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Explicit solution of a class of Riemann–Hilbert problems

Abstract. Analytical solutions to a special class of Riemann–Hilbert boundary value problems on multiply connected domains are presented. The solutions are expressed, up to a finite number of accessory parameters, as non-singular indefinite integrals whose integrands are expressed in terms of the Schottky–Klein prime function associated with the Schottky double of the planar domain.

1. A class of Riemann–Hilbert problems

The subject of this paper is a special class of Riemann–Hilbert problems (RH problems) on multiply connected planar domains. The study of general RH problems is a classical subject and discussions of it can be found in standard monographs on boundary value problems [9], [18], [13]. A solution of the general (Riemann)–Hilbert boundary value problem has been found, using successive iteration methods, by Mityushev [14]. Here we restrict attention to a special (but important) subclass of the same RH problems and find an analytical expression for the solutions, up to a finite set of accessory parameters, in terms of a transcendental function known as the Schottky–Klein prime function [3] associated with the multiply connected domain.

We define a *circular domain* D_ζ in a complex parametric ζ-plane to be a domain whose boundaries are all circles. Let D_ζ be the $M+1$ connected circular domain in a ζ-plane consisting of the unit disc with M smaller discs excised from its interior. The outer boundary of D_ζ is the unit circle which we label C_0. Label the M inner boundary circles of D_ζ as $C_1, \ldots, C_M$. For $k = 0, 1, \ldots, M$ let the centre and radius of C_k be δ_k and q_k respectively.

Consider the Riemann–Hilbert problem for the function $w(\zeta)$:

$$\mathrm{Re}\left[\overline{\lambda_k}\, w(\zeta)\right] = d_k \qquad \text{on } C_k,\ k = 0, 1, \ldots, M, \tag{1}$$

where $\{\lambda_k \in \mathbb{C} \mid |\lambda_k| = 1,\ k = 0, 1, \ldots, M\}$ is a set of complex constants with unit modulus and $\{d_k \in \mathbb{R} \mid k = 0, 1, \ldots, M\}$ is a set of real constants. We solve for

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$w(\zeta)$ satisfying (1) that is analytic, but not necessarily single-valued, in D_ζ except for a simple pole, with known residue, at some point $\zeta = \beta$ strictly inside D_ζ.

Circular domains are a canonical class of planar domains because every planar domain is conformally equivalent to some circular domain [10]. Because of this, and because the class of RH problems (1) is conformally invariant, it means that the solution scheme which follows is rather general. It applies, up to conformal mapping from the canonical class of circular domains, to *any* multiply connected planar region.

Problem (1) is a generalization of the classical Schwarz problem [9], [18], [13], a case of which is retrieved on making the choice, for example, that $\lambda_k = 1$ for all $k = 0, 1, \ldots, M$ in (1). This paper produces an analytical expression for the solution of (1) when the constants $\{\lambda_k \in \mathbb{C} \mid k = 0, 1, \ldots, M\}$ are generally distinct. The solution is expressed as a non-singular, indefinite integral whose integrand is written in terms of the Schottky–Klein prime function [3] associated with D_ζ. This integrand depends on a finite set of accessory parameters that can, in principle, be determined (for example, numerically) from the given data $\{\lambda_k, d_k \in \mathbb{C} \mid k = 0, 1, \ldots, M\}$.

The special form of RH problem (1) has been considered by other authors. Vekua [18] shows that, if it exists, the solution of the RH problem (1) is unique [18]. Wegmann & Nasser [19] study the doubly connected case $M = 1$ of (1) in a recent paper on numerical solutions of RH problems on multiply connected regions using integral equations based on the generalized Neumann kernel.

The class of RH problems appears in a variety of applications, especially in the more general (discontinuous) case when the value of the constant λ_k assumes different values on different segments of the circle C_k (the methods of this paper, presented for the continuous problem, can be generalized to this case). One of the more important applications is to free streamline theory in hydrodynamics. There, in the study of jets and cavities, it is traditional to study a function known as the Joukowski function [11], often written as

$$\Omega(\zeta) \equiv \log\left(\frac{1}{V_0}\frac{dw(z)}{dz}\right),$$

where $z = x + \mathrm{i}y$, V_0 is a constant scaling factor and $w(z)$ is an analytic function in the flow region (known as the complex velocity potential). On any solid boundaries in contact with the fluid, the imaginary part of $\Omega(\zeta)$ is constant; on any free streamlines, owing to the constancy of pressure in a cavity region on one side of the free streamline and Bernoulli's theorem, it is the real part of $\Omega(\zeta)$ that is constant. Since a single streamline in a real flow can, in part, be in contact with a solid boundary and then separate into a free streamline bounding a cavity, $\Omega(\zeta)$ turns out to satisfy a (discontinuous) Riemann–Hilbert problem of precisely the form (1). In the simply connected situation, Schwarz–Christoffel methods have proved to be very useful in problems of this kind [11]. Interestingly, there has been recent interest [2] in developing this nonlinear theory to flows involving multiple body-cavity systems. The theory presented here, for multiply connected situations, should find application in such studies.

2. Function theory

The investigation we now present borrows ideas from prior work by the author [5], [6] in which new analytical formulae for the Schwarz–Christoffel mappings to bounded and unbounded polygonal domains were constructed. Although this viewpoint is not the one taken in [5], [6], such Schwarz–Christoffel mappings can be viewed as satisfying a RH problem on a multiply connected domain of exactly the form (1). Here, the same constructive method is exploited to find explicit representations of the solution of broader classes of RH problems in multiply connected domains.

In this paper, for ease of exposition, we focus on the continuous case where the constant λ_k assumes the same value at all points on the circle C_k (in the discontinuous analogue, which is more akin to the usual Schwarz–Christoffel problem, the value of this constant is allowed to be different on different segments of C_k). A consequence of this assumption is that we effectively do not allow any branch point singularities of $w(\zeta)$ on any of the circles $\{C_j \mid j = 0, 1, \dots, M\}$. The method, however, can be readily generalized to the case where branch points *are* present.

We now construct some special functions associated with D_ζ. First, for $k = 0, 1, \dots, M$, define the Möbius transformation $\phi_k(\zeta)$ by

$$\phi_k(\zeta) = \overline{\delta_k} + \frac{q_k^2}{\zeta - \delta_k}, \qquad k = 0, 1, \dots, M. \tag{2}$$

It is straightforward to check that for ζ on circle C_k,

$$\phi_k(\zeta) = \overline{\zeta}.$$

We define the *reflection* of a point ζ in the circle C_k by $\overline{\phi_k(\zeta)}$. Then, for $k = 1, \dots, M$, introduce the Möbius transformation $\theta_k(\zeta)$ defined by

$$\theta_k(\zeta) = \overline{\phi_k\left(\overline{\zeta}^{-1}\right)}, \qquad k = 1, \dots, M. \tag{3}$$

It follows from (3) and (2) that

$$\theta_k(\zeta) = \delta_k + \frac{q_k^2 \zeta}{1 - \overline{\delta_k}\zeta}, \qquad k = 1, \dots, M.$$

For $k = 1, \dots, M$, let C_k' denote the reflection of C_k in C_0. It can be shown that $\theta_k(\zeta)$ maps C_k' onto C_k.

Let Θ denote the set of all compositions of the maps $\{\theta_k(\zeta) \mid k = 1, \dots, M\}$ and their inverses. It is an example of an infinite *Schottky group*. Further information on Schottky groups can be found in [3], [4]. We refer to the maps $\{\theta_k(\zeta) \mid k = 1, \dots, M\}$, together with their inverses, as the *generators* of Θ. A *fundamental region* of Θ is a connected region whose images under all maps in Θ tessellate the whole of the plane. Consider the region consisting of D_ζ and its reflection in C_0, i.e., the $2M$-connected region bounded by $\{C_k, C_k' \mid k = 1, \dots, M\}$. Label this region as F. F is a fundamental region of Θ.

Associated with Θ are M functions known as *integrals of the first kind* which we denote $\{\upsilon_k(\zeta) \mid k = 1, \ldots, M\}$. These are analytic, but not single-valued, in F. Indeed, for $j, k = 1, \ldots, M$ we have

$$[\upsilon_k(\zeta)]_{C_j} = -[\upsilon_k(\zeta)]_{C'_j} = \delta_{jk}, \tag{4}$$

where $[\upsilon_k(\zeta)]_{C_j}$ and $[\upsilon_k(\zeta)]_{C'_j}$ denote respectively the changes in $\upsilon_k(\zeta)$ on traversing C_j and C'_j with the interior of F on the right, and δ_{jk} denotes the Kronecker delta function. Furthermore, for $j, k = 1, \ldots, M$,

$$\upsilon_k(\theta_j(\zeta)) - \upsilon_k(\zeta) = \tau_{jk} \tag{5}$$

for some $\{\tau_{jk} \mid j, k = 1, \ldots, M\}$ which are constants, i.e., independent of ζ. The functions $\{\upsilon_k(\zeta) \mid k = 1, \ldots, M\}$ are uniquely determined (up to an additive constant) by their periods given by (4) and (5).

2.1. The Schottky–Klein prime function

Let α be some arbitrary point in F. It is established in [12] that there exists a unique function $X(\zeta, \alpha)$ defined by the properties:

(i) $X(\zeta, \alpha)$ is single-valued and analytic in F.

(ii) $X(\zeta, \alpha)$ has a second-order zero at each of the points $\theta(\alpha)$, $\theta \in \Theta$.

(iii) $\lim_{\zeta \to \alpha} \frac{X(\zeta,\alpha)}{(\zeta-\alpha)^2} = 1$.

(iv) For $k = 1, \ldots, M$,

$$X(\theta_k(\zeta), \alpha) = \exp\left(-2\pi \mathrm{i}(2\upsilon_k(\zeta) - 2\upsilon_k(\alpha) + \tau_{kk})\right) \frac{d\theta_k(\zeta)}{d\zeta} X(\zeta, \alpha).$$

The *Schottky–Klein prime function* (henceforth referred to as S–K prime function), which we denote $\omega(\zeta, \alpha)$, is defined as

$$\omega(\zeta, \alpha) = (X(\zeta, \alpha))^{1/2},$$

where the branch of the square root is chosen so that $\omega(\zeta, \alpha)$ behaves like $(\zeta - \alpha)$ as $\zeta \to \alpha$.

There are two known ways to evaluate the S–K prime function. One possibility is to use a classical infinite product formula for it as recorded, for example, in Baker [3]. It is given by

$$\omega(\zeta, \alpha) = (\zeta - \alpha) \prod_{\theta_k} \frac{(\theta_k(\zeta) - \alpha)(\theta_k(\alpha) - \zeta)}{(\theta_k(\zeta) - \zeta)(\theta_k(\alpha) - \alpha)}, \tag{6}$$

where the product is over all compositions of the basic maps $\{\theta_j, \theta_j^{-1} \mid j = 1, \ldots, M\}$ excluding the identity and all inverse maps. This product, even if it is convergent, can converge so slowly and require such a large number of terms in the product, that its use in many circumstances is impractical. An alternative numerical scheme has recently been put forward by Crowdy & Marshall [8]; it is much more computationally efficient than methods based on the infinite product (6) over the Schottky group.

3. The circular slit domain

To proceed with the construction, we introduce an intermediate η-plane. Consider a conformal mapping, denoted $\eta(\zeta;\alpha)$, taking the multiply connected circular domain D_ζ to a conformally equivalent circular slit domain called D_η. α is the point in D_ζ mapping to $\eta = 0$ in D_η, i.e., $\eta(\alpha;\alpha) = 0$. Figure 1 shows a schematic in a triply connected case. Let the image of C_0 under this mapping be the unit circle in the η-plane which will be called L_0. The M circles $\{C_j \mid j = 1,\ldots,M\}$ will be taken to have circular-slit images, centred on $\eta = 0$, and labelled $\{L_j \mid j = 1,\ldots,M\}$. Let the circular arc L_j be characterized by the conditions

$$|\eta| = r_j, \qquad \arg[\eta] \in [\phi_1^{(j)}, \phi_2^{(j)}].$$

There will be two pre-image points on the circle C_j corresponding to the two endpoints of the circular-slit L_j. These two pre-image points, labelled $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$, satisfy the conditions

$$\eta(\gamma_1^{(j)};\alpha) = r_j e^{i\phi_1^{(j)}}, \qquad \eta_\zeta(\gamma_1^{(j)},\alpha) = 0,$$
$$\eta(\gamma_2^{(j)};\alpha) = r_j e^{i\phi_2^{(j)}}, \qquad \eta_\zeta(\gamma_2^{(j)},\alpha) = 0.$$

These two zeros of $\eta_\zeta(\zeta)$ on C_j are simple zeros.

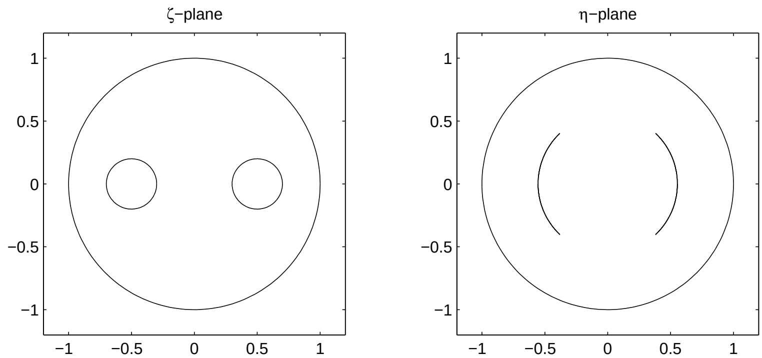

Figure 1: A typical circular slit mapping from a triply connected circular region D_ζ in a ζ-plane to a triply connected circular slit domain D_η in a η-plane.

It is shown in [5] and [7] that an explicit expression for the conformal slit mapping from D_ζ to D_η can be found in terms of the S–K prime function of D_ζ. It is given by

$$\eta(\zeta;\alpha) = \frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\overline{\alpha}^{-1})}. \tag{7}$$

Formula (7) will be crucial in the solution scheme to follow.

4. Solution scheme

The required function $w(\zeta)$ is analytic in D_ζ. One can also consider the composed function $W(\eta)$, analytic in D_η, defined by

$$W(\eta(\zeta;\alpha)) \equiv w(\zeta).$$

The boundary conditions (1), expressed in terms of this new function $W(\eta)$, are

$$\mathrm{Re}\left[\overline{\lambda_k}\, W(\eta)\right] = d_k \qquad \text{on } L_k,\ k = 0, 1, \ldots, M.$$

These can be rewritten in the form

$$\overline{\lambda_k}\, W(\eta) + \lambda_k \overline{W(\eta)} = 2d_k \qquad \text{on } L_k,\ k = 0, 1, \ldots, M,$$

or, on use of the fact that $\overline{\eta} = r_k^2 \eta^{-1}$ on L_k,

$$\overline{\lambda_k}\, W(\eta) + \lambda_k \overline{W}(r_k^2 \eta^{-1}) = 2d_k \qquad \text{on } L_k,\ k = 0, 1, \ldots, M. \tag{8}$$

Using $W'(\eta)$ to denote the derivative of W with respect to its argument, differentiation of (8) with respect to η gives

$$\overline{\lambda_k}\, W'(\eta) - \frac{r_k^2}{\eta^2} \lambda_k \overline{W}'(r_k^2 \eta^{-1}) = 0 \qquad \text{on } L_k,\ k = 0, 1, \ldots, M,$$

which can be rewritten as

$$\frac{\eta W'(\eta)}{\overline{\eta W'(\eta)}} = \frac{\lambda_k}{\overline{\lambda_k}} \qquad \text{on } L_k,\ k = 0, 1, \ldots, M.$$

This is a statement of the fact that the argument of $\eta W'(\eta)$ is constant on L_k.

Let us now suppose that we seek a solution for which there are precisely two zeros of the derivative $dw/d\zeta$ on each of the boundary components $\{C_j \mid j = 0, 1, \ldots, M\}$. Let the positions of the two zeros on C_j be at points a_j and c_j, i.e.,

$$\frac{dw}{d\zeta}(a_j) = 0 = \frac{dw}{d\zeta}(c_j).$$

These zero positions will not be known *a priori* but will enter our representation of the solution as accessory parameters.

4.1. Building block functions

A set of "building block" functions will be used to construct the required solutions. Their characterizing feature is that they all have constant argument on the boundary circles $\{C_j \mid j = 0, 1, \ldots, M\}$. These functions were introduced in [5] and their properties established there.

It is shown in [5] that functions of the form

$$R_1(\zeta; \zeta_1, \zeta_2) = \frac{\omega(\zeta, \zeta_1)}{\omega(\zeta, \zeta_2)}, \tag{9}$$

where ζ_1 and ζ_2 are any two points on the *same* circle C_k (for $k = 0, 1, \ldots, M$) has constant argument on each of the boundary circles $\{C_j \mid j = 0, 1, \ldots, M\}$. Also, functions of the form

$$R_2(\zeta; \zeta_1, \zeta_2) = \frac{\omega(\zeta, \zeta_1)\omega(\zeta, \overline{\zeta_1}^{-1})}{\omega(\zeta, \zeta_2)\omega(\zeta, \overline{\zeta_2}^{-1})}, \tag{10}$$

where ζ_1 and ζ_2 are any two ordinary points of the Schottky group (these points need not be points on the boundary circles) similarly have constant argument on each of the boundary circles $\{C_j \mid j = 0, 1, \ldots, M\}$.

Let γ_0 be some point on C_0 that is distinct from a_0 and c_0. Consider the function

$$R_1(\zeta; a_0, \gamma_0)R_1(\zeta; c_0, \gamma_0)R_2(\zeta; \gamma_0, \beta)R_2(\zeta; \alpha, \beta) \\ \times \prod_{k=1}^{M} R_1(\zeta; a_k, \gamma_k^{(1)})R_1(\zeta; c_k, \gamma_k^{(2)}). \tag{11}$$

First, since it is a product of the building block functions just introduced, the function in (11) has constant argument on the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. As for its singularities, it is a meromorphic function in D_ζ with a second order pole at $\zeta = \beta$ (and at $\overline{\beta}^{-1}$), simple poles at the points $\{\gamma_k^{(1)}, \gamma_k^{(2)} \mid k = 1, \ldots, M\}$, simple zeros at $\zeta = \alpha$ and $\overline{\alpha}^{-1}$ and simple zeros at the points $\{a_k, c_k \mid k = 0, 1, \ldots, M\}$. It has no other singularities in D_ζ. Let the function (11), considered now as a function of η, be called $U(\eta)$.

Now consider the function $\eta W'(\eta)$ which, we have already established, must have constant argument on the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. By the chain rule we have

$$\eta W'(\eta) = \eta \frac{dw/d\zeta}{d\eta/d\zeta}.$$

This function is analytic everywhere in D_η except for simple poles at the zeros of $d\eta/d\zeta$, i.e., at the points $\{\gamma_k^{(1)}, \gamma_k^{(2)} \mid k = 1, \ldots, M\}$. It also has second order poles at $\zeta = \beta$ and $\overline{\beta}^{-1}$. It has a simple zero at $\zeta = \alpha$ since $\eta(\zeta; \alpha)$ has a simple zero there and, as can be seen after making use of (7), it also has a simple zero at $\overline{\alpha}^{-1}$. By assumption, it also has $2(M+1)$ simples zeros at the points $\{a_k, c_k \mid k = 0, 1, \ldots, M\}$. In short, it has all the same zeros and poles in D_ζ as the function $U(\eta)$.

We are thus led to consider the ratio

$$V(\eta) \equiv \frac{\eta W'(\eta)}{U(\eta)}$$

in the domain D_η. Since we know that $U(\eta)$ and $\eta W'(\eta)$ have the same poles and zeros inside and on the boundaries of D_ζ, the function $V(\eta)$ can be deduced to be analytic everywhere in the domain D_η, as well as on its boundaries. This means that $V(\eta)$ is analytic everywhere in $|\eta| \leq 1$. Moreover, it is known that the arguments of both $U(\eta)$ and $\eta W'(\eta)$ are constant on L_0. Thus,

$$\overline{V(\eta)} = \epsilon V(\eta) \qquad \text{on } L_0,$$

for some constant ϵ implying that

$$\overline{V}(\eta^{-1}) = \epsilon V(\eta) \qquad \text{on } L_0.$$

This equation furnishes the analytic continuation of $V(\eta)$ into $|\eta| > 1$ and, in particular, shows that it is analytic there (and bounded at infinity). Since $V(\eta)$ is analytic everywhere in the complex η-plane, and bounded as $\eta \to \infty$, Liouville's theorem implies $V(\eta) = B$, where B is some complex constant.

On use of (9) and (10), and after some cancellations, we deduce that

$$\frac{dw(\zeta)}{d\zeta} = \frac{BS(\zeta;\alpha)}{\omega(\zeta,\beta)^2\omega(\zeta,\overline{\beta}^{-1})^2}\prod_{k=0}^{M}\omega(\zeta,a_k)\omega(\zeta,c_k),$$

where

$$S(\zeta;\alpha) \equiv \left(\frac{\omega(\zeta,\overline{\alpha}^{-1})\omega_\zeta(\zeta,\alpha) - \omega(\zeta,\alpha)\omega_\zeta(\zeta,\overline{\alpha}^{-1})}{\prod_{k=1}^{M}\omega(\zeta,\gamma_k^{(1)})\omega(\zeta,\gamma_k^{(2)})}\right).$$

Hence, the required solution can be written as the indefinite integral

$$w(\zeta) = A + B\int_1^{\zeta}\frac{S(\zeta';\alpha)}{\omega(\zeta',\beta)^2\omega(\zeta',\overline{\beta}^{-1})^2}\prod_{k=0}^{M}\omega(\zeta',a_k)\omega(\zeta',c_k)\,d\zeta', \qquad (12)$$

where A is some complex constant. Formula (12) is the main result of this paper.

It is demonstrated in the appendix that for any two distinct choices of α_1 and α_2, $S(\zeta;\alpha_1) = CS(\zeta;\alpha_2)$, where C is some constant (independent of ζ). This means that making different choices of α in the representation (12) simply corresponds to making a different choice of the constant B.

5. The doubly connected case

As verification we consider two problems in the doubly connected case. Let D_ζ to be the concentric annulus $\rho < |\zeta| < 1$ for some real ρ. Any doubly connected domain is conformally equivalent to some such annulus. The solutions to the following two problems can, it turns out, be found in analytical form using alternative arguments which allows us to check our analysis.

Problem 1

We specialize to the case where $\lambda_0 = \lambda_1 = 1$ with $c_0 = 0$. The problem is then the classical Schwarz problem. One form of the solution is

$$w(\zeta) = \frac{U}{\zeta - \beta} + \tilde{A}\log\zeta + I(\zeta), \qquad (13)$$

where $\tilde{A}$ is a constant and the single-valued function $I(\zeta)$ can be written in terms of the classical Villat formula [1]:

$$I(\zeta) = \frac{1}{2\pi i}\oint_{|\zeta'|=1}\frac{d\zeta'}{\zeta'}\,(1 - 2K(\zeta/\zeta',\rho))\left[-\mathrm{Re}\left[\frac{U}{\zeta-\beta} + \tilde{A}\log\zeta\right]\right]$$

$$-\frac{1}{2\pi i}\oint\limits_{|\zeta'|=\rho}\frac{d\zeta'}{\zeta'}\left(2-2K(\zeta/\zeta',\rho)\right)\left[c_1-\mathrm{Re}\left[\frac{U}{\zeta-\beta}+\tilde{A}\log\zeta\right]\right],$$

where

$$K(\zeta,\rho)\equiv\frac{\zeta P_\zeta(\zeta,\rho)}{P(\zeta,\rho)} \tag{14}$$

and

$$P(\zeta,\rho)\equiv(1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{15}$$

Alternatively, the same solution can be written in the form

$$w(\zeta)=\frac{U}{\beta}\left(K(\zeta\beta^{-1},\rho)+K(\zeta\overline{\beta},\rho)\right)+C\log\zeta+D,$$

where

$$C=\frac{1}{\log\rho}\left(c_1-\frac{U}{\beta}\left(K(\rho\beta^{-1},\rho)+K(\rho\overline{\beta},\rho)-K(\beta^{-1},\rho)-K(\overline{\beta},\rho)\right)\right)$$

and

$$D=-\frac{U}{\alpha}\left(K(\beta^{-1},\rho)+K(\overline{\beta},\rho)\right).$$

The new solution method given earlier provides a third representation of the same solution:

$$w(\zeta) = A+B\int\limits_1^{\zeta}\frac{S(\zeta';\alpha)}{\omega(\zeta',\beta)^2\omega(\zeta',\overline{\beta}^{-1})^2}\omega(\zeta',a_0)\omega(\zeta',c_0)\omega(\zeta',a_1)\omega(\zeta',c_1)\,d\zeta', \tag{16}$$

where, in this doubly connected case, it can be shown (see [5] for details) that

$$S(\zeta;\alpha)\propto\frac{1}{\zeta^2}.$$

To check (16) we use the solution (13) to numerically compute (using Newton's method) the two points on C_0 at which $dw/d\zeta=0$. These are substituted into (16) as the values of a_0 and c_0. Similarly, we find the two points on C_1 at which $dw/d\zeta=0$ and take these as the values of a_1 and c_1. Next, we set $A=w(1)$, where the right hand side is computed using the known solution (13). We also fix B by ensuring that

$$w(\rho)=A+B\int\limits_1^{\rho}\frac{S(\zeta')}{\omega(\zeta',\beta)^2\omega(\zeta',\overline{\beta}^{-1})^2}\omega(\zeta',a_0)\omega(\zeta',c_0)\omega(\zeta',a_1)\omega(\zeta',c_1)\,d\zeta',$$

where the left hand side of this equation is evaluated using the known solution (13). With all the parameters in (16) now determined, we check the value of the integral (16) against the values given by (13) for different (arbitrary) choices of ζ in the annulus (the integral (16) is computed using the trapezoidal rule). The values are found to be in agreement (to within the accuracy of the numerical method) thereby confirming that (16) is indeed a representation of the required solution.

PROBLEM 2
We now specialize to the case where $\lambda_0 = 1$, $\lambda_1 = e^{i\pi/2}$ with no restrictions on d_0 and d_1. This is no longer a classical Schwarz problem so the Villat formula cannot be used here. An analytical formula for the solution can, however, be found:

$$\begin{aligned} w(\zeta) = \frac{U}{\beta}\left[K(\zeta\beta^{-1},\rho^2) - K(\zeta\beta,\rho^2) - K(\zeta\beta^{-1}\rho^{-2},\rho^2) + K(\zeta\beta\rho^{-2},\rho^2)\right] \\ + d_0 + id_1, \end{aligned} \tag{17}$$

where the special function defined in (14) again appears. A derivation of (17) is given in appendix B. In a manner akin to that used in Problem 1, the expression (17) was used to find the locations of the zeros of $dw/d\zeta$ on both C_0 and C_1 (there are two on each circle). These are then used as the values of a_0, c_0, a_1 and c_1 in an expression of the form (16). The values of A and B are determined in the same way as in Problem 1 and the values of the integral (16) for arbitrary values of ζ checked against the values given by (17). They are found to be in agreement.

6. Discussion

This paper describes a constructive method for finding solutions to Riemann–Hilbert problems of the special form (1) on multiply connected domains. The solution having two zeros of the derivative on each of the boundary circles is given in (12) as a non-singular indefinite integral containing a finite set of accessory parameters. In general, these parameters must be determined from a set of equations obtained by substituting the form (12) into the boundary conditions (1). In other words, given the $2M+2$ real parameters associated with the set $\{\lambda_k, d_k \mid k = 0,1,\ldots,M\}$ it is possible to determine the $2M+2$ real parameters associated with the set of zeros $\{a_k, c_k \mid k = 0,1,\ldots,M\}$. How to determine these accessory parameters numerically in an efficient manner remains a subject for future research.

In principle, it is possible to extend the constructive method herein to find representations to solutions of the discontinuous analogues of the special RH problems considered here where the constant λ_k is allowed to assume different piecewise constant values on different segments of circle C_k. In such cases, one must generally introduce branch point singularities in the derivative $w_\zeta(\zeta)$ but this just requires the incorporation of appropriate non-integer powers of the building block functions when performing the construction described herein. It is very similar to what is done in constructing multiply connected Schwarz–Christoffel formulae [5], [6].

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A. The function $S(\zeta;\alpha)$

In this appendix we establish the fact that $S(\zeta;\alpha_1) = CS(\zeta;\alpha_2)$, where C is some constant (independent of ζ). To this end, consider the ratio

$$R(\zeta) \equiv \frac{S(\zeta;\alpha_1)}{S(\zeta;\alpha_2)}, \tag{18}$$

where α_1 and α_2 are two distinct values in D_ζ. First, notice that $S(\zeta;\alpha_1)$ can be rewritten in the form

$$S(\zeta;\alpha_1) = \left(\frac{\omega_\zeta(\zeta,\alpha_1)}{\omega(\zeta,\alpha_1)} - \frac{\omega_\zeta(\zeta,\overline{\alpha_1}^{-1})}{\omega(\zeta,\overline{\alpha_1}^{-1})}\right) \frac{\omega(\zeta,\alpha_1)\omega(\zeta,\overline{\alpha_1}^{-1})}{\prod_{k=1}^{M}\omega(\zeta,\gamma_k^{(1)})\omega(\zeta,\gamma_k^{(2)})}, \tag{19}$$

where $\{\gamma_k^{(1)},\gamma_k^{(2)} \mid k=1,\ldots,M\}$ are the zeros of the slit map $\eta(\zeta;\alpha_1)$. Similarly

$$S(\zeta;\alpha_2) = \left(\frac{\omega_\zeta(\zeta,\alpha_2)}{\omega(\zeta,\alpha_2)} - \frac{\omega_\zeta(\zeta,\overline{\alpha_2}^{-1})}{\omega(\zeta,\overline{\alpha_2}^{-1})}\right) \frac{\omega(\zeta,\alpha_2)\omega(\zeta,\overline{\alpha_2}^{-1})}{\prod_{k=1}^{M}\omega(\zeta,\tilde{\gamma}_k^{(1)})\omega(\zeta,\tilde{\gamma}_k^{(2)})}, \tag{20}$$

where $\{\tilde{\gamma}_k^{(1)},\tilde{\gamma}_k^{(2)} \mid k=1,\ldots,M\}$ are the zeros of the slit map $\eta(\zeta;\alpha_2)$. It is also easy to check that, for $j=1,2$,

$$\frac{\omega_\zeta(\zeta,\alpha_j)}{\omega(\zeta,\alpha_j)} - \frac{\omega_\zeta(\zeta,\overline{\alpha_j}^{-1})}{\omega(\zeta,\overline{\alpha_j}^{-1})} = \frac{\eta_\zeta(\zeta;\alpha_j)}{\eta(\zeta;\alpha_j)}.$$

Next, observe that on C_j (and for *any* α),

$$\eta(\zeta;\alpha)\overline{\eta(\zeta;\alpha)} = r_j^2,$$

where r_j is some real constant. A differentiation with respect to ζ yields

$$\frac{\eta_\zeta(\zeta;\alpha)}{\eta(\zeta;\alpha)} = -\left(\frac{d\overline{\zeta}}{d\zeta}\right)\overline{\left(\frac{\eta_\zeta(\zeta;\alpha)}{\eta(\zeta;\alpha)}\right)}.$$

It follows that the *ratio* of any two such functions, that is,

$$T(\zeta) \equiv \frac{\eta_\zeta(\zeta;\alpha_1)/\eta(\zeta;\alpha_1)}{\eta_\zeta(\zeta;\alpha_2)/\eta(\zeta;\alpha_2)}$$

will be real (and, in particular, have constant argument) on all the circles $\{C_j \mid j = 0,1,\ldots,M\}$.

Substitution of (19) and (20) into (18) then produces

$$R(\zeta) = T(\zeta)R_2(\zeta;\alpha_1,\alpha_2)\prod_{j=1}^{M} R_1(\zeta;\tilde{\gamma}_j^{(1)},\gamma_j^{(1)})R_1(\zeta;\tilde{\gamma}_j^{(2)},\gamma_j^{(2)}).$$

The important observation is that this is a product of functions that all have constant argument on the circles $\{C_j \mid j = 0, 1, \ldots, M\}$. These conditions can be written as

$$\overline{R(\zeta)} = \kappa_j R(\zeta) \qquad \text{on } C_j,\ j = 0, 1, \ldots, M, \tag{21}$$

for some set of complex constants $\{\kappa_j \mid j = 0, 1, \ldots, M\}$. $R(\zeta)$ can be shown to be a constant. One way to do this is to use arguments similar to those used in §4.1. to show that $V(\eta)$ is constant, but it is instructive to present an alternative argument based on RH methods. The function $R(\zeta)$ is known to be analytic and single-valued everywhere in the fundamental region of the group Θ. Consider the real part of equation (21); it can be written in the standard form of a RH problem:

$$\mathrm{Re}[\overline{\mu_j}\, R(\zeta)] = 0 \qquad \text{on } C_j,\ j = 0, 1, \ldots, M, \tag{22}$$

for some set of complex constants $\{\mu_j \mid j = 0, 1, \ldots, M\}$. The (homogeneous) Riemann–Hilbert problem (22) has been well studied and it is known (see, for example, p. 257 of Vekua [18]) that it admits no solution for $R(\zeta)$ unless all the constants $\{\mu_j \mid j = 0, 1, \ldots, M\}$ are identical. In this case, the unique solution is $R(\zeta) = C$, where C is a constant. Thus, we have established that $S(\zeta; \alpha_1) = CS(\zeta; \alpha_2)$ for some constant C that is independent of ζ.

B. Derivation of (17)

To find solution (17), consider the following boundary value problem for $w(\zeta)$:

$$\begin{aligned} \mathrm{Re}[w(\zeta)] &= 0 \qquad \text{on } |\zeta| = 1, \\ \mathrm{Im}[w(\zeta)] &= 0 \qquad \text{on } |\zeta| = \rho. \end{aligned}$$

These imply that

$$\begin{aligned} w(\zeta) + \overline{w}(\zeta^{-1}) &= 0 \qquad \text{on } |\zeta| = 1, \\ w(\zeta) - \overline{w}(\rho^2\zeta^{-1}) &= 0 \qquad \text{on } |\zeta| = \rho. \end{aligned} \tag{23}$$

The relations (23) can be analytically continued off the respective circles and imply that $w(\zeta)$ satisfies the functional relation

$$w(\rho^4\zeta) = w(\zeta). \tag{24}$$

Now $P(\zeta, \rho)$ can be shown, directly from its definition (15), to satisfy the functional relations

$$P(\zeta^{-1}, \rho) = -\zeta^{-1}P(\zeta, \rho), \qquad P(\rho^2\zeta, \rho) = -\zeta^{-1}P(\zeta, \rho),$$

from which it also follows that

$$K(\zeta^{-1}, \rho) = 1 - K(\zeta, \rho), \qquad K(\rho^2\zeta, \rho) = K(\zeta, \rho) - 1.$$

Furthermore, near $\zeta = 1$, $K(\zeta, \rho)$ has a simple pole with unit residue, i.e.,

$$K(\zeta, \rho) = \frac{1}{\zeta - 1} + \text{analytic}.$$

We can therefore use $K(\zeta, \rho^2)$ to construct a function $w(\zeta)$ satisfying (24) and having a simple pole at $\zeta = \beta$. The relations (23) imply that $w(\zeta)$ also has simple poles at $\zeta = \beta^{-1}, \rho^2\beta, \rho^2\beta^{-1}$ (and at all points equivalent to these under $\zeta \mapsto \rho^4\zeta$). The required form of solution can now easily be deduced to be that given in (17).

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