Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

P.A. Krutitskii, N.Ch. Krutitskaya

The harmonic Dirichlet problem in a planar domain with cracks

Abstract. The harmonic Dirichlet problem in a planar domain with smooth cracks of an arbitrary shape is considered in case, when the solution is not continuous at the ends of the cracks. The well-posed formulation of the problem is given, theorems on existence and uniqueness of a classical solution are proved, the integral representation for a solution is obtained. With the help of the integral representation, the properties of the solution are studied. It is proved that a weak solution of the Dirichlet problem in question does not typically exist, though the classical solution exists.

1. Introduction

Boundary value problems in planar domains with cracks are widely used in physics and in mechanics, and not only in mechanics of solids, but in fluid mechanics as well, where cracks (or cuts) model wings or screens in fluids. Integral representation of a classical solution to the harmonic Dirichlet problem in a planar domain with cracks of an arbitrary shape has been obtained by the method of integral equations in [5, 4, 3, 2, 6] in case when the solution is assumed to be continuous at the ends of the cracks. In the present paper this problem is considered in case when the solution is not continuous at the ends of the cracks. The well-posed formulation of the boundary value problem is given, theorems on existence and uniqueness of a classical solution are proved, the integral representation for a classical solution is obtained. Moreover, properties of the solution are studied with the help of this integral representation. It appears that the classical solution to the Dirichlet problem considered in the present paper exists, while the weak solution typically does not exist, though both the cracks and the functions specified in the boundary conditions are smooth enough. This result follows from the fact that the square of the gradient of a classical solution basically is not itegrable near the ends of the cracks, since singularities of the gradient are rather strong there. This

AMS (2000) Subject Classification: 35J05, 35J25, 31A25.

The research has been partly supported by the RFBR grants 08-01-00082, 09-01-00025.

result is very important for numerical analysis, when finite element and finite difference methods are used to obtain numerical solution. To use difference methods for numerical analysis one has to localize all strong singularities first and next to use difference method in a domain excluding the neighbourhoods of the singularities.

2. Formulation of the problem

By an open curve we mean a simple smooth non-closed arc of finite length without self-intersections [8].

In a plane with Cartesian coordinates $x = (x_1, x_2) \in \mathbb{R}^2$ we consider a connected domain $\mathcal{D}$ bounded by simple closed curves $\Gamma_1^2, \ldots, \Gamma_{N_2}^2$ of class $C^{2,\lambda}$, $\lambda \in (0,1]$. It is assumed that the curves $\Gamma_1^2, \ldots, \Gamma_{N_2}^2$ do not have common points. We set $\Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2$, therefore $\partial\mathcal{D} = \Gamma^2$. We will consider both the case of an exterior domain $\mathcal{D}$ and the case of an interior domain $\mathcal{D}$, when the curve Γ_1^2 encloses all others. In the domain $\mathcal{D}$ we consider disjoint open curves $\Gamma_1^1, \ldots, \Gamma_{N_1}^1$ of class $C^{2,\lambda}$. We set $\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1$, so $\Gamma^1 \subset \mathcal{D}$. We assume that points of the curves Γ^1, including endpoints, are interior points of the domain $\mathcal{D}$. In other words, it is assumed that the closed curves Γ^2 and the open curves Γ^1 do not have any common points, moreover, endpoints of Γ^1 do not belong to Γ^2. We set $\Gamma = \Gamma^1 \cup \Gamma^2$.

We assume that each curve Γ_n^j is parametrized by the arc length s:

$$\Gamma_n^j = \{x : \ x = x(s) = (x_1(s), x_2(s)), \ s \in [a_n^j, b_n^j]\}, \qquad n = 1, \ldots, N_j, \ j = 1, 2,$$

so that $a_1^1 < b_1^1 < \ldots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \ldots < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is placed to the right when the parameter s increases on Γ_n^2. The points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except the points a_n^2, b_n^2, which correspond to the same point x for $n = 1, \ldots, N_2$. Further on, the sets of the intervals

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \qquad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \qquad \bigcup_{j=1}^{2} \bigcup_{n=1}^{N_j} [a_n^j, b_n^j]$$

on the Os-axis will be denoted by Γ^1, Γ^2 and Γ too.

For $j = 0, 1$ and $r \in [0,1]$ set

$$C^{j,r}(\Gamma_n^2) = \{\mathcal{F}(s) : \mathcal{F}(s) \in C^{j,r}[a_n^2, b_n^2], \ \mathcal{F}^{(m)}(a_n^2) = \mathcal{F}^{(m)}(b_n^2), \ m = 0, \ldots, j\}$$

and

$$C^{j,r}(\Gamma^2) = \bigcap_{n=1}^{N_2} C^{j,r}(\Gamma_n^2).$$

The tangent vector to Γ at the point $x(s)$, in the direction of growth of the parameter of s, will be denoted by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, while the normal vector coinciding with τ_x after counterclockwise rotation by the angle of $\frac{\pi}{2}$, will be denoted by $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$. According to the chosen parametrization $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Thus, $\mathbf{n}_x$ is the interior normal to $\mathcal{D}$ on Γ^2. By X we denote the point set consisting of the endpoints of Γ^1:

$$X = \bigcup_{n=1}^{N_1} (x(a_n^1) \cup x(b_n^1)).$$

Let the plane be cut along Γ^1. We consider Γ^1 as a set of cracks (or cuts). The side of the crack Γ^1, which is situated on the left when the parameter s increases, will be denoted by $(\Gamma^1)^+$, while the opposite side will be denoted by $(\Gamma^1)^-$.

We say that the function $u(x)$ belongs to the smoothness class $\mathbf{K}_1$, if

1. $u \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, $\nabla u \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$;
2. in the neighbourhood of any point $x(d) \in X$ the equality

$$\lim_{r\to+0} \int\limits_{\partial S(d,r)} u(x) \frac{\partial u(x)}{\partial \mathbf{n}_x}\, dl = 0 \tag{1}$$

holds, where the curvilinear integral of the first kind is taken over a circle $\partial S(d,r)$ of radius r with the center in the point $x(d)$, $\mathbf{n}_x$ is a normal in the point $x \in \partial S(d,r)$, and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

Remark 1

By $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X)$ we denote the class of functions continuous in $\overline{\mathcal{D}} \setminus \Gamma^1$, which are continuously extendable to the sides of the cracks $\Gamma^1 \setminus X$ from the left and from the right, but their limit values on $\Gamma^1 \setminus X$ can be different from the left and from the right, so that these functions may have a jump on $\Gamma^1 \setminus X$. To obtain the definition of the class $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$ we have to replace $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X)$ by $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$ and $\overline{\mathcal{D}} \setminus \Gamma^1$ by $\mathcal{D} \setminus \Gamma^1$ in the previous sentence.

Problem $\mathbf{D}_1$

Find a function $u(x)$ from $\mathbf{K}_1$, so that $u(x)$ satisfies Laplace equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0, \tag{2a}$$

in $\mathcal{D} \setminus \Gamma^1$ and satisfies the boundary conditions

$$u(x)\big|_{x(s)\in(\Gamma^1)^+} = F^+(s),\ u(x)\big|_{x(s)\in(\Gamma^1)^-} = F^-(s),\ u(x)\big|_{x(s)\in\Gamma^2} = F(s). \tag{2b}$$

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity:

$$|u(x)| \le \text{const}, \qquad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{2c}$$

All conditions of the Problem $\mathbf{D}_1$ must be satisfied in a classical sense. The boundary conditions (2b) on Γ^1 must be satisfied in the interior points of Γ^1, their validity at the ends of Γ^1 is not required.

THEOREM 1
If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, *then there is no more than one solution to the problem* $\mathbf{D}_1$.

It is enough to prove that the homogeneous Problem $\mathbf{D}_1$ admits the trivial solution only. The proof will be given for an interior domain $\mathcal{D}$. Let $u^0(x)$ be a solution to the homogeneous Problem $\mathbf{D}_1$ with $F^+(s) \equiv F^-(s) \equiv 0$, $F(s) \equiv 0$. Let $S(d,\varepsilon)$ be a disc of small enough radius ε, with the center in the point $x(d)$ ($d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$). Let $\Gamma^1_{n,\varepsilon}$ be a set consisting of such points of the curve Γ^1_n which do not belong to discs $S(a_n^1,\varepsilon)$ and $S(b_n^1,\varepsilon)$. We choose a number ε_0 so small that the following conditions are satisfied:

1) for any $0 < \varepsilon \le \varepsilon_0$ the set of points $\Gamma^1_{n,\varepsilon}$ is a unique non-closed arc for each $n = 1, ..., N_1$;
2) the points belonging to $\Gamma \setminus \Gamma^1_n$ are placed outside the discs $S(a_n^1,\varepsilon_0)$, $S(b_n^1,\varepsilon_0)$ for any $n = 1, ..., N_1$;
3) discs of radius ε_0 with centers in different ends of Γ^1 do not intersect.

Set

$$\Gamma^{1,\varepsilon} = \bigcup_{n=1}^{N_1} \Gamma^1_{n,\varepsilon}, \qquad S_\varepsilon = \bigcup_{n=1}^{N_1} [S(a_n^1,\varepsilon) \cup S(b_n^1,\varepsilon)], \qquad \mathcal{D}_\varepsilon = \mathcal{D} \setminus \Gamma^{1,\varepsilon} \setminus S_\varepsilon \,.$$

Since $\Gamma^2 \in C^{2,\lambda}$, $u^0(x) \in C^0(\overline{\mathcal{D}} \setminus \Gamma^1)$ (remind that $u^0(x) \in \mathbf{K}_1$), and since $u^0|_{\Gamma^2} = 0 \in C^{2,\lambda}(\Gamma^2)$, and due to the theorem on regularity of solutions of elliptic equations near the boundary [1], we obtain: $u^0(x) \in C^1(\overline{\mathcal{D}} \setminus \Gamma^1)$. Since $u^0(x) \in \mathbf{K}_1$, we observe that $u^0(x) \in C^1(\overline{\mathcal{D}}_\varepsilon)$ for any $\varepsilon \in (0,\varepsilon_0]$. By $C^1(\overline{\mathcal{D}}_\varepsilon)$ we mean $C^1(\mathcal{D}_\varepsilon \cup \Gamma^2 \cup (\Gamma^{1,\varepsilon})^+ \cup (\Gamma^{1,\varepsilon})^- \cup \partial S_\varepsilon)$. Since the boundary of the domain $\mathcal{D}_\varepsilon$ is piecewise smooth, we write down Green's formula [10, p. 328] for the function $u^0(x)$:

$$\|\nabla u^0\|^2_{L_2(\mathcal{D}_\varepsilon)} = \int\limits_{\Gamma^{1,\varepsilon}} (u^0)^+ \left(\frac{\partial u^0}{\partial \mathbf{n}_x}\right)^+ ds - \int\limits_{\Gamma^{1,\varepsilon}} (u^0)^- \left(\frac{\partial u^0}{\partial \mathbf{n}_x}\right)^- ds$$
$$- \int\limits_{\Gamma^2} u^0 \frac{\partial u^0}{\partial \mathbf{n}_x}\, ds + \int\limits_{\partial S_\varepsilon} u^0 \frac{\partial u^0}{\partial \mathbf{n}_x}\, dl.$$

The exterior (with respect to $\mathcal{D}_\varepsilon$) normal on ∂S_ε at the point $x \in \partial S_\varepsilon$ is denoted by $\mathbf{n}_x$. By the superscripts $+$ and $-$ we denote the limit values of functions on $(\Gamma^1)^+$ and on $(\Gamma^1)^-$, respectively. Since $u^0(x)$ satisfies the homogeneous boundary condition (2b) on Γ, we observe that $u^0|_{\Gamma^2} = 0$ and $(u^0)^{\pm}|_{\Gamma^{1,\varepsilon}} = 0$ for any $\varepsilon \in (0, \varepsilon_0]$. Therefore

$$\|\nabla u^0\|^2_{L_2(\mathcal{D}_\varepsilon)} = \int\limits_{\partial S_\varepsilon} u^0 \frac{\partial u^0}{\partial \mathbf{n}_x}\, dl, \qquad \varepsilon \in (0, \varepsilon_0].$$

Setting $\varepsilon \to +0$, taking into account that $u^0(x) \in \mathbf{K}_1$ and using the relationship (1), we obtain:

$$\|\nabla u^0\|^2_{L_2(\mathcal{D}\setminus\Gamma^1)} = \lim_{\varepsilon\to+0} \|\nabla u^0\|^2_{L_2(\mathcal{D}_\varepsilon)} = 0.$$

From the homogeneous boundary conditions (2b) we conclude that $u^0(x) \equiv 0$ in $\mathcal{D} \setminus \Gamma^1$, where $\mathcal{D}$ is an interior domain. If $\mathcal{D}$ is an exterior domain, then the proof is analogous, but we have to use the condition (2c) and the theorem on behaviour of the gradient of a harmonic function at infinity [10, p. 373]. The maximum principle cannot be used for the proof of the theorem even in the case of the interior domain $\mathcal{D}$, since the solution to the problem may not satisfy the boundary condition (2b) at the ends of the cracks, and it may not be continuous at the ends of the cracks.

3. Existence of a classical solution

Let us turn to solving the Problem $\mathbf{D}_1$. Consider the double layer harmonic potential with the density $\mu(s)$ specified at the open arcs Γ^1:

$$w[\mu](x) = -\frac{1}{2\pi} \int\limits_{\Gamma^1} \mu(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x - y(\sigma)|\, d\sigma. \tag{3}$$

THEOREM 2
Let $\Gamma^1 \in C^{1,\lambda}$, $\lambda \in (0, 1]$. Let $S(d, \varepsilon)$ be a disc of a small enough radius ε with the center in the point $x(d)$ ($d = a^1_n$ or $d = b^1_n$, $n = 1, ..., N_1$).

I. *If $\mu(s) \in C^{0,\lambda}(\Gamma^1)$, then $w[\mu](x) \in C^0(\overline{\mathbb{R}^2 \setminus \Gamma^1} \setminus X)$ and for any $x \in S(d, \varepsilon)$, such that $x \notin \Gamma^1$, the inequality holds: $|w[\mu](x)| \le \mathrm{const}$.*

II. *If $\mu(s) \in C^{1,\lambda}(\Gamma^1)$, then*

1) *$\nabla w[\mu](x) \in C^0(\overline{\mathbb{R}^2 \setminus \Gamma^1} \setminus X)$;*

2) *for any $x \in S(d, \varepsilon)$, such that $x \notin \Gamma^1$, the formulae hold*

$$\frac{\partial w[\mu](x)}{\partial x_1} = \frac{1}{2\pi} \frac{\mp\mu(d)}{|x - x(d)|} \sin\psi(x, x(d)) + \Omega_1(x),$$

$$\sin\psi(x, x(d)) = \frac{x_2 - x_2(d)}{|x - x(d)|},$$

$$\frac{\partial w[\mu](x)}{\partial x_2} = \frac{1}{2\pi}\frac{\pm\mu(d)}{|x-x(d)|}\cos\psi(x,x(d)) + \Omega_2(x),$$

$$\cos\psi(x,x(d)) = \frac{x_1 - x_1(d)}{|x-x(d)|},$$

$$|\Omega_j(x)| \le \mathrm{const}\cdot\ln\frac{1}{|x-x(d)|}, \qquad j=1,2,$$

the upper sign in the formulae is taken if $d = a_n^1$, *while the lower sign is taken if* $d = b_n^1$;

3) *for* $w[\mu](x)$ *the relationship holds*

$$\lim_{\varepsilon\to+0}\int\limits_{\partial S(d,\varepsilon)} w[\mu](x)\frac{\partial w[\mu](x)}{\partial \mathbf{n}_x}\,dl = 0,$$

where the curvilinear integral of the first kind is taken over the circle $\partial S(d,\varepsilon)$; *in addition,* $\mathbf{n}_x = (-\cos\psi(x,x(d)), -\sin\psi(x,x(d)))$ *is the normal at* $x\in\partial S(d,\varepsilon)$, *directed to the center of the circle;*

4) $|\nabla w[\mu](x)|$ *belongs to* $L_2(S(d,\varepsilon))$ *for any small* $\varepsilon > 0$ *if and only if* $\mu(d) = 0$.

Class $C^0(\overline{\mathbb{R}^2\setminus\Gamma^1}\setminus X)$ is defined in the remark to the definition of the class $\mathbf{K}_1$ (Remark 1), if we set $\mathcal{D} = \mathbb{R}^2$. The proof of the theorem is based on the representation of a double layer potential in the form of the real part of the Cauchy integral with the real density $\mu(\sigma)$:

$$w[\mu](x) = -\operatorname{Re}\Phi(z), \qquad \Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma^1}\mu(\sigma)\frac{dt}{t-z}, \qquad z = x_1 + ix_2\,,$$

where $t = t(\sigma) = (y_1(\sigma) + iy_2(\sigma)) \in \Gamma^1$. If $\mu(\sigma)\in C^{1,\lambda}(\Gamma^1)$, then for $z\notin\Gamma^1$:

$$\begin{aligned}\frac{d\Phi(z)}{dz} &= -w'_{x_1} + iw'_{x_2}\\ &= -\frac{1}{2\pi i}\left(\sum_{n=1}^{N_1}\left\{\frac{\mu(b_n^1)}{t(b_n^1)-z} - \frac{\mu(a_n^1)}{t(a_n^1)-z}\right\} - \int\limits_{\Gamma^1}\frac{e^{-i\alpha(\sigma)}\mu'(\sigma)}{t-z}\,dt\right).\end{aligned}$$

Points I, II.1) and II.2) of Theorem 2 follow from these formulae and from the properties of Cauchy integrals, presented in [8]. Points II.3) and II.4) can be proved by direct verification using points I, II.1) and II.2).

We will construct a solution to the Problem $\mathbf{D}_1$ in assumption that $F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1)$, $\lambda\in(0,1]$, $F(s)\in C^0(\Gamma^2)$. We will look for a solution to the Problem $\mathbf{D}_1$ of the form

$$u(x) = -w[F^+ - F^-](x) + v(x), \tag{4}$$

where $w[F^+ - F^-](x)$ is the double layer potential (3), in which

$$\mu(\sigma) = F^+(\sigma) - F^-(\sigma).$$

The potential $w[F^+ - F^-](x)$ satisfies the Laplace equation (2a) in $\mathcal{D} \setminus \Gamma^1$ and belongs to the class $\mathbf{K}_1$ according to Theorem 2. Limit values of the potential $w[F^+ - F^-](x)$ on $(\Gamma^1)^\pm$ are given by the formula

$$w[F^+ - F^-](x)\big|_{x(s)\in(\Gamma^1)^\pm} = \mp \frac{F^+(s) - F^-(s)}{2} + w[F^+ - F^-](x(s)),$$

where $w[F^+ - F^-](x(s))$ is the direct value of the potential on Γ^1.

The function $v(x)$ in (4) must be a solution to the following problem.

PROBLEM **D**

Find a function $v(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, which satisfies the Laplace equation (2a) in the domain $\mathcal{D} \setminus \Gamma^1$ and satisfies the boundary conditions

$$v(x)\big|_{x(s)\in\Gamma^1} = \frac{F^+(s) + F^-(s)}{2} + w[F^+ - F^-](x(s)) = f(s),$$
$$v(x)\big|_{x(s)\in\Gamma^2} = F(s) + w[F^+ - F^-](x(s)) = f(s).$$

If $x(s) \in \Gamma^1$, then $w[F^+ - F^-](x(s))$ is the direct value of the potential on Γ^1.

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity:

$$|v(x)| \le \text{const}, \qquad |x| = \sqrt{x_1^2 + x_2^2} \to \infty.$$

All conditions of the Problem **D** have to be satisfied in the classical sense. Obviously, $w[F^+ - F^-](x(s)) \in C^0(\Gamma^2)$. It follows from [7, Lemma 4(1)] that $w[F^+ - F^-](x(s)) \in C^{1,\frac{\lambda}{4}}(\Gamma^1)$ (here by $w[F^+ - F^-](x(s))$ we mean the direct value of the potential on Γ^1). So, $f(s) \in C^{1,\frac{\lambda}{4}}(\Gamma^1)$ and $f(s) \in C^0(\Gamma^2)$.

We will look for the function $v(x)$ in the smoothness class $\mathbf{K}$. We say that the function $v(x)$ belongs to the smoothness class $\mathbf{K}$ if

1. $v(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, $\nabla v \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$, where X is the set consisting of the endpoints of Γ^1;

2. in a neghbourhood of any point $x(d) \in X$ the inequality

$$|\nabla v| \le \mathcal{C}|x - x(d)|^\delta$$

holds for some constants $\mathcal{C} > 0$, $\delta > -1$, where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

The definition of the functional class $C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X)$ is given in the remark to the definition of the smoothness class $\mathbf{K}_1$ (Remark 1). Clearly, $\mathbf{K} \subset \mathbf{K}_1$.

It can be verified directly that if $v(x)$ is a solution to the Problem $\mathbf{D}$ in the class $\mathbf{K}$, then the function (4) is a solution to the Problem $\mathbf{D}_1$.

Theorem 3
Let $\Gamma \in C^{2,\frac{\lambda}{4}}$, $f(s) \in C^{1,\frac{\lambda}{4}}(\Gamma^1)$, $\lambda \in (0,1]$, $f(s) \in C^0(\Gamma^2)$. Then the solution to the Problem $\mathbf{D}$ in the smoothness class $\mathbf{K}$ exists and is unique.

Theorem 3 has been proved in the following papers: 1) in [5, 4], if $\mathcal{D}$ is an interior domain; 2) in [3], if $\mathcal{D}$ is an exterior domain and $\Gamma^2 \neq \emptyset$; 3) in [2, 6], if $\Gamma^2 = \emptyset$ and so $\mathcal{D} = \mathbb{R}^2$ is an exterior domain. In all these papers, the integral representations for the solution to the Problem $\mathbf{D}$ in the class $\mathbf{K}$ are obtained in the form of potentials, densities of which are defined by the uniquely solvable Fredholm integro-algebraic equations of the second kind and index zero. Uniqueness of a solution to the Problem $\mathbf{D}$ is proved either by the maximum principle or by the method of energy (integral) identities. In the latter case we take into account that a solution to the problem belongs to the class $\mathbf{K}$. Note that the Problem $\mathbf{D}$ is a particular case of more general boundary value problems studied in [4, 3, 2, 6].

Note that Theorem 3 holds if $\Gamma \in C^{2,\lambda}$, $F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1)$, $\lambda \in (0,1]$, $F(s) \in C^0(\Gamma^2)$. From Theorems 2, 3 we obtain the solvability of the problem $\mathbf{D}_1$.

Theorem 4
Let $\Gamma \in C^{2,\lambda}$, $F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1)$, $\lambda \in (0,1]$, $F(s) \in C^0(\Gamma^2)$. Then a solution to the Problem $\mathbf{D}_1$ exists and is given by the formula (4), where $v(x)$ is a unique solution to the Problem $\mathbf{D}$ in the class $\mathbf{K}$, ensured by Theorem 3.

Remark 2
Let us check that the solution to the Problem $\mathbf{D}_1$ given by formula (4) satisfies condition (1). Let $d = a_n^1$ or $d = b_n^1$ $(n = 1, ..., N_1)$ and r be small enough. Then substituting (4) in the integral in (1) we obtain

$$\int\limits_{\partial S(d,r)} u(x)\frac{\partial u(x)}{\partial \mathbf{n}_x}\,dl = \int\limits_{\partial S(d,r)} w(x)\frac{\partial w(x)}{\partial \mathbf{n}_x}\,dl - \int\limits_{\partial S(d,r)} w(x)\frac{\partial v(x)}{\partial \mathbf{n}_x}\,dl$$
$$- \int\limits_{\partial S(d,r)} v(x)\frac{\partial w(x)}{\partial \mathbf{n}_x}\,dl + \int\limits_{\partial S(d,r)} v(x)\frac{\partial v(x)}{\partial \mathbf{n}_x}\,dl.$$

If $r \to 0$, then the first term tends to zero by Theorem 2(II.3). As mentioned above, $v(x) \in \mathbf{K} \subset \mathbf{K}_1$, therefore the condition (1) holds for the function $v(x)$, so the fourth term tends to zero as $r \to 0$. The second term tends to zero as $r \to 0$, since $w(x)$ is bounded at the ends of Γ^1 according to Theorem 2(I), and since $v(x)$ satisfies condition 2) in the definition of the class $\mathbf{K}$. Noting that $v(x)$ is continuous at the ends of Γ^1 due to the definition of the class $\mathbf{K}$, and using Theorem 2(II.2) for calculation of $\frac{\partial w(x)}{\partial \mathbf{n}_x}$ in the third term, we deduce that the third term tends to zero when $r \to 0$ as well. Consequently, the equality (1) holds for the solution to the Problem $\mathbf{D}_1$ constructed in Theorem 4.

Uniqueness of a solution to the Problem $\mathbf{D}_1$ follows from Theorem 1. The solution to the Problem $\mathbf{D}_1$ found in Theorem 4 is, in fact, a classical solution. Let us discuss, under which conditions this solution to the Problem $\mathbf{D}_1$ is not a weak solution.

4. Non-existence of a weak solution

Let $u(x)$ be a solution to the Problem $\mathbf{D}_1$ defined in Theorem 4 by the formula (4). Consider the disc $S(d,\varepsilon)$ with the center in the point $x(d) \in X$ and of radius $\varepsilon > 0$ ($d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$). In doing so, ε is a fixed positive number, which can be taken small enough. Since $v(x) \in \mathbf{K}$, we have $v(x) \in L_2(S(d,\varepsilon))$ and $|\nabla v(x)| \in L_2(S(d,\varepsilon))$ (this follows from the definition of the smoothness class $\mathbf{K}$). Let $x \in S(d,\varepsilon)$ and $x \notin \Gamma^1$. It follows from (4) that $|\nabla w[\mu](x)| \le |\nabla u(x)| + |\nabla v(x)|$, whence

$$\begin{aligned} |\nabla w[\mu](x)|^2 &\le |\nabla u(x)|^2 + |\nabla v(x)|^2 + 2|\nabla u(x)| \cdot |\nabla v(x)| \\ &\le 2(|\nabla u(x)|^2 + |\nabla v(x)|^2). \end{aligned}$$

Assume that $|\nabla u(x)|$ belongs to $L_2(S(d,\varepsilon))$; then, integrating this inequality over $S(d,\varepsilon)$, we obtain

$$\|\nabla w\|^2\big|_{L_2(S(d,\varepsilon))} \le 2\big(\|\nabla u\|^2\big|_{L_2(S(d,\varepsilon))} + \|\nabla v\|^2\big|_{L_2(S(d,\varepsilon))}\big).$$

Consequently, if $|\nabla u(x)| \in L_2(S(d,\varepsilon))$, then $|\nabla w| \in L_2(S(d,\varepsilon))$. However, according to Theorem 2, if $F^+(d) - F^-(d) \neq 0$, then $|\nabla w|$ does not belong to $L_2(S(d,\varepsilon))$. Therefore, if $F^+(d) \neq F^-(d)$, then our assumption that $|\nabla u| \in L_2(S(d,\varepsilon))$ does not hold, i.e., $|\nabla u| \notin L_2(S(d,\varepsilon))$. Thus, if among numbers $a_1^1, ..., a_{N_1}^1$, $b_1^1, ..., b_{N_1}^1$ there exists such a number d that $F^+(d) \neq F^-(d)$, then for some $\varepsilon > 0$ we have $|\nabla u| \notin L_2(S(d,\varepsilon)) = L_2(S(d,\varepsilon) \setminus \Gamma^1)$, so $u \notin W_2^1(S(d,\varepsilon) \setminus \Gamma^1)$, where W_2^1 is a Sobolev space of functions from L_2, which have generalized derivatives from L_2. We have proved the following result.

Theorem 5

Let conditions of Theorem 4 be satisfied and assume that there exists a number $d \in \{a_1^1, .., a_{N_1}^1, b_1^1, ..., b_{N_1}^1\}$ such that $F^+(d) \neq F^-(d)$. Then the solution to the

Problem $\mathbf{D}_1$ *, ensured by Theorem 4, does not belong to* $W_2^1(S(d,\varepsilon)\setminus\Gamma^1)$ *for some* $\varepsilon > 0$*, whence it follows that it does not belong to* $W_{2,loc}^1(\mathcal{D}\setminus\Gamma^1)$*. Here* $S(d,\varepsilon)$ *is a disc of a radius* ε *with the center in the point* $x(d)\in X$*.*

By $W_{2,loc}^1(\mathcal{D}\setminus\Gamma^1)$ we denote the class of functions which belong to W_2^1 on any bounded subdomain of $\mathcal{D}\setminus\Gamma^1$. If conditions of Theorem 5 hold, then the unique solution to the Problem $\mathbf{D}_1$, constructed in Theorem 4, does not belong to $W_{2,loc}^1(\mathcal{D}\setminus\Gamma^1)$, and so it is not a weak solution. We arrive to

Corollary
Let conditions of Theorem 5 be satisfied; then a weak solution to the Problem $\mathbf{D}_1$ *in the class of functions* $W_{2,loc}^1(\mathcal{D}\setminus\Gamma^1)$ *does not exist.*

Remark 3
Even if the number d, mentioned in Theorem 5, does not exist, then the solution $u(x)$ to the Problem $\mathbf{D}_1$, ensured by Theorem 4, may not be a weak solution to the Problem $\mathbf{D}_1$. The Hadamard example of a non-existence of a weak solution to the harmonic Dirichlet problem in a disc with continuous boundary data is given in [9, § 12.5] (the classical solution exists in this example).

Clearly, $L_2(\mathcal{D}\setminus\Gamma^1) = L_2(\mathcal{D})$, since Γ^1 is a set of zero measure.

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KIAM
Miusskaya Sq. 4
Moscow 117234
Russia

Dept. of Mathematics
Faculty of Physics
Moscow State University
Moscow 117234
Russia
E-mail: conf@math.phys.msu.su

Received: 25 June 2008; final version: 29 September 2008;
available online: 20 November 2008.