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Optimal design of unbounded 2D composite materials with circular inclusions of different radii

Abstract. Optimal design problem for 2D composite materials with different circular inclusions is studied on the base of the potential method combined with functional equation method. Exact geometric description of the optimal distribution of the inclusions is determined.

1. Introduction

The article is devoted to the constructive analysis of mathematical models arising at the study of optimal design of 2D composite materials (see e.g. [5]).

The optimal design problem in the considered case is the problem of the determination of a distribution of circular inclusions in a matrix of homogeneous material in such a way that the obtained inhomogeneous material possesses an extremal (minimal or maximal) effective conductivity in a given direction.

Potential analysis is used in combination with the method of functional equations (for wider description of the approach see [6] and [7]). Such approach makes possible to discover general properties of composite materials on the base of an explicit representation of the effective conductivity functional. Besides, in certain special cases we have found an exact geometric description of an optimal (in the above sense) distribution of inclusions.

The paper continues the authors' study of behind optimal design problems which were previously devoted to the case of 2D composite materials with equal circular inclusions (see [5]). In particular, in [5] we studied the problem of optimal design of 2D unbounded composite materials in the case of small Bergmann parameter. The corresponding boundary conditions are simplified, namely only their main parts are considered. Such model situation allows us to give a complete geometric description of the optimal distribution of circular inclusions of equal radius. It was shown that the solution of the simplified boundary value problems gives the minimal or maximal value to the functional of the effective conductivity if each inclusion touches at least one of others. For small number of inclusions an exact description of the optimal distribution of

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inclusions is given and exact optimal value of the changing part of the functional of the effective conductivity is calculated. Such model problem can be used for an approximation of the optimal design problem with sufficiently small concentration of inclusions ν.

Here we concern with the case of non-equal circular inclusions. We use the same argument. It should be noted that the constructive approach applied here differs from the recently studied models of optimal design based mainly on the homogenization technique (see e.g. [2], [1]).

2. A model

Let us consider 2D unbounded composite materials with circular inclusions of different radii. Let the matrix of a composite be geometrically modelled by an unbounded multiply connected circular domain, namely, an exterior of finite number of discs of different radii. These discs correspond to inclusions for which the radii are given but the position on the complex plane are subject of further determination. We suppose that the matrix is filled in by the homogeneous material of a constant (thermal) conductivity $\lambda_m = 1$, and the inclusions are filled in by another material of a constant conductivity $\lambda_i = \lambda$. We suppose additionally that the Bergmann parameter $\rho = \frac{\lambda-1}{\lambda+1}$ is sufficiently small ($|\rho| \ll 1$). The composite material is placed into the steady (thermal) field. In order to avoid indeterminancies, we consider only the case of positive Bergmann parameter, i.e., the conductivity of inclusions is greater than the conductivity of matrix.

The question is to determine the distribution of the inclusions for which the considered inhomogeneous composite material possesses an extremal (minimal or maximal) effective conductivity in a given direction (say in the direction of the positive real line). In the case of a small Bergmann parameter we use the same approach as in [5], namely, we simplify the boundary conditions by considering only the main part of them (with respect to the power of ρ) and then minimize or maximize the only changing part of the functional of the effective conductivity. Such simplification gives us possibility to obtain an analytic solution to the model problem. The later can be considered as an approximation to the starting optimal design problem. The model problem is studied by the reduction to a finite-dimensional extremal problem with centers of inclusions as unknown variables.

We have to note that our approach does not depend on the type of the considered physical field neither on the direction in which we determine the effective conductivity.

3. Solution to the optimal design problem in the case of a small Bergmann parameter

Let $D_k := \{z \in \mathbb{C} : \ |z - a_k| < r_k\}, \ k = 1, \dots, n, \ |a_j - a_k| \geq r_j + r_k$ and $a_j \neq a_k$, for $k \neq j$, be a finite number of disjoint discs and $L_k := \{z \in \mathbb{C} : \ |z - a_k| = r_k\}$ be their boundary circles. We consider an optimal design problem in the potential case, i.e., there are thermal potentials u_k in each disc D_k , $k = 1, \dots, n$, as well as a potential u in the domain $D = \mathbb{C} \setminus \bigcup_{k=1}^{n} \overline{D_k}$. We suppose that these potentials satisfy the ideal contact conditions on the boundary of inclusions $L = \bigcup_{k=1}^{n} L_k$. By introducing the complex potentials

$$\begin{aligned} \varphi(z) &= u(z) + iv(z), && z \in D, \\ \varphi_k(z) &= u_k(z) + iv_k(z), && z \in D_k, \ k = 1, \dots, n, \end{aligned} \tag{1}$$

we arrive at the $\mathbb{R}$-linear boundary value conditions on each circle L_k

$$\varphi(t) = \varphi_k(t) - \rho\overline{\varphi_k(t)} + g(t), \tag{2}$$

where $g(z)$ is a given function representing an external thermal field. It is well-known (see e.g. [6], [7]) that for a general domain problem, (2) does not admit an analytic solution. In the case of a multiply connected circular domain an analytic solution to the $\mathbb{R}$-linear boundary value problem with constant coefficients is obtained (see e.g. [7]) in the form of series with summations depending on behind certain group of symmetries. Anyway, even in this case we cannot use such representation in order to get an exact description of the optimal distribution of (circular) inclusions. That is why certain simplification of the problem is made.

Here we consider the case of unbounded composite materials with finite number of circular inclusions and with conductivities of matrix and inclusions close to each other (i.e., with small Bergmann parameter). Thus the concentration ν is equal to 0 and the second term in the right hand-side of (2) is sufficiently small. Thus we replace the starting optimal design problem by model one. The later consists in optimization of the changing part of the standard functional of the effective conductivity

$$\frac{\lambda_e}{\lambda_m} = 1 + \frac{2\nu\rho}{n} \sum_{k=1}^{n} \int\limits_{L_k} \operatorname{Re} \varphi_k^{-}(t)\, dy, \qquad t = x + iy, \tag{3}$$

on the set of solutions to a simplified boundary value problem (which depend on position of the inclusions in the composite).

Therefore, our model optimal design problem is considered in the form: to find positions of the discs D_k , $k = 1, \dots, n$, such that the following functional σ possesses an optimal value, namely

$$\sigma := \sum_{k=1}^{n} \int_{L_k} \operatorname{Re} \varphi_k^-(t)\, dy \quad \longrightarrow \quad \min(\max), \tag{4}$$

under constrains (boundary conditions)

$$\varphi^+(t) - \varphi_k^-(t) = g(t), \qquad t \in L = \bigcup_{k=1}^{n} L_k\,, \tag{5}$$

where $t = x + iy$.

It is known (see [3]) that the solution of the problem (5) can be represented in the following form

$$\varphi^\pm(z) = \frac{1}{2\pi i} \int_L \frac{g(t)\, dt}{t - z}, \qquad z \in D^\pm, \tag{6}$$

where each of the circles L_k is clock-wise oriented. Thus the extremal distribution of domains D_k is determined by their centers and also by the values of a given function g. The determination of the extremal value of the functional (4) is in general rather complicated problem. The complete (and exact) solution is possible only in the case when the function g is explicitly given. Here we confine ourself to the important for mechanics of composite materials case $g(z) = \overline{z}$. In this case, the integrals (6) are calculated explicitly and the general solution to the problem (5) has the following form

$$\varphi(z) = \begin{cases} -\displaystyle\sum_{m=1}^{n} \frac{r_m^2}{z - a_m}, & z \in D^-, \\ \overline{a_k} - \displaystyle\sum_{m \neq k}^{n} \frac{r_m^2}{z - a_m}, & z \in D_k\,. \end{cases} \tag{7}$$

We calculate the value of the functional (4) by using the mean value theorem for harmonic functions (see [4])

$$\sigma = \int_L \operatorname{Re} \varphi^-(t)\, dy = \sum_{k=1}^{n} \int_{L_k} \operatorname{Re} \varphi^-(t)\, dy = \pi \sum_{k=1}^{n} r_k^2 \operatorname{Re} \left(\varphi^-\right)'(a_k). \tag{8}$$

Let us find the values $\left(\varphi_k^-\right)'(a_k)$:

$$\left(\varphi_k^-\right)'(a_k) = \sum_{m \neq k} \frac{r_m^2}{(z - a_m)^2}\Bigg|_{z = a_k} = \sum_{m \neq k} \frac{r_m^2}{(a_k - a_m)^2}\,.$$

Substituting these values into (8), we get $\sigma = \operatorname{Re} \mu$, where

$$\mu = \pi \sum_{k=1}^{n} \sum_{m \neq k} \frac{r_m^2 r_k^2}{(a_k - a_m)^2} \,. \tag{9}$$

Therefore, the analysis of the initial extremal problem is reduced now to the study of the complex valued function μ of n complex variables $a_1, a_2, \ldots, a_n$. Since the value of the function μ is independent on the translation, we can fix one of the points a_k .

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Assume that the function $\sigma = \operatorname{Re}\mu$ attains its maximum on the set of points $A := \{a_1, a_2, \ldots, a_n\}$. Then

1) $\mu(A) \in \mathbb{R}$*;*
2) *each disc D_k is touched by at least one of other discs D_m , so that the closure of the domain $\overline{D^+}$ is a connected set on the complex plane $\mathbb{C}$.*

Proof. 1) Let us represent the functional μ given by (9) in the form $\mu = \mu_1 + i\mu_2$. Denote by $\mu(A)$ the value of the functional μ corresponding to the set of points A.

Consider the value of the function μ after rotation of the plane by a certain angle θ, i.e., corresponding to the set of points $A' = \{e^{i\theta}a_1, e^{i\theta}a_2, \ldots, e^{i\theta}a_n\}$:

$$\begin{aligned}\mu(A') &= \pi \sum_{k=1}^{n} \sum_{m \neq k} \frac{r_m^2 r_k^2}{e^{2i\theta}(a_k - a_m)^2} = e^{-2i\theta}(\mu_1(A) + i\mu_2(A)) \\ &= \mu_1(A)\cos 2\theta + \mu_2(A)\sin 2\theta + i(\mu_2(A)\cos 2\theta - \mu_1(A)\sin 2\theta).\end{aligned}$$

Therefore, $\sigma = \operatorname{Re}\mu = \mu_1 \cos 2\theta + \mu_2 \sin 2\theta$. One can choose the value of θ in such a way that

$$\mu(A') = |\mu(A)| = \sigma(A') \leq \sigma(A) = \operatorname{Re}\mu(A).$$

Hence $\operatorname{Im}\mu(A) = 0$ [1].

It follows that on the extremal set of points A the function

$$\mu(A) = \mu_1(A) = \pi \sum_{k=1}^{n} \sum_{m \neq k} \frac{r_m^2 r_k^2}{(a_k - a_m)^2}$$

has a real value.

2) Let $A := \{a_1, a_2, \ldots, a_n\}$ be an optimal set of centers, i.e., the functional σ attains its maximal value on A. Consider the corresponding optimal set of

[1] The authors are thankful to the referee who shows a shorter proof of the first assertion to Lemma.

discs with the centers $a_1, a_2, \ldots, a_n$ (for simplicity we can denote this set by A too). Introduce the following function

$$u(z) := \mathrm{Re} \sum_{m=2}^{n} \frac{r_m^2 r_1^2}{(z - a_k)^2} \, .$$

The sum $u(a_1)$ is the part of the sum σ, which is changing when the disc D_1 is moving. Under assumption, the function $u(z)$ attains its maximal value for $z = a_1$. Moreover, $|a_1 - a_k| > r_1 + r_k$, $k = 2, 3, \ldots, n$. But the function $u(z)$ is harmonic in the (in general, multiply connected) domain

$$\{z : \ |z - a_k| > r_1 + r_k, \ k = 2, 3, \ldots, n\}$$

and continuous in

$$\{z : \ |z - a_k| \geq r_1 + r_k, \ k = 2, 3, \ldots, n\},$$

vanishing at infinity. Therefore, the maximal value of the function $u(z)$ is attained at a boundary point of the above domain, i.e., when $|a_1 - a_k| = r_1 + r_k$ for certain k. Hence the optimal discs are touching each other.

Let us show now that the closure of the discs corresponding to the set A is a connected set. If not, then $A = A_1 \cup A_2$, with none disc from A_1 touching any disc from A_2. Let us fix one of the centers $a_p \in A_1$, and one of the centers $a_q \in A_2$. Represent another centers $a_k \in A_1$ in the form $a_k = a_p + b_{kp}$, and the centers $a_m \in A_2$ in the form $a_m = a_q + b_{qm}$, respectively. We consider the following function

$$u(z) := \mathrm{Re} \sum_{k,m} \frac{r_m^2 r_k^2}{(a_p - z + b_{kp} - b_{mq})^2} \, ,$$

where k, m are those values of indices for which $a_k \in A_1$, $a_m \in A_2$. The sum $u(a_q)$ is a part of the sum σ, which is changing when the mutual position of the sets A_1, A_2 is changing for fixed elements inside these sets. The variable z is modelling such changing. This variable is running along a compact set K in $\hat{\mathbb{C}}$, which described all possible changing of the mutual position of discs corresponding to A_1, A_2, up to the touching of these sets. The function $u(z)$ is harmonic in $\mathrm{int}\, K$ and continuous in K. By the Maximum Principle for harmonic functions this function has to attain its maximum on the boundary of K. The latter corresponds to the touching of certain discs from A_1 with certain discs from A_2. It contradicts our assumption and the Lemma is proved.

It follows from the Lemma that the optimal distribution of the discs always corresponds to the percolation situation, i.e., to the case when the discs are touching and constitute a connected set.

4. Exact geometric description of the solution in certain special cases

Let $n = 2$, $D_1 = \{z \in \mathbb{C} : |z - a_1| < r\}$, $D_2 = \{z \in \mathbb{C} : |z - a_2| < R\}$, $r < R$. Then $\mu_1 = \frac{2\pi r^2 R^2}{(a_1 - a_2)^2}$. Hence, $\mathrm{Im}\frac{1}{(a_1-a_2)^2} = 0$. It is possible in the following two cases:

a) $a_1 - a_2 = -(r + R)i$ (see Fig. 1.1), or $a_1 - a_2 = (r + R)i$ (see Fig. 1.2). In this case $\mu_1 = -\frac{2\pi r^2 R^2}{(R+r)^2}$. Therefore, the minimal value of the functional σ (which is the changing part of the effective conductivity functional) for composites with two circular inclusions is attained when the centers of these inclusions lay on the straight line parallel to imaginary axes, and these discs are touching each other, of course.

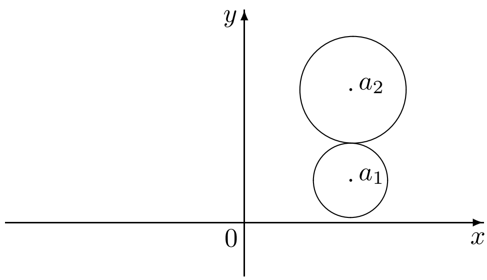

Fig. 1.1. Position of two inclusions corresponding to the minimal value of the functional (4)

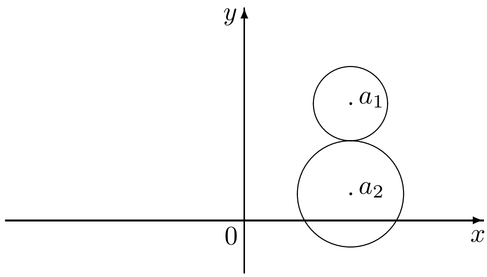

Fig. 1.2. Position of two inclusions corresponding to the minimal value of the functional (4)

b) $a_1 - a_2 = -(R + r)$ (see Fig. 1.3), or $a_1 - a_2 = R + r$ (see Fig. 1.4), and thus $\mu_1 = \frac{2\pi r^2 R^2}{(R+r)^2}$. Maximal value of the effective conductivity functional corresponds to the horizontal position of inclusions.

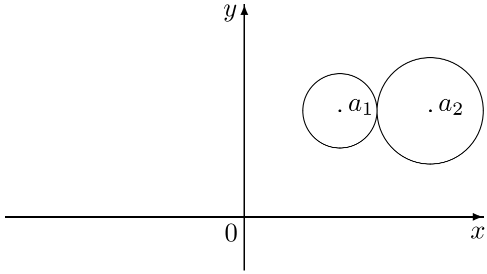

Fig. 1.3. Position of two inclusions corresponding to the maximal value of the functional (4)

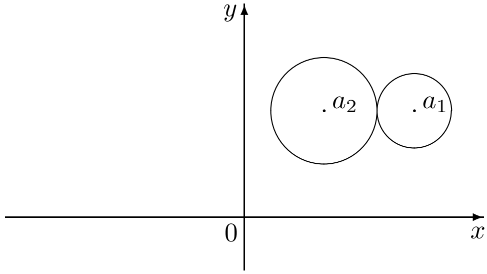

Fig. 1.4. Position of two inclusions corresponding to the maximal value of the functional (4)

Example
In the case of three inclusions an optimal configuration is of the cluster type, i.e., three inclusions are touching each other.

For instance, let us consider three discs of radii $r = 1, 2, 4$, respectively. Put an origin of the coordinate system at the focal point of the triangle with edges at the centers of the touching discs. Then the value of the functional μ is equal to

$$\mu = -\frac{28\pi}{r^2} e^{i(\frac{\pi}{3} - 2\alpha)},$$

where $r = \frac{45}{4\sqrt{14}}$ is the circumradius of the triangle, and α is a rotation angle. The minimal (maximal) value of this functional is achieved at $\alpha = \frac{\pi}{3}$ ($\alpha = \frac{4}{3}\pi$), and is equal to $\mu = -\frac{6272}{2025}\pi$ ($\mu = +\frac{6272}{2025}\pi$, respectively).

For comparison, if we consider the chain of inclusions of the same radii, then minimal (maximal) value of μ is equal to $\mu = -\frac{80}{81}\pi$ ($\mu = +\frac{80}{81}\pi$, respectively).

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