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On some equations stemming from quadrature rules

Abstract. We deal with functional equations of the type

$$F(y) - F(x) = (y - x) \sum_{k=1}^{n} f_k \left((1 - \lambda_k)x + \lambda_k y \right),$$

connected to quadrature rules and, in particular, we find the solutions of the following functional equation

$$f(x) - f(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)].$$

We also present a solution of the Stamate type equation

$$yf(x) - xf(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)].$$

All results are valid for functions acting on integral domains.

1. Introduction

We deal with some equations connected to quadrature rules. Having a function $f\colon \mathbb{R} \to \mathbb{R}$ we may approximate its integral using the following expression

$$F(y) - F(x) \approx (y - x) \sum_{k=1}^{n} \alpha_k f((1 - \lambda_k)x + \lambda_k y)$$

(where F is a primitive function for f), which is satisfied exactly for polynomials of certain degree. One of the simplest functional equations connected to quadrature rules is an equation stemming from Simpson's rule

$$F(y) - F(x) = (y - x) \left[\frac{1}{6} f(x) + \frac{2}{3} f\left(\frac{x + y}{2} \right) + \frac{1}{6} f(y) \right].$$

Another example is given by the equation

$$F(y) - F(x) = (y - x) \left[\frac{1}{8} f(x) + \frac{3}{8} f\left(\frac{x + 2y}{3} \right) + \frac{3}{8} f\left(\frac{2x + y}{3} \right) + \frac{1}{8} f(y) \right],$$

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which is satisfied by polynomials of degree not greater than 3. The generalized version of this equation

$$g(x) - f(y) = (x - y)[h(x) + k(sx + ty) + k(tx + sy) + h(y)] \tag{1}$$

was considered during the 44th ISFE held in Louisville, Kentucky, USA by P.K. Sahoo [7]. The solution has been given in the class of functions f, g, h, k mapping $\mathbb{R}$ into $\mathbb{R}$ and such that g and f are twice differentiable, and k is four times differentiable.

On the other hand, M. Sablik [5] during the $7th$ Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities presented the general solution of this equation in the case $s, t \in \mathbb{Q}$ without any regularity assumptions concerning the functions considered.

We deal with a special case of (1) (with $s = 1$, $t = 2$) for functions acting on integral domains. However, it is easy to observe that if we take $x = y$ in (1), then we immediately obtain that $f = g$. Thus we shall find the solutions of the following functional equation

$$f(x) - f(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)]. \tag{2}$$

Using the obtained result we will also present a solution of a similar Stamate type equation

$$yf(x) - xf(y) = (x - y)[g(x) + h(2x + y) + h(x + 2y) + g(y)]. \tag{3}$$

In the proof of Lemma 1 below we use the lemma established by M. Sablik [6] and improved by I. Pawlikowska [3]. First we need some notations. Let G, H be Abelian groups and $SA^0(G, H) := H$, $SA^1(G, H) := \mathrm{Hom}(G, H)$ (i.e., the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G, H)$ be the group of all i-additive and symmetric mappings from G^i into H. Furthermore, let $\mathcal{P} := \{(\alpha, \beta) \in \mathrm{Hom}(G, G)^2 : \ \alpha(G) \subset \beta(G)\}$. Finally, for $x \in G$ let $x^i = (\underbrace{x, \ldots, x}_{i})$, $i \in \mathbb{N}$.

Lemma 1

Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0, \ldots, I_N$ be finite subsets of $\mathcal{P}$. Suppose that H is uniquely divisible by $N!$ and let the functions $\varphi_i : G \to SA^i(G, H)$ and $\psi_{i,(\alpha,\beta)} : G \to SA^i(G, H)$ ($(\alpha, \beta) \in I_i$, $i = 0, \ldots, N$) satisfy

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(y^i)$$

for every $x, y \in G$. Then φ_N is a polynomial function of order at most $k - 1$, where

$$k = \sum_{i=0}^{N} \mathrm{card}\left(\bigcup_{s=i}^{N} I_s\right).$$

Now we will state a simplified version of this lemma. We take $N = 1$ and we consider functions acting on an integral domain P. Moreover, we consider only homomorphisms of the type $x \mapsto yx$, where $y \in P$ is fixed.

Lemma 2
Let P be an integral domain and let I_0, I_1 be finite subsets of P^2 such that for all $(a,b) \in I_i$ the ring P is divisible by b. Let $\varphi_i, \psi_{i,(\alpha,\beta)}\colon P \to P$ satisfy

$$\varphi_1(x)y + \varphi_0(x) = \sum_{(a,b)\in I_0} \psi_{0,(a,b)}(ax+by) + y \sum_{(a,b)\in I_1} \psi_{1,(a,b)}(ax+by)$$

for all $x, y \in P$. Then φ_1 is a polynomial function of order at most equal to $\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1$.

In the above lemmas a *polynomial function of order* n means a solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where Δ_h^n stands for the n-th iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$. Observe that a continuous polynomial function of order n is a polynomial of degree at most n (see [2, Theorem 4, p. 398]).

It is also well known that if P is an integral domain uniquely divisible by $n!$ and $f\colon P \to P$ is a polynomial function of order n, then

$$f(x) = c_0 + c_1(x) + \ldots + c_n(x), \qquad x \in P,$$

where $c_0 \in P$ is a constant and

$$c_i(x) = C_i(x, x, \ldots, x), \qquad x \in P$$

for some i-additive and symmetric function $C_i\colon P^i \to P$.

2. Results

We begin with the following lemma which will be usefull in the proof of the main result. However, we state it a bit more generally.

Lemma 3
Let P be an integral domain and let $f, f_k\colon P \to P$, $k = 0, \ldots, n$, be functions satisfying the equation

$$f(y) - f(x) = (y-x)\sum_{k=0}^{n} f_k(a_k x + b_k y), \tag{4}$$

where $a_k, b_k \in P$ are given numbers such that for every $k \in \{0, \ldots, n\}$ we have $a_k \neq 0$ or $b_k \neq 0$.

Let $i \in \{0, \ldots, n\}$ be fixed. If P is divisible by a_i, b_i and also by $a_i b_k - a_k b_i$, $k = 0, \ldots, n$; $k \neq i$, then the function

$$\tilde{f}(x) := (a_i + b_i) f_i((a_i + b_i)x)$$

is a polynomial function of degree at most $2n+1$.

Moreover, if there exists $k_1 \in \{0, 1, \dots, n\}$ such that $a_{k_1} = 0$ or $b_{k_1} = 0$, then function $\tilde{f}$ is a polynomial function of order at most $2n$ and if there exist $k_1, k_2 \in \{0, \dots, n\}$ such that $a_{k_1} = b_{k_2} = 0$, then $\tilde{f}$ is a polynomial function of order at most $2n - 1$.

Proof. Fix an $i \in \{0, \dots, n\}$, put in (4) $x - b_i y$ and $x + a_i y$ instead of x and y, respectively, to obtain

$$\begin{aligned} &f(x + a_i y) - f(x - b_i y) \\ &\quad = (a_i + b_i)y[f_0((a_0 + b_0)x + (a_i b_0 - a_0 b_i)y) + \dots \\ &\qquad + f_i((a_i + b_i)x) + \dots + f_n((a_n + b_n)x + (a_i b_n - a_n b_i)y)]. \end{aligned} \tag{5}$$

There are two possibilities:

1. $a_i, b_i \neq 0$,
2. $a_i = 0$ or $b_i = 0$.

Let us consider the first case. Then from (5) we obtain

$$\begin{aligned} y(a_i + b_i)f_i((a_i + b_i)x) = f(x + a_i y) - f(x - b_i y) \\ - (a_i + b_i)y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y), \end{aligned}$$

which means that

$$\begin{aligned} y\tilde{f}(x) &= f(x + a_i y) - f(x - b_i y) \\ &\quad -(a_i + b_i)y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y). \end{aligned} \tag{6}$$

Now we are in position to use Lemma 2 with

$$I_0 = \{(1, -b_i), (1, a_i)\}$$

and

$$I_1 = \{(a_k + b_k, a_i b_k - a_k b_i) : \ k = 0, \dots, n; \ k \neq i\}.$$

We clearly obtain that $\tilde{f}$ is a polynomial function of order at most equal to

$$\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 \leq (n + 2) + n - 1 = 2n + 1.$$

Further, if for example $a_{k_1} = 0$ for some $k_1 \in \{0, \dots, n\}$, $k_1 \neq i$, then we have a summand

$$f_{k_1}(b_{k_1} x + a_i b_{k_1} y) = f_{k_1}(b_{k_1}(x + a_i y))$$

on the right-hand side of (6). Thus we put $\tilde{f}_{k_1}(x) := f_{k_1}(b_{k_1} x)$ and (6) takes form

$$\begin{aligned} &y\tilde{f}(x) \\ &\quad = f(x - b_i y) - f(x + a_i y) \\ &\qquad - (a_i + b_i)y \left[\sum_{k=0, k\neq i, k_1}^{n} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y) + \tilde{f}_{k_1}(x + a_i y) \right]. \end{aligned}$$

Similarly as before we take

$$I_0 = \{(1, -b_i), (1, a_i)\}$$

and

$$I_1 = \{(a_k + b_k, a_i b_k - a_k b_i) : \; k = 0, \dots, n; \; k \neq i, k_1\} \cup \{(1, a_i)\}.$$

In this case we have $I_0 \cap I_1 = \{(1, a_i)\}$, i.e.,

$$\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 \leq (n+1) + n - 1 = 2n.$$

The proof in the case $a_{k_1} = b_{k_2} = 0$ is similar.

Now we consider the case $a_i = 0$ or $b_i = 0$. Let for example $a_i = 0$, then from (6) we have

$$y(b_i) f_i(b_i x) - f(x) = -f(x - b_i y) - b_i y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x - a_k b_i y),$$

i.e.,

$$y b_i \tilde{f}(x) - f(x) = -f(x - b_i y) - b_i y \sum_{k=0, k\neq i}^{n} f_k((a_k + b_k)x - a_k b_i y).$$

In this case we take

$$I_0 = \{(1, -b_i)\}$$

and

$$I_1 = \{(a_k + b_k, -a_k b_i) : \; k = 0, \dots, n; \; k \neq i\}.$$

Thus similarly as before $\tilde{f}$ is a polynomial function of degree not greater than

$$\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 \leq (n+1) + n - 1 = 2n.$$

It is easy to see that if for some $k_2 \in \{0, \dots, n\}$, $b_{k_2} = 0$, then $\tilde{f}$ is a polynomial function of order at most $2n - 1$.

Now we are in position to state the most important result of this paper. Namely, we give a general solution of (2) for functions acting on integral domains satisfying some assumptions.

Theorem 1

Let P be an integral domain with unit element $\mathbb{1}$, uniquely divisible by $5!$ and such that for every $n \in \mathbb{N}$ we have $n\mathbb{1} \neq 0$. The functions $f, g, h \colon P \to P$ satisfy the equation (2) if and only if there exist $a, b, c, d, \bar{d}, e \in P$ and an additive function $A \colon P \to P$ such that

$$\begin{aligned} f(x) &= 18ax^4 + 8bx^3 + cx^2 + 2dx + e, && x \in P, \\ g(x) &= 9ax^3 + 3bx^2 + cx - 3A(x) + d - \bar{d}, && x \in P, \\ h(x) &= ax^3 + bx^2 + A(x) + \bar{d}, && x \in P. \end{aligned}$$

Proof. Assume that $f, g, h\colon P \to P$ satisfy the equation (2). From Lemma 3 we know that g and h are polynomial functions of order at most 5. Therefore

$$g(x) = c_0 + c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x), \qquad x \in P \tag{7}$$

and

$$h(x) = d_0 + d_1(x) + d_2(x) + d_3(x) + d_4(x) + d_5(x), \qquad x \in P, \tag{8}$$

where $c_i, d_i\colon P \to P$ are diagonalizations of some i-additive and symmetric functions $C_i, D_i\colon P^i \to P$, respectively. Taking in (2) $y = 0$, we obtain the following formula

$$f(x) = x[g(x) + h(x) + h(2x) + g(0)] + f(0), \qquad x \in P, \tag{9}$$

which used in (2) gives us

$$\begin{aligned} &x[g(x) + h(x) + h(2x) + g(0)] - y[g(y) + h(y) + h(2y) + g(0)] \\ &\quad = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)], \qquad x, y \in P. \end{aligned}$$

After some simple calculations we get

$$\begin{aligned} &x[h(2x) + h(x) - h(x + 2y) - h(2x + y) - g_0(y)] \\ &\quad = y[h(2y) + h(y) - h(x + 2y) - h(2x + y) - g_0(x)], \qquad x, y \in P, \end{aligned} \tag{10}$$

where $g_0(x) := g(x) - g(0)$, $x \in P$.

Further, putting $2x$ instead of y in (10), we have

$$h(5x) - h(4x) - h(2x) + h(x) = g_0(2x) - 2g_0(x), \qquad x \neq 0,$$

which is also satisfied for $x = 0$, since $g_0(0) = 0$. Thus

$$h(5x) - h(4x) - h(2x) + h(x) = g_0(2x) - 2g_0(x), \qquad x \in P. \tag{11}$$

By (7) we obtain

$$g_0(2x) - 2g_0(x) = 2c_2(x) + 6c_3(x) + 14c_4(x) + 30c_5(x) \tag{12}$$

and similarly from (8) we have

$$h(5x) - h(4x) - h(2x) + h(x) = 6d_2(x) + 54d_3(x) + 354d_4(x) + 2070d_5(x). \tag{13}$$

Using (13) and (12) in (11) we may write

$$6d_2(x) + 54d_3(x) + 354d_4(x) + 2070d_5(x) = 2c_2(x) + 6c_3(x) + 14c_4(x) + 30c_5(x).$$

Comparing the corresponding terms on both sides of this equality we get

$$\begin{aligned} c_2(x) &= 3d_2(x), \\ c_3(x) &= 9d_3(x), \\ 7c_4(x) &= 177d_4(x), \\ c_5(x) &= 69d_5(x). \end{aligned}$$

Using these equations in (7) we have

$$g(x) = c_0 + c_1(x) + 3d_2(x) + 9d_3(x) + c_4(x) + 69d_5(x), \qquad x \in P, \tag{14}$$

where

$$7c_4(x) = 177d_4(x), \qquad x \in P. \tag{15}$$

Substitute in (10) $-x$ in place of y. Then

$$h(2x) + h(-2x) - [h(x) + h(-x)] = g_0(x) + g_0(-x), \qquad x \in P.$$

This, in view of (8) and (14), means that

$$6d_2(x) + 30d_4(x) = 6d_2(x) + 2c_4(x), \qquad x \in P,$$

i.e,

$$c_4(x) = 15d_4(x), \qquad x \in P$$

and from (15) we have

$$d_4(x) = 0, \qquad x \in P \tag{16}$$

and also $c_4 = 0$.

Now we shall show that $d_5(x) = 0$ for all $x \in P$. To this end we put in (10) in places of x and y, respectively $-x$ and $2x$. Thus

$$-2h(4x) + 3h(3x) - 2h(2x) - h(-2x) - h(-x) + 3h(0) = -g_0(2x) - 2g_0(-x)$$

for $x \in P$. Similarly as before, using (8), (14) and (16), we have

$$-18d_2(x) - 54d_3(x) - 1350d_5(x) = -18d_2(x) - 54d_3(x) - 2070d_5(x), \qquad x \in P,$$

which means that

$$d_5(x) = 0, \qquad x \in P.$$

Now formulas (14) and (8) take forms

$$g(x) = c_0 + c_1(x) + 3d_2(x) + 9d_3(x), \qquad x \in P \tag{17}$$

and

$$h(x) = d_0 + d_1(x) + d_2(x) + d_3(x), \qquad x \in P. \tag{18}$$

Using these equalities in (10), we get

$$\begin{aligned} x[-c_1(y) &- 3d_1(y) + 5d_2(x) - 3d_2(y) - d_2(x+2y) - d_2(2x+y) \\ &+ 9d_3(x) - 9d_3(y) - d_3(x+2y) - d_3(2x+y)] \\ &= y[-c_1(x) - 3d_1(x) + 5d_2(y) - 3d_2(x) - d_2(x+2y) - d_2(2x+y) \\ &+ 9d_3(y) - 9d_3(x) - d_3(x+2y) - d_3(2x+y)]. \end{aligned}$$

Now, since the ring P is divisible by 3 and 2, the functions d_i are diagonalizations of symmetric and i-additive functions $D_i\colon P^i \to P$, i.e., $d_i(x) = D_i(x^i)$, $x \in P$. Using these forms of d_i in the above equation we obtain

$$\begin{aligned} 2(x-y)[4D_2(x,y) &+ 9D_3(x,x,y) + 9D_3(x,y,y)] \\ &= y[c_1(x) + 3d_1(x) + 8d_2(x) + 18d_3(x)] \\ &\quad -x[c_1(y) + 3d_1(y) + 8d_2(y) + 18d_3(y)] \end{aligned} \tag{19}$$

for all $x, y \in P$. Put in (19) $-y$ instead of y. Then for all $x, y \in P$ we have

$$\begin{aligned} 2(x+y)[-4D_2(x,y) - 9D_3(x,x,y) + 9D_3(x,y,y)] \\ = -y[c_1(x) + 3d_1(x) + 8d_2(x) + 18d_3(x)] \\ -x[-c_1(y) - 3d_1(y) + 8d_2(y) - 18d_3(y)]. \end{aligned} \tag{20}$$

Adding the equations (19) and (20) we arrive at

$$9xD_3(x,y,y) - y[4D_2(x,y) + 9D_3(x,x,y)] = -4xd_2(y), \qquad x, y \in P,$$

and, consequently,

$$9xD_3(x,y,y) - 9yD_3(x,x,y) = 4yD_2(x,y) - 4xd_2(y), \qquad x, y \in P. \tag{21}$$

Interchanging in these equations x with y and using the symmetry of both D_2 and D_3 we may write

$$9yD_3(x,x,y) - 9xD_3(x,y,y) = 4xD_2(x,y) - 4yd_2(x), \qquad x, y \in P. \tag{22}$$

Now, we add (21) and (22) to get

$$(x+y)D_2(x,y) = xd_2(y) + yd_2(x), \qquad x, y \in P.$$

Put here $x + y$ in place of x, then

$$(x+2y)D_2(x+y,y) = (x+y)d_2(y) + yd_2(x+y), \qquad x, y \in P,$$

which yields

$$xD_2(x,y) = yd_2(x), \qquad x, y \in P \tag{23}$$

and changing the roles of x and y

$$yD_2(x,y) = xd_2(y), \qquad x, y \in P. \tag{24}$$

Now, we multiply (23) by y and (24) by x to obtain

$$xyD_2(x,y) = y^2 d_2(x), \qquad x, y \in P$$

and

$$xyD_2(x,y) = x^2 d_2(y), \qquad x, y \in P.$$

Thus

$$y^2 d_2(x) = x^2 d_2(y), \qquad x, y \in P,$$

which after substituing $y = \mathbb{1}$ gives the formula

$$d_2(x) = bx^2, \qquad x \in P, \tag{25}$$

where $b := d_2(\mathbb{1})$. Thus from (24) we obtain

$$D_2(x,y) = bxy, \qquad x, y \in P. \tag{26}$$

Using the formulas (25) and (26) in (21) we have

$$yD_3(x,x,y) = xD_3(x,y,y), \qquad x,y \in P. \tag{27}$$

Putting $x+y$ in place of x (27), we get

$$yD_3(x+y,x+y,y) = (x+y)D_3(x+y,y,y),$$

which after some calculations gives

$$yD_3(x,x,y) - (x-y)D_3(x,y,y) = xd_3(y), \qquad x,y \in P.$$

We use here the condition (27). Then

$$xD_3(x,y,y) - (x-y)D_3(x,y,y) = xd_3(y), \qquad x,y \in P,$$

i.e.,

$$yD_3(x,y,y) = xd_3(y), \qquad x,y \in P. \tag{28}$$

Clearly we also have

$$xD_3(x,x,y) = yd_3(x), \qquad x,y \in P. \tag{29}$$

Now, multiply the equation (28) by x and (29) by y^2. Then we have

$$xyD_3(x,y,y) = x^2d_3(y), \qquad x,y \in P \tag{30}$$

and

$$xy^2D_3(x,x,y) = y^3d_3(x). \tag{31}$$

On the other hand, we multiply (27) by y. We obtain

$$y^2D_3(x,x,y) = xyD_3(x,y,y), \qquad x,y \in P. \tag{32}$$

Using (32) in (30) we arrive at

$$x^2d_3(y) = y^2D_3(x,x,y), \qquad x,y \in P,$$

which multiplied by x yields

$$x^3d_3(y) = xy^2D_3(x,x,y), \qquad x,y \in P. \tag{33}$$

Comparing the equation (31) and (33) we obtain

$$y^3d_3(x) = x^3d_3(y), \qquad x,y \in P,$$

i.e.,

$$d_3(x) = ax^3, \qquad x \in P, \tag{34}$$

where $a := d_3(\mathbb{1})$. Now equalities (28) and (29) take forms

$$D_3(x,y,y) = axy^2, \qquad x,y \in P \tag{35}$$

and

$$D_3(x,x,y) = ax^2y, \qquad x,y \in P. \tag{36}$$

Using the formulas (25), (26), (34), (35) and (36) in (19) we have

$$y[c_1(x) + 3d_1(x)] = x[c_1(y) + 3d_1(y)], \qquad x,y \in P.$$

Substituting here $y = \mathbb{1}$ we obtain

$$c_1(x) + 3d_1(x) = x[c_1(\mathbb{1}) + 3d_1(\mathbb{1})], \qquad x \in P,$$

which means that

$$c_1(x) = cx - 3d_1(x), \qquad x \in P,$$

where $c := c_1(\mathbb{1}) + 3d_1(\mathbb{1})$.

Thus we have shown that the formulas (17) and (18) may be written in the form

$$g(x) = 9ax^3 + 3bx^2 + cx - 3d_1(x) + c_0, \qquad x \in P$$

and

$$h(x) = ax^3 + bx^2 + d_1(x) + d_0, \qquad x \in P,$$

where d_1 is a given additive function. Now it suffices to use the obtained expressions in (9), to get the desired formula for f.

It is an easy calculation to show that these functions f, g, h satisfy the equation (2).

With the aid of this theorem we may prove also a Stamate-kind result.

Corollary 1

Let P be an integral domain with unit element $\mathbb{1}$, uniquely divisible by $5!$ and such that for every $n \in \mathbb{N}$ we have $n\mathbb{1} \neq 0$. Functions $f,g,h\colon P \to P$ satisfy the equation (3) if and only if there exist $a,\bar{a},b,c,d,\bar{d} \in P$ and an additive function $A\colon P \to P$ such that

$$f(x) = \begin{cases} 18ax^3 + 8bx^2 + cx + 2d, & x \neq 0 \\ \bar{a}, & x = 0 \end{cases},$$

$$g(x) = \begin{cases} -9ax^3 - 5bx^2 - 3A(x) - d - \bar{d}, & x \neq 0 \\ d - \bar{d} - \bar{a}, & x = 0 \end{cases},$$

$$h(x) = ax^3 + bx^2 + A(x) + \bar{d}, \qquad x \in P.$$

Conversely, $f,g,h\colon P \to P$ given by the above equalities satisfy (2).

Proof. First we write the equation (3) in the form

$$\begin{aligned}(y-x)f(y) - yf(y) + (y-x)f(x) + xf(x)\\ = (x-y)[g(x) + h(2x+y) + h(x+2y) + g(y)]\end{aligned}$$

and, consequently,

$$xf(x) - yf(y) = (x-y)[g(x) + f(x) + h(2x+y) + h(x+2y) + g(y) + f(y)].$$

Putting here $k(t) := g(t) + f(t)$ and $F(t) := tf(t)$ for all $t \in P$ we obtain

$$F(x) - F(y) = (x-y)[k(x) + h(2x+y) + h(x+2y) + k(y)], \qquad x, y \in P.$$

Thus, using Theorem 1, we get

$$xf(x) = 18ax^4 + 8bx^3 + cx^2 + 2dx + e, \qquad x \in P, \tag{37}$$

$$g(x) + f(x) = 9ax^3 + 3bx^2 + cx - 3A(x) + d - \bar{d}, \qquad x \in P, \tag{38}$$

$$h(x) = ax^3 + bx^2 + A(x) + \bar{d}, \qquad x \in P.$$

Now, from (37) it easily follows that $e = 0$ and furthermore

$$xf(x) = 18ax^4 + 8bx^3 + cx^2 + 2dx,$$

i.e.,

$$f(x) = 18ax^3 + 8bx^2 + cx + 2d, \qquad x \neq 0,$$

which gives us

$$g(x) = -9ax^3 - 5bx^2 - 3A(x) - d - \bar{d}, \qquad x \neq 0.$$

Moreover, from (38) we get $g(0) + f(0) = d - \bar{d}$, thus putting $\bar{a} := f(0)$ we obtain that $g(0) = d - \bar{d} - \bar{a}$.

On the other hand, it is easy to see that functions given by the above formulae yield a solution of the equation (3).

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