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Bergman kernel functions for planar domains and conformal equivalence of domains

Abstract. The Bergman kernels of multiply connected domains are related with proper holomorphic maps onto the unit disc. We study multiply connected planar domains and represent conformal equivalence of the Bell representative domains with annuli or any doubly connected domains by explicit formulae. We study the expression for the Bergman kernels of circular multiply connected planar domains.

1. Introduction

In this paper, we study the Bergman kernels of multiply connected domains and their Bell representations and circular multiply connected domains.

Let Ω be a bounded domain in $\mathbb{C}$. The Bergman projection P is the orthogonal projection of $L^2(\Omega)$ onto its subspace $H^2(\Omega)$ of holomorphic functions. The Bergman kernel $K_\Omega(\cdot,\cdot)$ is the kernel for P in the sense that for $f \in L^2(\Omega)$

$$Pf(z) = \int_\Omega K_\Omega(z,\zeta) f(\zeta)\, dA, \qquad z \in \Omega.$$

Let U be the unit disc in $\mathbb{C}$ with the area measure $dA = dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}$. Then the Bergman kernel for U is given by

$$K_U(z,\zeta) = \frac{1}{\pi} \frac{1}{\left(1 - z\overline{\zeta}\right)^2}, \qquad z,\zeta \in U. \tag{1.1}$$

Let $\Omega \neq \mathbb{C}$ be a simply connected planar domain and $f\colon \Omega \longrightarrow U$ be the Riemann map with $f(a) = 0$ and $f'(a) > 0$. The transformation formula for the Bergman kernel is

$$K_\Omega(z,\zeta) = f'(z) K_U(f(z), f(\zeta)) \overline{f'(\zeta)}. \tag{1.2}$$

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It implies that

$$K_\Omega(z,\zeta) = \frac{1}{\pi}\frac{f'(z)\overline{f'(\zeta)}}{\big(1-f(z)\overline{f(\zeta)}\,\big)^2} \qquad z,\zeta\in\Omega. \tag{1.3}$$

Hence, $K_\Omega(a,a) = \frac{1}{\pi}f'(a)^2$. Therefore, the derivative of $f(z)$ is determined through the Bergman kernel by the formula

$$f'(z) = K_\Omega(z,a)\sqrt{\frac{\pi}{K_\Omega(a,a)}}. \tag{1.4}$$

The transformation formula (1.2) for the Bergman kernel holds under any biholomorphic map between two domains. Let us determine the Bergman kernel for $\{z\in\mathbb{C}:\ |z|<2\}$.

Example 1.1
Let $U'=\{z\in\mathbb{C}:\ |z|<2\}$. Let $f(z)=\frac{i}{2}z$ be a biholomorphic map from U' to the unit disc. Then the transformation formula for the Bergman kernels implies that

$$\begin{aligned} K_{U'}(z,\zeta) &= \frac{1}{\pi}\frac{f'(z)\overline{f'(\zeta)}}{\big(1-f(z)\overline{f(\zeta)}\,\big)^2} \\ &= \frac{1}{\pi}\frac{\frac{i}{2}\frac{-i}{2}}{\left(1-\frac{iz}{2}\frac{-i\overline{\zeta}}{2}\right)^2} \\ &= \frac{1}{\pi}\frac{4}{(4-z\overline{\zeta}\,)^2}. \end{aligned} \tag{1.5}$$

Let $\Omega_\rho=\{z\in\mathbb{C}:\ \rho<|z|<1\}$ be a circular annulus. The orthonormal complete set for $H^2(\Omega)$ is given by

$$\begin{aligned} \varphi_{2n-1}(z) &= z^{n-1}\left(\frac{n}{\pi(1-\rho^{2n})}\right)^{\frac{1}{2}}, \qquad n=1,2,\dots, \\ \varphi_2(z) &= \frac{1}{z}\left(\frac{1}{-2\pi\ln\rho}\right)^{\frac{1}{2}}, \\ \varphi_{2n}(z) &= \frac{1}{z^n}\left(\frac{1-n}{\pi(1-r^{2(n-1)})}\right)^{\frac{1}{2}}, \qquad n=2,\dots. \end{aligned}$$

Hence, we have

$$\begin{aligned} K_{\Omega_\rho}(z,\zeta) &= \sum_{n=1}^{\infty}\varphi_n(z)\varphi_n(\overline{\zeta}\,) \\ &= \frac{1}{\pi z\overline{\zeta}}\left(\mathcal{P}(\ln z\overline{\zeta}) + \frac{\eta_1}{\pi i} - \frac{1}{2\ln\rho}\right), \end{aligned} \tag{1.6}$$

where $\mathcal{P}$ is the Weierstrass function with the periods $\omega_1 = \pi i$, $\omega_2 = \ln \rho$, and $2\eta_1$ is the increment of the Weierstrass ζ-function related to the period ω_1 (see [5]).

On the other hand, the Bergman kernels for domains in $\mathbb{C}^n$ are known in special cases such as the unit ball, the polydisc, the Thullen domain [8], convex domains [6], the Lie ball [10], the minimal ball [17] and so on. For example, the Bergman kernel for the unit ball B in $\mathbb{C}^n$ is

$$K_B(z,\zeta) = \frac{n!}{\pi^n} \frac{1}{(1 - z\overline{\zeta})^{n+1}}.$$

Suppose that Ω is a bounded domain with C^∞ smooth boundary. The Green function $G_\Omega(z,w)$ and the Bergman kernel $K_\Omega(z,w)$ associated to Ω are related via the following formula, see [1]:

$$K_\Omega(z,\zeta) = -\frac{2}{\pi} \frac{\partial^2 G_\Omega(z,\zeta)}{\partial z \partial \overline{\zeta}}. \tag{1.7}$$

2. Bell representations

A holomorphic function $A(z,w)$ on an open set in $\mathbb{C} \times \mathbb{C}$ is called algebraic if there exists a polynomial $P(A(z,w),z,w) = 0$.

The kernel $K_\Omega(z,w)$ is algebraic if and only if $K_\Omega(z,w) = R(z,\bar{w})$ where R is a holomorphic algebraic function of $\{(z,\bar{w}) :\ (z,w) \in \Omega \times \Omega\}$. It is the same as for fixed $b \in \Omega$, $K_\Omega(z,b)$ is an algebraic function of z.

In this section we study the Bell representative domains where the Bergman kernels are algebraic. One can see from (1.3) that it is possible to represent the Bergman kernel for simply connected planar domains via the Riemann map. It is rational if and only if the corresponding Riemann map is rational. For a bounded n-connected domain, the Bergman kernel cannot be rational if $n > 1$ (see [2]). Hence, for n-connected domains, it is interesting to study the question, when the Bergman kernel is algebraic even though we cannnot express it explicitly.

Let Ω be an n-connected planar domain and let $f_a\colon \Omega \longrightarrow U$ be the Ahlfors map with $f_a(a) = 0$, $f_a'(a) > 0$. Then

$$\sum_{k=1}^{n} K_\Omega(z, F_k(\zeta))\overline{F_k'(\zeta)} = f_a'(z) K_U(f_a(z),\zeta)$$

for $z \in \Omega$, $\zeta \in U - f_a(V)$ where $V = \{z \in \Omega :\ f_a'(z) = 0\}$ (see [1]).

The following theorem in [3] tells us when the Bergman kernel is algebraic.

Theorem 2.1
Let Ω be an n-connected non-degenerate planar domain. The following statements are equivalent:

1) *The Bergman kernel* $K_\Omega(\cdot,\cdot)$ *is algebraic.*

2) *The Szegő kernel* $S_\Omega(\cdot,\cdot)$ *is algebraic.*

3) *There exists a proper holomorphic map* $f\colon\Omega \longrightarrow U$ *which is algebraic.*

4) *Every proper holomorphic map from* Ω *onto* U *is algebraic.*

Let us consider an example. Let

$$A_r = \left\{z \in \mathbb{C} : \ |z + \tfrac{1}{z}| < r\right\}$$

for $r > 2$. Then A_r is a 2-connected domain with real analytic boundary if $r > 2$. The algebraic function

$$f_r(z) = \frac{1}{r}\left(z + \frac{1}{z}\right)$$

defines a proper holomorphic map from A_r to U which is a 2-sheeted branched covering map and it is algebraic. By the above theorem, the Bergman kernel for A_r is algebraic.

Additionally, the mapping f_r which is a 2-to-1 map from A_r to U extends to a 1-to-1 biholomorphic from every connected component of A_r^c in $\overline{\mathbb{C}}$ onto U^c in $\overline{\mathbb{C}}$. The modulus of A_r is a continuous increasing function of r that approaches to 0 as $r \to 2^+$ and to ∞ as $r \to \infty$. Hence, every 2-connected domain is biholomorphic to one of A_r (see [3]).

This result leads to the conjecture (see [3]) that any n-connected non-degenerate planar domain Ω is biholomorphic to a domain

$$\left\{z \in \mathbb{C} : \ \left|z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}\right| < r\right\}$$

with $a_k, b_k \in \mathbb{C}$, $r > 0$. Such a domain is called Bell representation and this conjecture is solved in [13]. Let

$$(a, b) = (a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}) \in \mathbb{C}^{2n-2}$$

and the corresponding domain

$$W_{a,b} = \left\{z \in \mathbb{C} : \ \left|z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}\right| < 1\right\}, \qquad a_k, b_k \in \mathbb{C}.$$

THEOREM 2.2 ([13])
Let Ω *be a non-degenerate* n*-connected planar domain with* $n > 1$. *Then* Ω *is biholomorphic to a domain* $W_{a,b}$ *.*

The Bergman kernel associated with $W_{a,b}$ is algebraic since $f\colon W_{a,b} \longrightarrow U$ defined by

$$f_{a,b}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is an algebraic proper holomorphic map. To describe domains which possess algebraic proper holomorphic maps onto the unit disc is an important task in the problem of the equivalence between domains. Let

$$B_n = \{(a,b) \in \mathbb{C}^{2n-2} : \ W_{a,b} \text{ is an } n\text{-connected planar domain}\}.$$

B_n is called the coefficient body for n-connected canonical domains. In [14], B_n is explicitly figured out.

Theorem 2.3 ([14])
For $a \in \mathbb{C}$, let $a' \in \mathbb{C}$ be such that $(a')^2 = a$. Then,

$$B_2 = \{(a,b) \in \mathbb{C}^2 : \ a \neq 0, \ |b + 2a'| < 1, \ |b - 2a'| < 1\}.$$

Fix $(a,b) \in B_n$ and let $W_{a,b}$ be the corresponding n-connected canonical domain. Let $E(W_{a,b})$ be the leaf in B_n for $W_{a,b}$ consisting of all the points which correspond to n-connected canonical domains biholomorphically equivalent to $W_{a,b}$.

Theorem 2.4 ([14])
For $r > 2$,

$$E(A_r) = \left\{ (a,b) \in B_2 : \ \left| \frac{4a'}{1 - (\overline{b + 2a'})(b - 2a')} \right| = \frac{4r}{4 + r^2} \right\}.$$

In particular, $E(A_r) \cap \{(a,0) \in \mathbb{C}^2\} = \{(a,0) \in \mathbb{C}^2 : \ |a| = r^{-2}\}$.

Now, we give two examples of points in $E(A_r)$ explaining the above theorems.

Example 2.5
For any real θ, let $a = r^{-2} e^{i\theta}$ and $a' = r^{-1} e^{i\frac{\theta}{2}}$ be so that $(a')^2 = a$ and $(a,0) \in E(A_r)$. Let f be defined by $f(z) = a'z$. Take $z \in A_r$ so that $\left|z + \frac{1}{z}\right| < r$. Then $f(z) = w$ satisfies

$$\begin{aligned} \left| w + \frac{a}{w} \right| &= \left| a'z + \frac{a}{a'z} \right| = |a'| \left| z + \frac{1}{z} \right| \\ &< |a'| r = 1. \end{aligned}$$

So, f is a biholomorphic map from A_r onto $W_{a,0}$.

Example 2.6
Let $r = 3$, $a = \frac{9}{169}$, and $a' = \frac{3}{13}$ so that $(a')^2 = a$. Then $(a, 2a') \in B_2$ by Theorem 2.3. Also, since $4a' = \frac{12}{13}$, it belongs to $E(A_3)$ by Theorem 2.4.

For $n > 2$ we have the following theorem suggesting the basic idea for describing B_n.

THEOREM 2.7 ([15])
B_n is the set of (a,b) such that equation $f'_{a,b}(z) = 0$ has $2n-2$ solutions $c_1, c_2, \ldots, c_{2n-2}$ counted with multiplicities such that $|f_{a,b}(c_j)| < 1$ for every j. In particular, B_n is an open subset of $\mathbb{C}^{2n-2}$.

Now, we give an example for a point in B_3.

EXAMPLE 2.8
Let $a_1 = a_2 = \frac{-2+\sqrt{20}}{16^2}$ and $b_1 = -b_2 = \frac{1}{16}$. Then $(a_1, a_2, b_1, b_2) \in B_3$. In fact $\left\{ \pm \frac{\sqrt{3+\sqrt{20}}}{16}, \pm \frac{\sqrt{5-\sqrt{20}}}{16} i \right\}$ is the set of critical points of $f_{a,b}$ and $|f_{a,b}| < 1$ at each critical point.

3. Conformal equivalence between domains

In the previous section, we get the biholomorphic equivalence of any n-connected domain and a Bell representation while we studied the algebraicity property of the Bergman kernel. For 2-connected domains, annuli Ω_ρ and Bell representations A_r are two canonical domains. So, it is interesting to demonstrate the equivalence of these domains. In order to check the conformal equivalence of them, we project them onto the unit disc.

Note that Ω_ρ is biholomorphic to A_r for some $r > 2$ if and only if there is a biholomorphic map $T: U \longrightarrow U$ with $T(\{\pm i c_\rho\}) = \{\pm \frac{2}{r}\}$. The Ahlfors map $f_\rho: \Omega_\rho \longrightarrow U$ with $f_\rho(\sqrt{\rho}) = 0$ and $f'_\rho(\sqrt{\rho}) > 0$ maps $\{|z| = \sqrt{\rho}\}$ onto a line segment with endpoints $\pm i c_\rho$. Hence we get the following theorem.

THEOREM 3.1 ([12])
Let $\Omega_\rho = \{z \in \mathbb{C} : \ \rho < |z| < 1\}$ with $0 < \rho < 1$. Ω_ρ is conformally equivalent to A_r, $(r > 2)$ if and only if $r = \frac{2}{c_\rho}$, where

$$c_\rho = \frac{2\sqrt{\rho}\displaystyle\sum_{k=0}^{\infty}(-1)^{\frac{(k+1)}{2}}\frac{\rho^k}{1+\rho^{2k+1}}}{1+2\displaystyle\sum_{k=0}^{\infty}(-1)^{\frac{k+2}{2}}\frac{\rho^{2k+1}}{1+\rho^{2k+1}}}.$$

Also, Crowdy [7] got the relation between r and ρ and constructed a conformal mapping from Ω_ρ onto A_r using Schottky–Klein prime functions associated with Ω_ρ.

Deger [9] showed that when $J(z)=\frac{1}{2}(z+\frac{1}{z})$, $\frac{2}{r}J(z)$ is in fact the Ahlfors map for A_r with $\frac{2}{r}J(i)=0$ and expressed the Bergman kernel for A_r as

$$K_{A_r}(z,w)=C_1\frac{2k^2S(z,\bar{w})+kC(z,\bar{w})D(z,\bar{w})+C_2}{z\bar{w}\sqrt{1-k^2J(z)^2}\sqrt{1-k^2J(\bar{w})^2}}$$

where $k=(\frac{2}{r})^2$ and C_1, C_2 are constants that depend only on r and $S(z,w)$, $C(z,w)$, $D(z,w)$ are given.

In fact, $\frac{2}{r}J(z)=f_r(z)$ and so f_r is the Ahlfors map for A_r with $f_r(i)=0$, $f_r'(i)>0$. The following expression of the Bergman kernel for any 2-connected domain is given in [4].

Theorem 3.2
The Bergman kernel $K_\Omega(z,w)$ for any 2-connected planar domain Ω is given by

$$\Phi'(z)K_{A_r}(\Phi(z),\Phi(w))\overline{\Phi'(w)}$$

where the biholomorphic map Φ from Ω onto its representative domain A_r satisfies that $\frac{2}{r}J(\Phi(z))=\lambda f_a(z)$ where $f_a\colon\Omega\longrightarrow U$ is an Ahlfors map for a point a on the median of Ω, $|\lambda|=1$.

4. Circular multiply connected planar domain

Let the discs

$$D_k=\{z\in\mathbb{C}:\ |z-a_k|<r_k\},\qquad k=1,2,\dots,n$$

be mutually disjoint and let

$$D=\overline{\mathbb{C}}-\bigcup_{k=1}^{n}(D_k\cup\partial D_k)$$

be the complement of these discs to the extended complex plane. The domain D is called a circular multiply connected domain. Let $f\colon D\longrightarrow\Omega$ be a biholomorphic mapping of D onto a bounded domain Ω with C^∞ smooth boundary. Then

$$K_D(z,\zeta)=f'(z)K_\Omega(f(z),f(\zeta))\overline{f'(\zeta)}\qquad z,\zeta\in D.$$

In addition, the Green functions G_D and G_Ω associated with D and Ω respectively, satisfy the identity

$$G_D(z,\zeta)=G_\Omega(f(z),f(\zeta)),\qquad z,\zeta\in D. \tag{4.1}$$

Hence, the Bergman kernel K_D and the Green function $G_D(z,\zeta)$ associated to D are related via

$$K_D(z,\zeta) = -\frac{2}{\pi}\frac{\partial^2 G_D(z,\zeta)}{\partial z \partial \overline{\zeta}}. \tag{4.2}$$

Let $z^*_{(k)}$ denote the inversions with respect to the circles

$$\partial D_k = \{z : |z-a_k| = r_k\}, \qquad k = 1,2,\dots,n,$$

given by

$$z^*_{(k)} := \frac{r_k^2}{\overline{z - a_k}} + a_k\,. \tag{4.3}$$

We denote their compositions by:

$$z^*_{(k_s k_{s-1}\cdots k_1)} := \left(z^*_{(k_{s-1}\cdots k_1)}\right)^*_{(k_s)} \tag{4.4}$$

where two adjacent numbers k_j , k_{j+1} $(j = 1,2,\dots,s-1)$ are not equal. Here s represents the number of inversions and is called the level of the mapping.

These are Möbius transformations γ_j, $(j = 0,1,\dots)$ in z or $\overline{z}$ if s is even or odd, respectively. To be precise, they are defined by

$$\begin{aligned}
\gamma_0(z) &:= z,\\
\gamma_1(\overline{z}) &:= z^*_{(1)}, \qquad \gamma_2(\overline{z}) := z^*_{(2)}, \dots, \qquad \gamma_n(\overline{z}) := z^*_{(n)},\\
\gamma_{n+1}(z) &:= z^*_{(12)}, \quad \gamma_{n+2}(z) := z^*_{(13)}, \dots, \quad \gamma_{n^2}(z) := z^*_{(n,n-1)},\\
\gamma_{n^2+1}(\overline{z}) &:= z^*_{(121)}, \qquad \text{and so on.}
\end{aligned}$$

The level s of γ_j is not decreasing. The above functions generate a Schottky group $\mathcal{S}$ (see [16]). Let $\mathcal{S}_m = \{z^*_{(k_s k_{s-1}\dots k_1)} : \ k_s \neq m\} \subset \mathcal{S} - \{\gamma_0\}$.

Mityushev and Rogosin [16] constructed the explicit expression for the complex Green function $M_D(z,\zeta)$ associated to D using the above γ_j . The expression for the real Green function $G_D(z,\zeta)$ and the calculation of $\frac{\partial^2 G_D}{\partial z \partial \overline{\zeta}}$ leads to the following expression for the Bergman kernel $K_D(z,\zeta)$.

Theorem 4.1 ([11])
Let

$$\Psi_m^{(j)}(z) := \begin{cases} \dfrac{\gamma_j'(z)}{\gamma_j(z) - a_m} & \text{if level of } \gamma_j \text{ is even,}\\[2ex] -\dfrac{\overline{\gamma_j}'(z)}{\overline{\gamma_j(\overline{z}) - a_m}} & \text{if level of } \gamma_j \text{ is odd.} \end{cases} \tag{4.5}$$

The Bergman kernel $K_D(\cdot,\cdot)$ associated to a circular multiply connected planar domain D is given by

$$K_D(z,\zeta) = -\frac{1}{\pi}\sum_{k=1}^{n}\sum_{m=1}^{n} A_m \overline{\sum_{\gamma_j\in\mathcal{S}_m}^{\infty} \Psi_m^{(j)}(\zeta)} \sum_{\gamma_j\in\mathcal{S}_k} \Psi_k^{(j)}(z) \\ -\frac{1}{\pi}\sum_{\gamma_j\in\mathcal{F}} \frac{\overline{\gamma}_j'(z)}{(\overline{\zeta}-\overline{\gamma}_j(z))^2}. \tag{4.6}$$

where A_m are some real constants and $\mathcal{F}$ is the set of γ_j's of the odd level.

Example 4.2
We consider the simply connected domain

$$D = \{z \in \overline{\mathbb{C}} : \ |z| > 2\}.$$

Then we have two-element group of inversions

$$\gamma_0(z) = z, \qquad \gamma_1(\overline{z}) = \frac{2^2}{\overline{z}}\,.$$

The constant A_1 is equal to zero and

$$K_D(z,\zeta) = -\frac{1}{\pi}\frac{\overline{\gamma}_1'(z)}{(\overline{\zeta}-\overline{\gamma}_1(z))^2} = \frac{1}{\pi}\frac{2^2}{(2^2 - z\overline{\zeta}\,)^2}. \tag{4.7}$$

Similarly, for a general circular simply connected domain

$$D = \{z \in \overline{\mathbb{C}} : \ |z - a_1| > r_1\},$$

the Bergman kernel is given by

$$K_D(z,\zeta) = \frac{1}{\pi}\frac{r_1^2}{(r_1^2 - (z-a_1)\overline{\zeta}\,)^2}$$

and hence it is rational.

We find that the Bergman kernel in (4.6) for D matches with the result in (1.5). Therefore, we conclude that for $n = 1$, D is biholomorphic to U with rational biholomorphic map $f(z) = \frac{r_1}{z-a_1}$ and hence $K_D(z,\zeta)$ is rational.

Remarks
If $n > 1$, $K_D(z,\zeta)$ is not rational. But, $K_D(z,\zeta)$ is algebraic if there is an algebraic proper holomorphic map from D onto U.

Open questions

1. Find a precise description of B_3 in order to make corresponding Bell representations which are canonical 3-connected domains.
2. Find a relation between the expression (1.6) of the Bergman kernel associated with an annulus and the expression (4.6) of the Bergman kernel associated with a circular doubly connected planar domain.
3. Find relations between circular multiply connected planar domains and Bell representations.

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