto B_{Te_γ}. At the same time, we prove also that every onto isometry $T\colon(s)_p\to(s)_p$ (resp., $l^{(p_n)}\longrightarrow l^{(p_n)}$, in particular, $l_\beta(\Gamma)\longrightarrow l_\beta(\Gamma)$) is affine. For a number of results for isometric mappings one may see works of M. Day, Ding and Huang, and Th.M. Rassias.

Szymon Wąsowicz *On some inequalities between quadrature operators*

In the class of 3-convex functions we establish the order structure of the set of six well known operators connected with an approximate integration:

two-point and three-point Gauss–Legendre quadratures, Chebyshev quadrature, four-point and five-point Lobatto quadratures and the Simpson's Rule. We show that 12 (of 15 possible) inequalities are true while only 3 fail. For 5-convex functions the situation diametrally differs: only 3 inequalities hold and 12 fail. Among the considered inequalities at least one seems to be not trivial. To prove it we use some method connected with the spline approximation of convex functions of higher order.

Wirginia Wyrobek *Measurable orthogonally additive functions modulo a discrete subgroup*

Joint work with Tomasz Kochanek.

Under appropriate conditions on Abelian topological groups G and H, an orthogonality $\perp \subset G^2$ and a σ-algebra $\mathfrak{M}$ of subsets of G we decompose an $\mathfrak{M}$-measurable function $f\colon G \longrightarrow H$ which is orthogonally additive modulo a discrete subgroup K of H into its continuous additive and continuous quadratic part (modulo K).

Problems and Remarks

1. Remark. *On application of the multiplication formula to $\frac{1}{n}$-stable probability distributions*

From the multiplication formula for the gamma function we obtain for $n \in \mathbb{N}$, $x > 0$, that

$$\frac{n\Gamma(nx)}{\Gamma(x)} = (n^n)^x \cdot \frac{\Gamma(x+\frac{1}{n})}{\Gamma(\frac{1}{n})} \cdot \frac{\Gamma(x+\frac{2}{n})}{\Gamma(\frac{2}{n})} \cdot \ldots \cdot \frac{\Gamma(x+\frac{n-1}{n})}{\Gamma(\frac{n-1}{n})}. \tag{1}$$

If $\xi_{\frac{1}{n}}, \xi_{\frac{2}{n}}, \ldots, \xi_{\frac{n-1}{n}}$, are independent random variables with $\Gamma(\frac{1}{n}, 1), \ldots, \Gamma(\frac{n-1}{n}, 1)$ probability distributions, respectively, then for the right hand side we can write

$$RHS_{(1)}(n,x) = E(n^n \cdot \xi_{\frac{1}{n}} \cdot \xi_{\frac{2}{n}} \cdot \ldots \cdot \xi_{\frac{n-1}{n}})^x, \tag{2}$$

where E stands for the expectation.

On the other hand we can write

$$LHS_{(1)}(n,x) = E(\sigma_{1/n})^{-x}, \tag{3}$$

if $\sigma_{1/n}$ has the strictly stable probability distribution defined by its Laplace transform

$$E\left(e^{-s\sigma_{1/n}}\right) = e^{(-s^{\frac{1}{n}})}, \qquad \operatorname{Re} s \geq 0.$$

Indeed, by Fubini's theorem we obtain

$$E(\sigma_{1/n})^{-x} = \int_0^\infty v^{-x}\, dP_{\sigma_{\frac{1}{n}}}(v)$$

$$= \int_0^\infty \left(\int_0^\infty y^{x-1} e^{-vy} dy \right) \frac{1}{\Gamma(x)} dP_{\sigma_{\frac{1}{n}}}(v)$$

$$= \frac{1}{\Gamma(x)} \int_0^\infty y^{x-1} e^{-y^{\frac{1}{n}}} dy$$

$$= n \frac{\Gamma(nx)}{\Gamma(x)}, \qquad x > 0.$$

Thus, by the uniqueness theorem for the inverse two-sided Laplace transform, from (1)-(3) we obtain the equality of distributions

$$\sigma_{1/n} \stackrel{d}{=} \frac{1}{n^n \cdot \xi_{1/n} \cdot \xi_{2/n} \cdot \ldots \cdot \xi_{(n-1)/n}}, \qquad n = 1, 2, 3, \ldots. \tag{4}$$

For $n = 2$ it is known as a property of the $\Gamma(\frac{1}{2}, 1)$ probability distribution (P. Lévy).

Joachim Domsta

2. Problem. *Lipschitz perturbation of continuous linear functionals*

Let X be a normed space, $D \subseteq X$ be an open convex set and let $f\colon D \longrightarrow \mathbb{R}$ be a Lipschitz perturbation of a linear functional, i.e., let f be of the form

$$f = x^* + \ell,$$

where x^* is a continuous linear functional and $\ell\colon D \longrightarrow \mathbb{R}$ is an ε-Lipschitz function, i.e.,

$$|\ell(x) - \ell(y)| \le \varepsilon \|x - y\| \qquad (x, y \in D).$$

Then, for $x, y \in D$ and $t \in [0, 1]$, we have

$$\begin{aligned}
&\big|tf(x) + (1-t)f(y) - f(tx + (1-t)y)\big| \\
&\quad = \big|t\ell(x) + (1-t)\ell(y) - \ell(tx + (1-t)y)\big| \\
&\quad \le t\big|\ell(x) - \ell(tx + (1-t)y)\big| + (1-t)\big|\ell(y) - \ell(tx + (1-t)y)\big| \\
&\quad \le t\varepsilon\|x - (tx + (1-t)y)\| + (1-t)\varepsilon\|y - (tx + (1-t)y)\| \\
&\quad = 2\varepsilon t(1-t)\|x - y\|.
\end{aligned}$$

On the other hand, in the case $X = \mathbb{R}$, we have the following converse of the above observation.

Claim

Let I be an open interval and $\varepsilon \ge 0$. Assume that $f\colon I \longrightarrow \mathbb{R}$ satisfies, for all $x, y \in I$ and $t \in [0, 1]$, the inequality

$$\big|tf(x) + (1-t)f(y) - f(tx + (1-t)y)\big| \le 2\varepsilon t(1-t)|x - y|. \tag{1}$$

Then there exists a constant $c \in \mathbb{R}$ such that the function $\ell: I \longrightarrow \mathbb{R}$ defined by $\ell(x) := f(x) - cx$ is (2ε)-Lipschitz.

The proof is elementary and is left to the reader. However, the following more general and open problem seems to be of interest.

Problem

Does there exist a constant γ (that may depend on X and D) such that, whenever a function $f: D \longrightarrow X$ satisfies inequality (1) for all $x, y \in D$ and $t \in [0,1]$, then there exists a continuous linear functional x^* such that the function $\ell := f - x^*$ is $\gamma\varepsilon$-Lipschitz on D?

Zsolt Páles

3. Problems. 1. Find all mappings $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the following functional equation is satisfied:

$$\|T\vec{u} \times T\vec{v}\| = \|\vec{u} \times \vec{v}\|, \qquad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^3.$$

Geometrically the problem is asking for the determination of all mappings $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which preserve area of parallelograms in the Euclidean 3-dimensional space.

2. Find all mappings $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the following functional equation is satisfied

$$|(T\vec{u} \times T\vec{v}) \cdot T\vec{w}| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \qquad \text{for all } \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3.$$

Geometrically the problem is asking for the determination of all mappings $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, which preserve volume of parallelepipeds in the Euclidean 3-dimensional space.

Note. In the above two problems the symbols $\times, \cdot$ denote vector product and dot (scalar) product, respectively.

Remark. It will be interesting to formulate and solve the analogous functional equations for mappings $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ for the determination of all mappings which preserve area (resp. volume) of the surfaces of balls (resp. solid balls) in the Euclidean 3-dimensional space. The same problem remains open for ellipsoids in $\mathbb{R}^3$.

3. Examine whether there exists a mapping $f: B \longrightarrow \mathbb{R}^3$, that is at the same time *harmonic* as well as *homeomorphism.*

Remark. $f: B \longrightarrow \mathbb{R}^3$ is a harmonic mapping if its three coordinate real-valued functions on the unit ball B are harmonic functions, i.e., if each one of the three coordinate functions of f satisfies the Laplace equation in B. The mapping $f: B \longrightarrow \mathbb{R}^3$ is a homeomorphism if it is a bijective and bicontinuous mapping.

The same problem remains open for the n-dimensional case where $n = 4, 5, \ldots$.

Themistocles M. Rassias

4. Remark. *Some remarks on the talk of T. Miura*

Recently the number of papers whose title includes the word combination "Ulam stability" threateningly grows. Little by little these works begin to involve operators which are very far from the linear functional operators $\mathcal{P}$ in several variables (in the framework of these operators this notion have appeared). The interest to the type of stability in question is easily explained by its evident connection with the very important problem of the approximate solvability of the inequality $\|\mathcal{P}F\| < \varepsilon$. But to the late days any progress in the solvability of this problem in the class of functional equations in several variables was connected mainly with a success in guessing new types of the operators $\mathcal{P}$ to which it is possible (after a series of substitutions and arithmetic transformations) to apply the original construction of Hyers.

As to the classical operators of analysis (integral, differential, partial differential, etc.) to which from time to time some authors turn in order to cross them with the Ulam stability, for these operators the "Ulam problem" is successfully solving under different names during 70 years in the framework of functional analysis (Banach, Riesz, Schauder, Leray and others). I'll demonstrate this on the basis of the Miura's talk "A note on stability of Volterra type integral equation". The speaker delivered the following result.

> *Let $f\colon \mathbb{R} \longrightarrow B$ be a $C(\mathbb{R}, B)$-function, B a Banach space and*
>
> $$(Tf)(t) = f(0) + \alpha(t) \int_0^t p(s,t) f(s)\, ds$$
>
> *with α and p being continuous maps to $\mathbb{C}$. Then there is a function $\varphi\colon \mathbb{R} \longrightarrow B$ depending on f such that $\varphi - T\varphi = 0$ and $\|f-\varphi\|_C < m\varepsilon$ if $\|f - Tf\| \leq \varepsilon$ with m a constant depending neither on f nor ε.*

(As a matter of fact, the speaker dealt with the simplest case $\alpha = 1$, $p(s,t) = p(s)$.) From the point of view of functional analysis this is a standard exercise related to the invertibility of linear operators in B-space. No hint at stability! The following solution does not require any comment. Denote by E the identical operators in $C(I, B)$ with I – a compact interval in $\mathbb{R}$, and let $f \in C(I, B)$. Then

1. the operator T is compact in $C(I, B) \Rightarrow$
2. the range of the operator $E - T$ is closed $\Rightarrow$
3. the *a priori* estimate

   $$\inf_{\varphi \in \ker(E-T)} |f - \varphi|_{C(I,B)} \leq m|(E-T)f|_{C(I,B)}, \qquad f \in C(I, B)$$

   holds.

This completes the solution.

In the case $\alpha = 1$, $p(s,t) = p(s)$ the space $\ker(E - T)$ is one-dimensional and consists of the functions $\varphi = \lambda \exp(\int_0^t p(s)\,ds)$ with $\lambda \in B$.

In my opinion, the majority of results related to integro-differential operators with the reference to the stability, as a matter of fact, has the same nature: some usual property of an inverse operator is treated as the Ulam stability. But from this point of view any classical result in the theory of boundary problems for partial differential equations (the unique solvability of the Dirichlet problem for the Laplace operator, for example) may be treated as Ulam stability.

Boris Paneah

5. Remark. *Some remarks on the talk of Z. Kominek*

In his talk Prof. Z. Kominek considered the operator

$$\mathcal{P}\colon f(t) \longrightarrow f(x+2y) + f(x) - 2f(x+y) - 2f(y)$$

from $C(\mathbb{R})$ to $C(\mathbb{R}^2)$ and formulated the following proposition: there is a function $w(x,y)$ such that if $|(\mathcal{P}f)(x,y)| < |w(x,y)|$ for all points $(x,y) \in \mathbb{R}^2$ then the equation $\mathcal{P}F = 0$ is uniquely solvable, and for some function ψ the relation $|f - F| \le \psi(w)$ holds. No information about w and ψ had been mentioned.

It is easily seen that the above operator $\mathcal{P}\colon C(I) \longrightarrow C(D)$ with $D = \{(x,y) \mid\ x + 2y \le 1,\ x \ge 0,\ y \ge 0\}$ and $I = [0,1]$ satisfies all conditions formulated in my talk and providing solvability of the identifying problem for $\mathcal{P}$. According to the main result of this talk, if $|(\mathcal{P}f)(x,y)|_{\langle 2\rangle} < \varepsilon$ for all points (x,y) of the straight line $\Gamma = \{(x,y) \mid x = \frac{1}{3}t,\ y\frac{1}{3}t;\ 0 \le t \le 1\}$, then the inequality $|f(t) - \lambda t^2|_{\langle 2\rangle} < c\varepsilon$ holds for a constant λ and all points t, $0 \le t \le 1$. The constant c does not depend on f nor t, $|\cdot|_{\langle 2\rangle}$ is the norm in the space $C_{\langle 2\rangle}$ of continuous in I functions satisfying the 2-Hölder condition at $t = 0$. What is important here is that the initial condition of the smallness of $\mathcal{P}f$ is imposed only at points of an one-dimensional manifold Γ and the approximate solution f of the relation $|\mathcal{P}_\Gamma f| < \varepsilon$ is close not to the subspace $\ker \mathcal{P}$, but to the subspace $\ker \mathcal{P}_\Gamma = \{\varphi \mid\ \varphi(3t) - 2\varphi(2t) - \varphi(t) = 0,\ 0 \le t \le 1\}$, where $\mathcal{P}_\Gamma$ denotes the restriction of the operator $\mathcal{P}$ to Γ.

Boris Paneah

6. Remark. *On functional equations "of Kuczma's type"*

The first paper on functional equations written by Marek Kuczma (1935-1991) had appeared 50 years ago. Together with his colleagues and students he developed the theory of functional equations called "in a single variable" or "iterative" – later on.

Having this anniversary in mind I proposed to introduce in the title of our paper [1] the name "functional equation of Kuczma's type".

But, motivated by what have been said at the Conference on names assigned to stability problems, I have found this idea was not good. First of all, the late

Marek Kuczma himself would be against it. And all who knew him personally would confirm this prediction. Moreover, the new name is unprecise, may led to confusions, and the existing ones are satisfactory.

The aim of this remark is to declare that we decided to change the title of our paper, as indicated in [1].

[1] B. Choczewski, M. Czerni, *Special solutions of a linear functional equation of Kuczma's type*. New title: *Special solutions of a linear iterative functional equation*, Aequationes Math., to appear.

Bogdan Choczewski

7. Problem. *A functional equation with two complex variables*

The functional equation

$$\varphi(z+2\pi i)=\varphi(z), \qquad z\in\mathbb{C} \tag{1}$$

in the class of entire functions has the general solution of the form $\varphi(z) = \psi(\exp z)$, where ψ is an arbitrary entire function. Equation (1) characterizes the complex exponent.

Let $f(z)$ be a given polynomial or entire function. Consider now the functional equation

$$\varphi[z+2\pi i, f(z)]=\varphi[z, f(z)], \qquad z\in\mathbb{C}, \tag{2}$$

where $\varphi(z,w)$ is unknown entire function with respect to z and to w.

Conjecture

The general solution of (2) *has the form* $\varphi(z,w)=\psi(\exp z, w)$, *where* ψ *is an arbitrary entire function of two variables.*

It is worth noting that an artificial insert of the exponent in φ does not solve the problem. For instance, the function $\psi_0(u,w)=\ln u$ produces $\varphi_0(z,w) = \ln\exp z = z$ in the strip $0\le \operatorname{Im} z\le 2\pi$ periodically continued onto $\mathbb{C}$. The function φ_0 satisfies (2), however, φ_0 and ψ_0 are not entire functions.

One can see also that the functional equation $\varphi(z+2\pi i, w) = \varphi(z,w)$, $(z,w)\in\mathbb{C}^2$ (compare to equation (1)) has only exponent in z solutions. But the restriction $w=f(z)$ in (2) yields complications.

The case $f(z)=z$ and its application to Arnold's problem [1, p. 168-170] of topologically elementary functions were discussed in [2].

[1] V.I. Arnold (ed.), *Arnold's Problems*, Springer, Berlin, 2004.

[2] V. Mityushev, *Exponent in one of the variables*, Jan Długosz University of Częstochowa, Scientific Issues, Mathematics XII, Częstochowa, 2007.

Vladimir Mityushev

8. Problems. *Stability of the orthogonality preserving property and related problems*

1. As it was reminded in Prof. Aleksej Turnšek's talk, the orthogonality preserving property for linear mappings between Hilbert spaces is stable. Namely (cf. [1], [4]), if $f\colon X \longrightarrow Y$ is a *linear* mapping satisfying

$$x \perp y \implies fx \perp^{\varepsilon} fy, \qquad x, y \in X$$

(where $u \perp^{\varepsilon} v$ means that $|\langle u|v\rangle| \leq \varepsilon \|u\| \, \|v\|$), then there exists a *linear* mapping $g\colon X \longrightarrow Y$ satisfying

$$x \perp y \implies gx \perp gy, \qquad x, y \in X$$

and such that

$$\|fx - gx\| \leq \left(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right) \cdot \min\{\|fx\|, \|gx\|\}, \qquad x \in X.$$

Problem 1. The question is whether the linearity can be omitted in the above statement (both in assumption and in assertion).

2. Orthogonality preserving property can also be defined for mappings between normed spaces, with one of the possible notions of orthogonality. Attempting to solve the stability problem for this property, with respect to the *isosceles orthogonality* ($u \perp v \Leftrightarrow \|u+v\| = \|u-v\|$) I encountered the following problem concerning the stability of isometries.

Problem 2. Let $f\colon X \longrightarrow Y$ be a linear mapping between Banach spaces satisfying

$$\big|\, \|fx - fy\| - \|x - y\| \,\big| \leq \varepsilon \|x - y\|, \qquad x, y \in X.$$

Does there exists a linear isometry $I\colon X \longrightarrow Y$ such that

$$\|fx - Ix\| \leq \delta(\varepsilon)\|x\|, \qquad x \in X$$

(with some $\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\lim_{\varepsilon \longrightarrow 0^+} \delta(\varepsilon) = 0$)?

Without the linearity assumption, the question has a negative answer, as it was shown by G. Dolinar [3, Proposition 4].

Yet during the meeting Problem 2 has been solved by Prof. Vladimir Protasov. For finite-dimensional spaces the answer to the question is positive (some compactness argument is sufficient) whereas for infinite ones it is generally not true. Namely, for an increasing sequence of positive numbers $\{\alpha_k\}_{k\in\mathbb{N}}$ such that the series $\sum_{k\in\mathbb{N}}(1-\alpha_k^2)$ converges, consider the following norm in a Hilbert space l_2:

$$\|x\|_\alpha := \sup\left\{\|x\|_{l_2}, \left|\frac{x_1}{\alpha_1}\right|, \left|\frac{x_2}{\alpha_2}\right|, \dots\right\}, \qquad x = (x_1, x_2, \dots) \in l_2 .$$

Denote the space l_2 endowed with the new norm $\|\cdot\|_\alpha$ by H_α (this space is reflexive). Now, let $A_k\colon H_\alpha \to H_\alpha$ be an operator interchanging the kth and $(k+1)$st coordinates. It can be shown that A_k is an approximate isometry (with given ε provided that k is sufficiently big) but it cannot be approximated by a linear isometry, as the only linear isometries on the considered space are coordinate symmetries (i.e., $Te_i = \pm e_i$, $i = 1, 2, \dots$).

Afterwards, A. Turnšek pointed out that the problem has been already considered in the literature, e.g. in [2].

[1] J. Chmieliński, *Stability of the orthogonality preserving property in finite-dimensional inner product spaces*, J. Math. Anal. Appl. **318** (2006), 433-443.

[2] G.G. Ding, *The approximation problem of almost isometric operators by isometric operators*, Acta Math. Sci. (English Ed.), **8** (1988), 361-372.

[3] G. Dolinar, *Generalized stability of isometries*, J. Math. Anal. Appl. **242** (2000), 39-56.

[4] A. Turnšek, *On mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **336** (2007), 625-631.

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