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On functional bases of the first-order differential invariants for non-conjugate subgroups of the Poincaré group $P(1,4)$

Abstract. It is established which functional bases of the first-order differential invariants of the splitting and non-splitting subgroups of the Poincaré group $P(1,4)$ are invariant under the subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. The obtained sets of functional bases are classified according to dimensions.

1. Introduction

It is well known (see, for example, [10, 11, 12, 13]), that functional bases of differential invariants of Lie groups of the point transformations play an important role in group analysis of differential equations, theorethical and mathematical physics, geometry, etc.

The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. Some applications of this group in the theoretical and mathematical physics can be found in [7, 8, 9].

Continuous subgroups of the group $P(1,4)$ have been described in [3, 4, 6]. One of important consequences of the study of the non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ (group symmetry of relativistic physics) and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (group symmetry of non-relativistic physics) (see also [7]).

Recently the functional bases of the first-order differential invariants for all continuous subgroups of the group $P(1,4)$ have been constructed. Some of them can be found in [2, 1].

The present paper is devoted to the classification of the functional bases of the first-order differential invariants of continuous subgroups of the group $P(1,4)$. It is established which functional bases of the first-order differential

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invariants of the splitting and non-splitting subgroups of the group $P(1,4)$ are invariant under the subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. The obtained sets of functional bases are classified according to dimensions.

In order to present some of obtained results, we consider the Lie algebra of the group $P(1,4)$.

2. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\begin{aligned} \left[P'_\mu, P'_\nu\right] &= 0, \\ \left[M'_{\mu\nu}, P'_\sigma\right] &= g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu, \\ \left[M'_{\mu\nu}, M'_{\rho\sigma}\right] &= g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho}, \end{aligned}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

All non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ are divided into splitting and non-splitting ones.

Splitting subalgebras $P_{i,a}$ of the Lie algebra of the group $P(1,4)$ can be written in the following form:

$$P_{i,a} = F_i \mathbin{\overset{\circ}{+}} N_{ia},$$

where F_i are subalgebras of the Lie algebra of the group $O(1,4)$, N_{ia} are subalgebras of the Lie algebra of the translations group $T(5) \subset P(1,4)$ and $\overset{\circ}{+}$ is the semi-direct sum.

Non-splitting subalgebras $\widetilde{P}_{j,k}$ are subalgebras, for which a basis can be chosen in the form:

$$\widetilde{B}_k = B_k + \sum_i c_{ki}X_i, \qquad \sum_j d_{rj}X_j,$$

where c_{ki} and d_{rj} are fixed real constants (not equal zero simultaneously). B_k are bases of subalgebras of the Lie algebra of the group $O(1,4)$, X_i are bases of subalgebras of the Lie algebra of the group $T(5)$.

We consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3}, \quad P'_4 = -\frac{\partial}{\partial x_4},$$

$$M'_{\mu\nu} = -(x_\mu P'_\nu - x_\nu P'_\mu).$$

Further, we will use the following basis elements:

$$
\begin{aligned}
&G = M'_{40}\,,\\
&L_1 = M'_{32}\,, \qquad L_2 = -M'_{31}\,, \qquad L_3 = M'_{21}\,,\\
&P_a = M'_{4a} - M'_{a0}\,, \quad (a = 1, 2, 3),\\
&C_a = M'_{4a} + M'_{a0}\,, \quad (a = 1, 2, 3),\\
&X_0 = \frac{1}{2}(P'_0 - P'_4), \qquad X_k = P'_k \quad (k = 1, 2, 3), \qquad X_4 = \frac{1}{2}(P'_0 + P'_4).
\end{aligned}
$$

The Lie algebra of the group $\widetilde{G}(1,3)$ is generated by the following basis elements:

$$L_1\,,\ L_2\,,\ L_3\,,\ P_1\,,\ P_2\,,\ P_3\,,\ X_0\,,\ X_1\,,\ X_2\,,\ X_3\,,\ X_4\,.$$

3. The first-order differential invariants of splitting subgroups of the group $P(1,4)$

The first-order differential invariants J of any non-conjugate k-parametrical subgroup of the group $P(1,4)$ can be obtained as solutions of the following systems of differential equations:

$$
\begin{cases}
\widetilde{X}_1 J(x_0, x_1, x_2, x_3, x_4, u, u_0, u_1, u_2, u_3, u_4) = 0,\\
\widetilde{X}_2 J(x_0, x_1, x_2, x_3, x_4, u, u_0, u_1, u_2, u_3, u_4) = 0,\\
\qquad\vdots\\
\widetilde{X}_k J(x_0, x_1, x_2, x_3, x_4, u, u_0, u_1, u_2, u_3, u_4) = 0,
\end{cases}
$$

where $\{\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_k,\ (k = 1, \ldots, 12, 15)\}$ are one times prolonged basis operators of any k-dimensional subalgebras of the Lie algebra of group $P(1,4)$, u is an arbitrary smooth function on $M(1,4)$, $u_\mu \equiv \frac{\partial u}{\partial x_\mu}$, $\mu = 0, 1, 2, 3, 4$.

Any solution of this system can be written in the following form:

$$J(x_0, x_1, x_2, x_3, x_4, u, u_0, u_1, u_2, u_3, u_4) = F(J_1, J_2, \ldots, J_t),$$

where $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of the considered subalgebra, F is an arbitrary smooth function. In this formula

$$J_i = J_i(x_0, x_1, x_2, x_3, x_4, u, u_0, u_1, u_2, u_3, u_4), \qquad i = 1, \ldots, t.$$

More details about solutions construction of the above mentioned type systems as well as the solved examples can be found in [10, 12, 13].

Using the prolongation theory (see, for example, [12, 13]) we have constructed the first prolongation for basis operators of the Lie algebra of the group $P(1,4)$. One times prolonged bases operators of the Lie algebra of the group $P(1,4)$ have the following form:

$$\widetilde{G} = -x_4\frac{\partial}{\partial x_0} - x_0\frac{\partial}{\partial x_4} + u_4\frac{\partial}{\partial u_0} + u_0\frac{\partial}{\partial u_4},$$

$$\widetilde{L}_1 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} + u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3},$$

$$\widetilde{L}_2 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3} - u_3\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_3},$$

$$\widetilde{L}_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2},$$

$$\widetilde{P}_1 = x_1\frac{\partial}{\partial x_0} + (x_0 + x_4)\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_4} - u_1\frac{\partial}{\partial u_0} - (u_0 - u_4)\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_4},$$

$$\widetilde{P}_2 = x_2\frac{\partial}{\partial x_0} + (x_0 + x_4)\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_4} - u_2\frac{\partial}{\partial u_0} - (u_0 - u_4)\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_4},$$

$$\widetilde{P}_3 = x_3\frac{\partial}{\partial x_0} + (x_0 + x_4)\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4} - u_3\frac{\partial}{\partial u_0} - (u_0 - u_4)\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4},$$

$$\widetilde{C}_1 = -x_1\frac{\partial}{\partial x_0} - (x_0 - x_4)\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_4} + u_1\frac{\partial}{\partial u_0} + (u_0 + u_4)\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_4},$$

$$\widetilde{C}_2 = -x_2\frac{\partial}{\partial x_0} - (x_0 - x_4)\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_4} + u_2\frac{\partial}{\partial u_0} + (u_0 + u_4)\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_4},$$

$$\widetilde{C}_3 = -x_3\frac{\partial}{\partial x_0} - (x_0 - x_4)\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4} + u_3\frac{\partial}{\partial u_0} + (u_0 + u_4)\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4},$$

$$\widetilde{X}_0 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_4}\right), \qquad \widetilde{X}_1 = -\frac{\partial}{\partial x_1}, \qquad \widetilde{X}_2 = -\frac{\partial}{\partial x_2},$$

$$\widetilde{X}_3 = -\frac{\partial}{\partial x_3}, \qquad \widetilde{X}_4 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_4}\right).$$

In the mentioned above denotations this basis can be written as $\{\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_{15}\}$.

Going through the list of all non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ presented in [5] one derives that the set of functional bases of the first-order differential invariants of the splitting subgroups of the group $P(1,4)$ contains 99 ones which are invariant under the splitting subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. It is impossible to present all these bases here. Therefore, below we give only a short review of the results obtained. In each example we write the basis elements of the splitting subalgebras of the Lie algebra of the group $\widetilde{G}(1,3)$ and their functional basis.

1. There is 1 three-dimensional functional basis

$$\langle X_1 \equiv P_1\,,\ X_2 \equiv P_2\,,\ X_3 \equiv P_3\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_3 - P_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv P_3\,,\ X_5 \equiv X_0\,,\ X_6 \equiv X_1\,,\ X_7 \equiv X_2\,,\ X_8 \equiv X_3\,,\ X_9 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_1\,,\ X_2 \equiv L_2\,,\ X_3 \equiv L_3\,,\ X_4 \equiv P_1\,,\ X_5 \equiv P_2\,,\ X_6 \equiv P_3\,,\ X_7 \equiv X_0\,,\ X_8 \equiv X_1\,,\ X_9 \equiv X_2\,,\ X_{10} \equiv X_3\,,\ X_{11} \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,, \qquad J_3 = u_0 - u_4\,;$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \qquad (\mu = 0,1,2,3,4).$$

2. There are 4 four-dimensional functional bases. For example

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv P_3\,,\ X_3 \equiv X_0\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_3\,,\ X_7 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,,$$
$$J_3 = u_0 - u_4\,, \qquad J_4 = u_1^2 + u_2^2\,.$$

3. There are 12 five-dimensional functional bases. For example

$$\langle X_1 \equiv P_1\,,\ X_2 \equiv P_2\,,\ X_3 \equiv X_1\,,\ X_4 \equiv X_2\,,\ X_5 \equiv X_3\,,\ X_6 \equiv X_4\rangle,$$

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_3\,,\ X_7 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,, \qquad J_3 = x_0 + x_4\,,$$
$$J_4 = u_3\,, \qquad J_5 = u_0 - u_4\,.$$

4. There are 19 six-dimensional functional bases. For example

$$\langle X_1 \equiv L_1\,,\ X_2 \equiv L_2\,,\ X_3 \equiv L_3\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,,$$
$$J_3 = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \qquad J_4 = x_1u_1 + x_2u_2 + x_3u_3\,,$$
$$J_5 = u_0\,, \qquad J_6 = u_4\,.$$

5. There are 26 seven-dimensional functional bases. For example

$$\langle X_1 \equiv P_3\,,\ X_2 \equiv X_0\,,\ X_3 \equiv X_3\,,\ X_4 \equiv X_4\rangle,$$

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,, \quad J_3 = x_1\,,$$
$$J_4 = x_2\,, \quad J_5 = u_1\,, \quad J_6 = u_2\,,$$
$$J_7 = u_0 - u_4\,.$$

6. There are 20 eight-dimensional functional bases. For example

$$\langle X_1 \equiv L_3\,,\ X_2 \equiv X_0\,,\ X_3 \equiv X_4\rangle,$$

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,, \quad J_3 = x_3\,,$$
$$J_4 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad J_5 = x_1u_2 - x_2u_1\,, \quad J_6 = u_0\,,$$
$$J_7 = u_3\,, \quad J_8 = u_4\,.$$

7. There are 11 nine-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 - P_3\,,\ X_2 \equiv X_4\rangle,$$

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,,$$
$$J_3 = x_0 + x_4\,, \quad J_4 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$
$$J_5 = \arctan\frac{x_1}{x_2} + \frac{x_3}{x_0 + x_4}\,, \quad J_6 = x_1u_2 - x_2u_1\,,$$
$$J_7 = \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4}\,, \quad J_8 = u_0 - u_4\,,$$
$$J_9 = u_1^2 + u_2^2\,.$$

8. There are 6 ten-dimensional functional bases. For example

$$\langle X_1 \equiv P_3\rangle,$$

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,,$$
$$J_3 = x_1\,, \quad J_4 = x_2\,,$$
$$J_5 = x_0 + x_4\,, \quad J_6 = (x_0^2 - x_3^2 - x_4^2)^{\frac{1}{2}},$$
$$J_7 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3\,, \quad J_8 = u_0 - u_4\,,$$
$$J_9 = u_1\,, \quad J_{10} = u_2\,.$$

4. The first-order differential invariants of the non-splitting subgroups of the group $P(1,4)$

As in the Section 3, it is established that the set of functional bases of the first-order differential invariants of the non-splitting subgroups of the group $P(1,4)$ contains 158 ones which are invariant under the non-splitting subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. It is impossible to present all these bases here. Therefore, below we give only a short review of the results obtained. In each example we write the basis elements of the non-splitting subalgebras of the Lie algebra of the group $\widetilde{G}(1,3)$ and their functional basis.

1. There is 1 three-dimensional functional basis

$$\langle X_1 \equiv L_3 - X_0\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv P_3\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,$$

$$\langle X_1 \equiv P_1\,,\ X_2 \equiv P_2\,,\ X_3 \equiv P_3 + X_0\,,\ X_4 \equiv L_3 + \beta X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_3\,,\ X_8 \equiv X_4\rangle,\ \beta < 0,$$

$$J_1 = u, \qquad J_2 = u_0 - u_4\,, \qquad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,;$$
$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad (\mu = 0,1,2,3,4).$$

2. There are 5 four-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 + d_3 X_3\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_0\,,\ X_5 \equiv X_1\,,\ X_6 \equiv X_2\,,\ X_7 \equiv X_4\rangle,\ d_3 < 0,$$

$$\langle X_1 \equiv L_3 - X_0\,,\ X_2 \equiv P_1\,,\ X_3 \equiv P_2\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_3\,,\ X_7 \equiv X_4\rangle,$$

$$J_1 = u, \qquad J_2 = u_3\,, \qquad J_3 = u_0 - u_4\,, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_4^2\,.$$

3. There are 12 five-dimensional functional bases. For example

$$\langle X_1 \equiv P_1 + \beta X_3\,,\ X_2 \equiv P_2\,,\ X_3 \equiv P_3 + X_0\,,\ X_4 \equiv X_1\,,\ X_5 \equiv X_2\,,\ X_6 \equiv X_4\rangle,\ \beta > 0,$$

$$J_1 = u, \qquad J_2 = (x_0 + x_4) + \frac{u_3}{u_0 - u_4}\,,$$
$$J_3 = (x_0 + x_4)^2 - 2x_3 + 2\beta\frac{u_1}{u_0 - u_4}\,, \qquad J_4 = u_0 - u_4\,,$$
$$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\,.$$

4. There are 34 six-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 + d X_3\,,\ X_2 \equiv P_3\,,\ X_3 \equiv X_1\,,\ X_4 \equiv X_2\,,\ X_5 \equiv X_4\rangle,\ d < 0,$$

$$J_1 = x_0 + x_4\,, \qquad J_2 = u, \qquad J_3 = x_3 + d\arctan\frac{u_1}{u_2} + u_3\frac{x_0 + x_4}{u_0 - u_4},$$
$$J_4 = u_0 - u_4\,, \qquad J_5 = u_1^2 + u_2^2\,, \quad J_6 = u_0^2 - u_3^2 - u_4^2\,.$$

5. There are 49 seven-dimensional functional bases. For example

$$\langle X_1 \equiv P_1 + \delta X_3\,,\ X_2 \equiv P_2 + X_3\,,\ X_3 \equiv X_1,\ X_4 \equiv X_4\rangle,\ \delta > 0,$$

$$J_1 = x_0 + x_4\,, \qquad J_2 = u, \qquad J_3 = \frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4},$$
$$J_4 = \frac{\delta u_1 + u_2}{u_0 - u_4} - x_3\,, \qquad J_5 = u_3\,, \qquad J_6 = u_0 - u_4\,,$$
$$J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2\,.$$

6. There are 32 eight-dimensional functional bases. For example

$$\langle X_1 \equiv P_3 + X_0\,,\ X_2 \equiv X_1\,,\ X_3 \equiv X_4\rangle,$$

$$J_1 = x_2\,, \qquad J_2 = (x_0 + x_4)^2 - 2x_3\,, \qquad J_3 = u,$$
$$J_4 = (x_0 + x_4) + \frac{u_3}{u_0 - u_4}, \qquad J_5 = u_1\,, \qquad J_6 = u_2\,,$$
$$J_7 = u_0 - u_4\,, \qquad J_8 = u_0^2 - u_3^2 - u_4^2\,.$$

7. There are 19 nine-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 - X_4\,,\ X_2 \equiv X_3\rangle,$$

$$J_1 = x_0 + x_4\,, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_3 = u,$$
$$J_4 = x_1 u_2 - x_2 u_1\,, \qquad J_5 = \arctan\frac{u_1}{u_2} + x_0 - x_4\,, \qquad J_6 = u_0\,,$$
$$J_7 = u_3\,, \qquad J_8 = u_4\,, \qquad J_9 = u_1^2 + u_2^2\,.$$

8. There are 6 ten-dimensional functional bases. For example

$$\langle X_1 \equiv L_3 - P_3 + \alpha_0 X_0\rangle,\ \alpha_0 < 0,$$

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_2 = (x_0 + x_4)^2 + 2\alpha_0 x_3\,,$$
$$J_3 = x_1 u_2 - x_2 u_1\,, \qquad J_4 = \alpha_0 \arctan\frac{x_1}{x_2} - x_0 - x_4\,,$$
$$J_5 = x_0 + x_4 - \alpha_0 \frac{u_3}{u_0 - u_4}, \qquad J_6 = 2(x_0 + x_4)^3 + 6\alpha_0 x_3 (x_0 + x_4) + 3\alpha_0^2 (x_0 - x_4),$$
$$J_7 = u, \qquad J_8 = u_0 - u_4\,,$$
$$J_9 = u_1^2 + u_2^2\,, \qquad J_{10} = u_0^2 - u_3^2 - u_4^2\,.$$

As we see there are not the functional bases with dimensions less than 3 as well as ones with dimensions bigger than 10. It follows from using of the theorem on invariants of Lie groups of the point transformations for all non-conjugate subgroups of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$. More details about this theorem can be found in [12, 13].

The results obtained can be used for the construction and investigation of classes of first–order differential equations (defined in the space $M(1,4) \times R(u)$)

invariant under continuous subgroups of the group $\widetilde{G}(1,3) \subset P(1,4)$. $R(u)$ is the axis of the dependent variable u.

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