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On a multivalued second order differential problem with Jensen multifunction

Abstract. The aim of this paper is to present a generalization of the results published in [5] and [8] for continuous Jensen multifunctions. In particular, we study a second order differential problem for multifunctions with the Hukuhara derivative.

Throughout this paper all vector spaces are supposed to be real. Let X be a vector space. We introduce the notations:

$$A + B := \{a + b : \ a \in A, \, b \in B\} \quad \text{and} \quad \lambda A := \{\lambda a : \ a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$.

A subset K of X is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

Let X and Y be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F: K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *additive* if

$$F(x + y) = F(x) + F(y) \qquad \text{for } x, y \in K$$

and F is *Jensen* if

$$F\left(\frac{x + y}{2}\right) = \frac{F(x) + F(y)}{2} \qquad \text{for } x, y \in K. \tag{1}$$

From now on, we assume that X is a normed vector space, $c(X)$ denotes the family of all compact members of $n(X)$ and $cc(X)$ stands for the family of all convex sets of $c(X)$.

Lemma 1 ([4], Theorem 5.6)
Let K be a convex cone with zero in X and Y be a topological vector space. A set-valued function $F: K \to c(Y)$ satisfies the equation (1) if and only if there exist an additive multifunction $A_F: K \to cc(Y)$ and a set $G_F \in cc(Y)$ such that

$$F(x) = A_F(x) + G_F \qquad \textit{for } x \in K.$$

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The *Hukuhara difference* $A - B$ of $A, B \in cc(X)$ is a set $C \in cc(X)$ such that $A = B + C$. By Rådström's Cancellation Lemma [9] it follows that if this difference exists, then it is unique.

For a multifunction $F\colon [a,b] \to cc(X)$ such that there exist the Hukuhara differences $F(t) - F(s)$ as $a \le s \le t \le b$, the *Hukuhara derivative* at $t \in (a,b)$ is defined by the formula

$$DF(t) = \lim_{k\to 0^+} \frac{F(t+k) - F(t)}{k} = \lim_{k\to 0^+} \frac{F(t) - F(t-k)}{k},$$

whenever both these limits exist with respect to the Hausdorff distance h (see [3]). Moreover,

$$DF(a) = \lim_{s\to a^+} \frac{F(s) - F(a)}{s-a}, \qquad DF(b) = \lim_{s\to b^-} \frac{F(b) - F(s)}{b-s}.$$

Let X be a Banach space and let $[a,b] \subset \mathbb{R}$. If a multifunction $F\colon [a,b] \to cc(X)$ is continuous, then there exists the Riemann integral of F (see [3]). We need the following properties of the Riemann integral.

Lemma 2 ([7], Lemma 10)
If $F\colon [a,b] \to cc(X)$ is continuous, then $H(t) = \int_a^t F(u)\,du$ for $a \le t \le b$ is continuous.

Lemma 3 ([10], Lemma 4)
If $F\colon [a,b] \to cc(X)$ is continuous and $H(t) = \int_a^t F(u)\,du$, then $DH(t) = F(t)$ for $a \le t \le b$.

Let $(K, +)$ be a semigroup. A one-parameter family $\{F_t : t \ge 0\}$ of set-valued functions $F_t\colon K \to n(K)$ is said to be a *cosine family* if

$$F_0(x) = \{x\} \qquad \text{for } x \in K$$

and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) := 2 \bigcup_{y\in F_s(x)} F_t(y)$$

for $x \in K$ and $0 \le s \le t$.

Let X be a normed space. A cosine family is called *regular* if

$$\lim_{t\to 0^+} h(F_t(x), \{x\}) = 0.$$

Example 1
Let $K = [0, +\infty)$ and $F_t(x) = [x\cosh at, x\cosh bt]$, where $0 \le a \le b$. Then $\{F_t : t \ge 0\}$ is a regular cosine family of continuous additive multifunctions.

Example 2
Let $K = [0, +\infty)$ and $F_t(x) = [x, x\cosh t + \cosh t - 1]$. Then $\{F_t : t \ge 0\}$ is a regular cosine family of continuous Jensen multifunctions.

We say that a cosine family $\{F_t : t \geq 0\}$ is *differentiable* if all multifunctions $t \mapsto F_t(x)$ $(x \in K)$ have the Hukuhara derivative on $[0, +\infty)$.

Lemma 4 ([8], Theorem)
Let X be a Banach space and let K be a closed convex cone with a nonempty interior in X. Suppose that $\{A_t : t \geq 0\}$ is a regular cosine family of continuous additive set-valued functions $A_t: K \to cc(K)$, $x \in A_t(x)$ for all $x \in K$, $t \geq 0$ and $A_t \circ A_s = A_s \circ A_t$ for all $s, t \geq 0$. Then this cosine family is twice differentiable and

$$DA_t(x)|_{t=0} = \{0\}, \qquad D^2 A_t(x) = A_t(A(x))$$

for $x \in K$, $t \geq 0$, where $DA_t(x)$ denotes the Hukuhara derivative of $A_t(x)$ with respect to t and $A(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

We would like to obtain a similar result to the above one for a cosine family of continuous Jensen multifunctions. For this purpose we remind some properties of such a family.

Lemma 5 ([6], Theorem 3)
Let X be a Banach space and let K be a closed convex cone in X such that $\operatorname{int} K \neq \emptyset$. A one-parameter family $\{F_t : t \geq 0\}$ is a regular cosine family of continuous Jensen multifunctions $F_t: K \to cc(K)$ such that $x \in F_t(x)$ for all $x \in K$, $t \geq 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$ if and only if there exist a regular cosine family $\{A_t : t \geq 0\}$ of continuous additive multifunctions $A_t: K \to cc(K)$ such that $x \in A_t(x)$ for all $x \in K$, $t \geq 0$, $A_t \circ A_s = A_s \circ A_t$ for all $s, t \geq 0$ and a set $D \in cc(K)$ with zero for which conditions

$$A_{t+s}(D) + A_{t-s}(D) = 2A_t(A_s(D)) \qquad \text{for } 0 \leq s \leq t,$$

$$F_t(x) = A_t(x) + \int_0^t \left(\int_0^s A_u(D)\, du \right) ds \qquad \text{for } t \geq 0$$

hold.

Using Lemmas 2, 3, 4 and 5 we obtain the following theorem.

Theorem 1
Let X be a Banach space and let K be a closed convex cone with a nonempty interior in X. Suppose that $\{F_t : t \geq 0\}$ is a regular cosine family of continuous Jensen set-valued functions $F_t: K \to cc(K)$, $x \in F_t(x)$ for all $x \in K$, $t \geq 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$. Then this cosine family is twice differentiable and

$$DF_t(x)|_{t=0} = \{0\}, \qquad D^2 F_t(x) = A_t(A(x) + D)$$

for $x \in K$, $t \geq 0$, where $DF_t(x)$ denotes the Hukuhara derivative of $F_t(x)$ with respect to t, $D \in cc(K)$ with zero, $A(x) = D^2 A_t(x)|_{t=0}$, $\{A_t : t \geq 0\}$ is a regular cosine family of continuous additive multifunctions (as in Lemma 5).

Let K be a closed convex cone with a nonempty interior in X. We consider a continuous multifunction $\Phi\colon [0,+\infty)\times K\to cc(K)$ Jensen with respect to the second variable. According to Lemma 1 there exist multifunctions $A_\Phi\colon[0,+\infty)\times K\to cc(X)$ additive with respect to the second variable and $G_\Phi\colon[0,+\infty)\to cc(X)$ such that

$$\Phi(t,x)=A_\Phi(t,x)+G_\Phi(t) \qquad \text{for } x\in K,\ t\in[0,+\infty). \tag{2}$$

Setting $x=0$ in (2) we have

$$\Phi(t,0)=G_\Phi(t)\in cc(K) \qquad \text{for } t\in[0,+\infty).$$

Since $A_\Phi(t,x)+\frac{1}{n}G_\Phi(t)=\frac{1}{n}\Phi(t,nx)\subset K$ for all $n\in\mathbb{N}$ and the set K is closed, $A_\Phi(t,x)\in cc(K)$ for $x\in K$, $t\in[0,+\infty)$. Moreover, multifunctions A_Φ, G_Φ are continuous. Indeed, $t\mapsto G_\Phi(t)=\Phi(t,0)$ is continuous. As Φ and G_Φ are continuous, the multifunction A_Φ is also continuous.

Theorem 1 is a motivation for studying existence and uniqueness of a solution $\Phi\colon[0,+\infty)\times K\to cc(K)$, which is Jensen with respect to the second variable, of the following differential problem

$$\begin{aligned} \Phi(0,x)&=\Psi(x),\\ D\Phi(t,x)|_{t=0}&=\{0\},\\ D^2\Phi(t,x)&=A_\Phi(t,H(x)), \end{aligned} \tag{3}$$

where $H,\Psi\colon K\to cc(K)$ are given continuous Jensen set-valued functions, $D\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t and A_Φ is the additive, with respect to the second variable, part of Φ.

Definition 1

A multifunction $\Phi\colon[0,+\infty)\times K\to cc(K)$ is said to be a solution of the problem (3) if it is continuous, twice differentiable with respect to t and Φ satisfies (3) everywhere in $[0,+\infty)\times K$ and in K, respectively, where $H,\Psi\colon K\to cc(K)$ are two given continuous Jensen multifunctions.

With the problem (3), we associate the following equation

$$\Phi(t,x)=\Psi(x)+\int_0^t\left(\int_0^s A_\Phi(u,H(x))\,du\right)ds \tag{4}$$

for $x\in K$, $t\in[0,+\infty)$, where $H,\Psi\colon K\to cc(K)$ are given continuous Jensen multifunctions and A_Φ is the additive, with respect to the second variable, part of Φ.

Definition 2

Let $H,\Psi\colon K\to cc(K)$ be two continuous Jensen set-valued functions. A map $\Phi\colon[0,+\infty)\times K\to cc(K)$ is said to be a solution of (4) if it is continuous and satisfies (4) everywhere.

THEOREM 2
Let K be a closed convex cone with a nonempty interior in a Banach space and let $H, \Psi\colon K \to cc(K)$ be two continuous Jensen multifunctions. Let $\Phi\colon [0, +\infty) \times K \to cc(K)$ be a given Jensen with respect to the second variable set-valued function. This Φ is a solution of the problem (3) if and only if it is a solution of (4).

The proof of Theorem 2 is the same as the proof of Theorem 1 in [5].

In the proof of the next theorem we use the following lemmas.

LEMMA 6 ([12], Theorem 3)
Let X and Y be two normed spaces and let K be a convex cone in X. Suppose that $\{F_i : i \in I\}$ is a family of superadditive lower semicontinuous in K and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i\colon K \to n(Y)$. If K is of the second category in K and $\bigcup_{i\in I} F_i(x) \in b(Y)$ for $x \in K$, then there exists a constant $M \in (0, +\infty)$ such that

$$\sup_{i\in I} \|F_i(x)\| \le M\|x\| \qquad \text{for } x \in K.$$

Let K be a closed convex cone in X. Applying Lemma 6 we can define the *norm* $\|F\|$ of a continuous additive multifunction $F\colon K \to n(K)$ to be the smallest element of the set

$$\{M > 0 : \ \|F(x)\| \le M\|x\|, \, x \in K\}.$$

LEMMA 7
Let K be a closed convex cone with a nonempty interior in a Banach space and let $H, \Psi\colon K \to cc(K)$ be two continuous Jensen multifunctions. Assume that a continuous multifunction $A\colon [0, T] \times K \to cc(K)$ is additive with respect to the second variable. Then the multifunction

$$F(t, x) := \Psi(x) + \int_0^t \left(\int_0^s A(u, H(x))\, du \right) ds, \qquad (t, x) \in [0, T] \times K \tag{5}$$

is Jensen with respect to the second variable and continuous.

Proof. The proof is based upon ideas found in the proof of Theorem 2 in the paper [5]. According to the proof of Theorem 1 in [5] we have that the multifunction $u \mapsto A(u, H(x))$ is continuous for all $x \in K$. We see that every set $F(t, x)$ belongs to $cc(K)$ and F is Jensen with respect to the second variable.

Next we show that F is continuous. Let $x, y \in K$ and $0 \le t_1 \le t_2 \le T$. The set

$$A([0, T], x) = \bigcup_{t\in[0,T]} A(t, x)$$

is compact (see [1], Ch. IV, p. 110, Theorem 3), so it is bounded. Therefore, by Lemma 6, there exists a positive constant M_A such that

$$\|A(u, a)\| \le M_A\|a\| \tag{6}$$

for $u \in [0,T]$ and $a \in K$. This implies that

$$\|A(u, H(x))\| \le M_A \|H(x)\|$$

for $u \in [0,T]$. Thus

$$\begin{aligned}\left\| \int_{t_1}^{t_2} \left(\int_0^s A(u, H(x))\, du \right) ds \right\| &\le \int_{t_1}^{t_2} \left(\int_0^s \|A(u, H(x))\|\, du \right) ds \\ &\le \int_{t_1}^{t_2} \left(\int_0^s M_A \|H(x)\|\, du \right) ds \\ &= \frac{t_2^2 - t_1^2}{2} M_A \|H(x)\|.\end{aligned} \tag{7}$$

From Lemma 5 in [11] and (6) there exists a positive constant M_0 such that

$$h(A(u,a), A(u,b)) \le M_0 \|A(u,\cdot)\| \|a-b\| \le M_0 M_A \|a-b\|$$

for $u \in [0,T]$ and $a, b \in K$. Therefore,

$$A(u,a) \subset A(u,b) + M_0 M_A \|a-b\| S$$

for $u \in [0,T]$ and $a, b \in K$.

Let $\varepsilon > 0$ and $a \in H(x)$. There exists $b \in H(y)$ for which

$$\|a-b\| < d(a, H(y)) + \frac{\varepsilon}{M_0 M_A}.$$

This shows that for every $a \in H(x)$ there exists $b \in H(y)$ such that

$$\begin{aligned}A(u,a) &\subset A(u,b) + M_0 M_A d(a, H(y)) S + \varepsilon S \\ &\subset A(u, H(y)) + M_0 M_A h(H(x), H(y)) S + \varepsilon S,\end{aligned}$$

thus

$$A(u, H(x)) \subset A(u, H(y)) + M_0 M_A h(H(x), H(y)) S + \varepsilon S$$

for $u \in [0,T]$. Since $\varepsilon > 0$ and $x, y \in K$ are arbitrary, we obtain

$$h(A(u, H(x)), A(u, H(y))) \le M_0 M_A h(H(x), H(y)).$$

Hence and by properties of the Riemann integral we have

$$\begin{aligned}&h\left(\int_0^t \left(\int_0^s A(u, H(x))\, du \right) ds, \int_0^t \left(\int_0^s A(u, H(y))\, du \right) ds \right) \\ &\quad \le \int_0^t \left(\int_0^s h(A(u, H(x)), A(u, H(y)))\, du \right) ds \\ &\quad \le \int_0^t \left(\int_0^s M_0 M_A h(H(x), H(y))\, du \right) ds \\ &\quad = \frac{t^2}{2} M_0 M_A h(H(x), H(y)).\end{aligned} \tag{8}$$

By (5), (7) and (8) we get

$$
\begin{aligned}
&h(F(t_1,x),F(t_2,y))\\
&\qquad \le h(\Psi(x),\Psi(y))\\
&\qquad\quad + h\left(\int_0^{t_1}\left(\int_0^s A(u,H(x))\,du\right)ds, \int_0^{t_2}\left(\int_0^s A(u,H(y))\,du\right)ds\right)\\
&\qquad \le h(\Psi(x),\Psi(y))\\
&\qquad\quad + h\left(\int_0^{t_1}\left(\int_0^s A(u,H(x))\,du\right)ds, \int_0^{t_1}\left(\int_0^s A(u,H(y))\,du\right)ds\right)\\
&\qquad\quad + h\left(\{0\}, \int_{t_1}^{t_2}\left(\int_0^s A(u,H(y))\,du\right)ds\right)\\
&\qquad \le h(\Psi(x),\Psi(y)) + \frac{t_1^2}{2}M_0M_A h(H(x),H(y)) + \frac{t_2^2-t_1^2}{2}M_A\|H(y)\|.
\end{aligned}
$$

This shows that F is a continuous set-valued function, because Ψ and H are continuous.

Theorem 3

Let K be a closed convex cone with a nonempty interior in a Banach space and let $H,\Psi\colon K\to cc(K)$ be two continuous Jensen multifunctions. Then there exists exactly one solution, Jensen with respect to the second variable, of the problem (3).

Proof. Fix $T>0$. Let E be the set of all continuous set-valued functions $\Phi\colon[0,T]\times K\to cc(K)$ such that $x\mapsto\Phi(t,x)$ are Jensen. As it was shown, for $\Phi\in E$ there exist continuous multifunctions $A_\Phi\colon[0,T]\times K\to cc(K)$ additive with respect to the second variable and $G_\Phi\colon[0,T]\to cc(K)$ such that $\Phi(t,x)=A_\Phi(t,x)+G_\Phi(t)$ for $x\in K$, $t\in[0,T]$.

Let $\Phi,\Pi\in E$ be given by

$$\Phi(t,x)=A_\Phi(t,x)+G_\Phi(t) \quad\text{and}\quad \Pi(t,x)=A_\Pi(t,x)+G_\Pi(t) \tag{9}$$

for $(t,x)\in[0,T]\times K$, where $A_\Phi,A_\Pi\colon[0,T]\times K\to cc(K)$ are additive with respect to the second variable and $G_\Phi(t),G_\Pi(t)\in cc(K)$. We define a functional ρ in $E\times E$ as follows

$$
\begin{aligned}
\rho(\Phi,\Pi)=\sup\{h(A_\Phi(t,B),A_\Pi(t,B))+h(G_\Phi(t),G_\Pi(t)):\\
0\le t\le T,\ B\in cc(K),\ \|B\|\le 1\}.
\end{aligned}
$$

We see that sets

$$A_i([0,T],x)=\bigcup_{t\in[0,T]}A_i(t,x),\qquad x\in K,$$

$$G_i([0,T])=\bigcup_{t\in[0,T]}G_i(t),$$

where $i \in \{\Phi, \Pi\}$ are compact (see [1], Ch. IV, p. 110, Theorem 3), so they are bounded. By Lemma 6 there exist positive constants M_{A_Φ} and M_{A_Π} such that

$$\|A_\Phi(t,x)\| \le M_{A_\Phi}\|x\|, \qquad \|A_\Pi(t,x)\| \le M_{A_\Pi}\|x\|$$

for $t \in [0,T]$ and $x \in K$. We note that

$$\begin{aligned} &h(A_\Phi(t,B), A_\Pi(t,B)) + h(G_\Phi(t), G_\Pi(t)) \\ &\quad \le \|A_\Phi(t,B)\| + \|A_\Pi(t,B)\| + \|G_\Phi([0,T])\| + \|G_\Pi([0,T])\| \\ &\quad \le M_{A_\Phi} + M_{A_\Pi} + \|G_\Phi([0,T])\| + \|G_\Pi([0,T])\| \end{aligned}$$

for $t \in [0,T]$ and $B \in cc(K)$ such that $\|B\| \le 1$. Thus

$$\rho(\Phi, \Pi) < +\infty,$$

so the functional ρ is finite. It is easy to verify that ρ is a metric in E.

As the space $(cc(K), h)$ is a complete metric space (see [2]), (E, ρ) is also a complete metric space.

We introduce the map Γ which associates with every $\Phi \in E$ the set-valued function $\Gamma\Phi$ defined by

$$(\Gamma\Phi)(t,x) := \Psi(x) + \int_0^t \left(\int_0^s A_\Phi(u, H(x))\, du \right) ds$$

for $(t,x) \in [0,T] \times K$. We see that every set $(\Gamma\Phi)(t,x)$ belongs to $cc(K)$. By Lemma 7 the multifunction $\Gamma\Phi$ is Jensen with respect to the second variable and continuous. Therefore, $\Gamma\colon E \to E$.

Now, we prove that Γ has exactly one fixed point. According to Lemma 1 we take the notations $\Psi(x) = A_\Psi(x) + G_\Psi$ and $H(x) = A_H(x) + G_H$, $x \in K$, where $A_\Psi, A_H\colon K \to cc(K)$ are additive and $G_\Psi, G_H \in cc(K)$. Let $\Phi, \Pi \in E$ be of the form (9) and let $(t,x) \in [0,T] \times K$. We observe that

$$\begin{aligned} (\Gamma\Phi)(t,x) &= \Psi(x) + \int_0^t \left(\int_0^s A_\Phi(u, H(x))\, du \right) ds \\ &= A_\Psi(x) + G_\Psi + \int_0^t \left(\int_0^s A_\Phi(u, A_H(x))\, du \right) ds \\ &\quad + \int_0^t \left(\int_0^s A_\Phi(u, G_H)\, du \right) ds, \end{aligned}$$

thus the additive part $A_{\Gamma\Phi}(t,x)$ of $\Gamma\Phi$ is equal to

$$A_\Psi(x) + \int_0^t \left(\int_0^s A_\Phi(u, A_H(x))\, du \right) ds$$

and similarly

$$A_{\Gamma\Pi}(t,x) = A_\Psi(x) + \int_0^t \left(\int_0^s A_\Pi(u, A_H(x))\, du \right) ds.$$

Hence and by properties of the Hausdorff metric we have

$$\begin{aligned}
&h(A_{\Gamma\Phi}(t,x), A_{\Gamma\Pi}(t,x)) + h(G_{\Gamma\Phi}(t), G_{\Gamma\Pi}(t)) \\
&\quad = h\left(\int_0^t \left(\int_0^s A_\Phi(u, A_H(x))\, du \right) ds, \int_0^t \left(\int_0^s A_\Pi(u, A_H(x))\, du \right) ds \right) \\
&\qquad + h\left(\int_0^t \left(\int_0^s A_\Phi(u, G_H)\, du \right) ds, \int_0^t \left(\int_0^s A_\Pi(u, G_H)\, du \right) ds \right) \\
&\quad \le \frac{t^2}{2!}\rho(\Phi,\Pi)\|A_H(x)\| + \frac{t^2}{2!}\rho(\Phi,\Pi)\|G_H\| \\
&\quad \le 2\frac{t^2}{2!}\rho(\Phi,\Pi)\max\{\|A_H(x)\|, \|G_H\|\}.
\end{aligned}$$

Suppose that

$$\begin{aligned}
h(A_{\Gamma^n\Phi}(t,x), A_{\Gamma^n\Pi}(t,x)) &+ h(G_{\Gamma^n\Phi}(t), G_{\Gamma^n\Pi}(t)) \\
&\le 2\frac{t^{2n}}{(2n)!}\rho(\Phi,\Pi)\max\{\|A_H(x)\|, \|G_H\|\}^n
\end{aligned} \tag{10}$$

for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
&h(A_{\Gamma^{n+1}\Phi}(t,x), A_{\Gamma^{n+1}\Pi}(t,x)) + h(G_{\Gamma^{n+1}\Phi}(t), G_{\Gamma^{n+1}\Pi}(t)) \\
&\quad = h\left(\int_0^t \left(\int_0^s A_{\Gamma^n\Phi}(u, A_H(x))\, du \right) ds, \int_0^t \left(\int_0^s A_{\Gamma^n\Pi}(u, A_H(x))\, du \right) ds \right) \\
&\qquad + h\left(\int_0^t \left(\int_0^s A_{\Gamma^n\Phi}(u, G_H)\, du \right) ds, \int_0^t \left(\int_0^s A_{\Gamma^n\Pi}(u, G_H)\, du \right) ds \right) \\
&\quad \le \int_0^t \left(\int_0^s 2\frac{u^{2n}}{(2n)!}\rho(\Phi,\Pi)\max\{\|A_H(x)\|, \|G_H\|\}^{n+1}\, du \right) ds \\
&\quad = 2\frac{t^{2n+2}}{(2n+2)!}\rho(\Phi,\Pi)\max\{\|A_H(x)\|, \|G_H\|\}^{n+1}.
\end{aligned}$$

This shows that (10) holds for all $n \in \mathbb{N}$. Therefore,

$$\rho(\Gamma^n\Phi, \Gamma^n\Pi) \le 2\frac{(T^2\max\{\|A_H\|, \|G_H\|\})^n}{(2n)!}\rho(\Phi,\Pi), \qquad n \in \mathbb{N}.$$

We observe that for every $T > 0$ there exists $n \in \mathbb{N}$ such that

$$2\frac{(T^2 \max\{\|A_H\|, \|G_H\|\})^n}{(2n)!} < 1.$$

By Banach Fixed Point Theorem we get that Γ^n has exactly one fixed point, whence it follows that Γ has exactly one fixed point. This means that there exists exactly one solution of the problem (3) for $(t, x) \in [0, T] \times K$.

Now we give an application. Let K be a closed convex cone with a nonempty interior in a Banach space. Suppose that $\{F_t : t \geq 0\}$ and $\{G_t : t \geq 0\}$ are regular cosine families of continuous Jensen multifunctions $F_t \colon K \to cc(K)$, $G_t \colon K \to cc(K)$ such that $x \in F_t(x)$, $x \in G_t(x)$, $F_t \circ F_s = F_s \circ F_t$, $G_t \circ G_s = G_s \circ G_t$ for $x \in K$, $s, t \geq 0$ and

$$H(x) := D^2 F_t(x)|_{t=0} = D^2 G_t(x)|_{t=0}.$$

Then multifunctions $(t, x) \mapsto F_t(x)$ and $(t, x) \mapsto G_t(x)$ are Jensen with respect to x and satisfy (3) with $\Psi(x) = \{x\}$. According to Theorem 3 we have $F_t(x) = G_t(x)$ for $(t, x) \in [0, +\infty) \times K$. This means that if two regular cosine family as above have the same second order infinitesimal generator, then there are equal.

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