Annales Academiae Paedagogicae Cracoviensis

Folia 63 Studia Mathematica VII

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The transmission problem for elliptic second order equations in a conical domain

Abstract. The present article is a survey of our last results. We establish best possible estimates of the weak solutions to the transmission problem near conical boundary point. We study this problem for the Laplace operator with N different media, for linear and quasi-linear (with semi-linear principal part) elliptic second order equations in divergence form. Boundary conditions in these problems are different: the Dirichlet, the Neumann, the Robin, as well as mixed boundary conditions.

The transmission problems often appear in different fields of physics and technics. For instance, one of the important problems of the electrodynamics of solid media is the electromagnetic processes research in ferromagnetic media with different dielectric constants. Such problems also appear in solid mechanics if a body consists of composite materials. Let us quote also vibrating folded membranes, composite plates, folded plates, junctions in elastic multi-structures etc.

The present article is a survey of our last results. We consider the best possible estimates of the weak solutions to the transmission problem near conical boundary point. Analogous results were established in [3] for the Dirichlet and Robin problems in a conical domain without interfaces.

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a conical surface with vertex at $\mathcal{O}$ and the opening ω_0. We assume that $G = \bigcup_{i=1}^{N} G_i$ is divided into $N \geq 2$ subdomains G_i, $i = 1, \dots, N$ by $(N-1)$ hyperplanes Σ_k, $k = 1, \dots, N-1$ (by hyperplane Σ_0 in the case $N = 2$), where $\mathcal{O}$ belongs to every $\overline{\Sigma_k}$ and $G_i \cap G_j = \emptyset$, $i \neq j$. We shall study the following elliptic transmission problems.

AMS (2000) Subject Classification: 35J25, 35J60, 35J85, 35B65.

This work was supported at the final stage by the Polish Ministry of Science and Higher Education through the grant Nr N201 381834..

Problem (*LN*). For the Laplace operator with N different media and mixed boundary condition

$$\begin{cases} \mathcal{L}_i[u] \equiv a_i \triangle u_i - p_i u_i(x) = f_i(x), & x \in G_i\,,\ i = 1, \dots, N; \\ [u]_{\Sigma_k} = 0, & k = 1, \dots, N-1; \\ \mathcal{S}_k[u] \equiv \left[a \dfrac{\partial u}{\partial \vec{n}_k} \right]_{\Sigma_k} + \dfrac{1}{|x|} \beta_k(\omega) u(x) = h_k(x), & x \in \Sigma_k\,, \\ & k = 1, \dots, N-1; \\ \mathcal{B}[u] \equiv \alpha(x) a \dfrac{\partial u}{\partial \vec{n}} + \dfrac{1}{|x|} \gamma(\omega) u(x) = g(x), & x \in \partial G \setminus \mathcal{O}, \end{cases}$$

where $\omega = \frac{x}{|x|}$, $a_i > 0$, $p_i \geq 0$, $(i = 1, \dots, N)$ are constants;

$$\alpha(x) = \begin{cases} 0, & \text{if } x \in \mathcal{D}; \\ 1, & \text{if } x \notin \mathcal{D}, \end{cases}$$

and $\mathcal{D} \subseteq \partial G$ is the part of the boundary ∂G where we consider the Dirichlet boundary condition; here $\vec{n}_k$ $(\vec{n}\,)$ denotes the unite outward with respect to G_k (G) normal to Σ_k $(\partial G \setminus \mathcal{O})$.

Problem (*L*). For linear equations

$$\begin{cases} \mathcal{L}[u] \equiv \dfrac{\partial}{\partial x_i} (a^{ij}(x) u_{x_j}) + a^i(x) u_{x_i} + a(x) u = f(x), & x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0; & \\ \mathcal{S}[u] \equiv \left[\dfrac{\partial u}{\partial \nu} \right]_{\Sigma_0} + \dfrac{\beta(\omega)}{|x|} u(x) = h(x), & x \in \Sigma_0; \\ \mathcal{B}[u] \equiv \dfrac{\partial u}{\partial \nu} + \dfrac{\gamma(\omega)}{|x|} u = g(x), & x \in \partial G \setminus \mathcal{O}. \end{cases}$$

Problem (*WL*). For weak nonlinear equations

$$\begin{cases} -\dfrac{d}{dx_i} (|u|^q a^{ij}(x) u_{x_j}) + b(x, u, \nabla u) = 0, & q \geq 0,\ x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0; & \\ \mathcal{S}[u] \equiv \left[\dfrac{\partial u}{\partial \nu} \right]_{\Sigma_0} + \dfrac{\beta(\omega)}{|x|} u|u|^q = h(x, u), & x \in \Sigma_0; \\ \mathcal{B}[u] \equiv \dfrac{\partial u}{\partial \nu} + \dfrac{\gamma(\omega)}{|x|} u|u|^q = g(x, u), & x \in \partial G \setminus \mathcal{O} \end{cases}$$

(the summation over repeated indices from 1 to n is understood; $\frac{\partial u}{\partial \nu}$ is the co-normal derivative of $u(x)$), i.e., $\frac{\partial u}{\partial \nu} = |u|^q a^{ij}(x) u_{x_j} \cos(\vec{n}, x_i)$.

The principal new feature of our work is the consideration of estimates of weak solutions for *linear* elliptic second-order equations with *minimal smooth coefficients* in *n-dimensional conical* domains. Our examples demonstrate this fact.

1. Problem (LN)

Let ϕ_i be openings at the vertex $\mathcal{O}$ in domains G_i. Let us define the value $\theta_k = \phi_1 + \phi_2 + \ldots + \phi_k$, thus $\omega_0 = \theta_N$. We introduce the following notations:

- Ω_i – a domain on the unit sphere S^{n-1} with boundary $\partial\Omega_i$ obtained by the intersection of the domain G_i with the sphere S^{n-1}, $(i = 1, \ldots, N)$; thus $\Omega = \bigcup_{i=1}^{N} \Omega_i$;
- $\Sigma = \sum_{k=1}^{N-1} \Sigma_k, \quad \Sigma_k = G \cap \{\omega_1 = \frac{\omega_0}{2} - \theta_k\}, \qquad k = 1, \ldots, N-1;$ $\sigma = \sum_{k=1}^{N-1} \sigma_k, \quad \sigma_k = \Sigma_k \cap \Omega;$
- $(G_i)_a^b = \{(r, \omega) \mid \ 0 \le a < r < b;\ \omega \in \Omega\} \cap G_i \qquad i = 1, \ldots, N;$
- $(\Sigma_k)_a^b = G_a^b \cap \Sigma_k, \qquad k = 1, \ldots, N-1;$
- $u(x) = u_i(x), \quad f(x) = f_i(x), \quad x \in G_i; \quad a\big|_{\overline{G_i}} = a_i$, etc.;
- $[u]_{\Sigma_k}$ denotes the saltus of the function $u(x)$ on crossing Σ_k, i.e.,

$$[u]_{\Sigma_k} = u_k(\overline{x})\big|_{\Sigma_k} - u_{k+1}(\overline{x})\big|_{\Sigma_k},$$
$$u_k(\overline{x})\big|_{\Sigma_k} = \lim_{G_k \ni x \to \overline{x} \in \Sigma_k} u(x), \qquad u_{k+1}(\overline{x})\big|_{\Sigma_k} = \lim_{G_{k+1} \ni x \to \overline{x} \in \Sigma_k} u(x);$$

- $\left[a\frac{\partial u}{\partial \vec{n}_k}\right]_{\Sigma_k}$ denotes the saltus of the co-normal derivative of the function $u(x)$ on crossing Σ_k, i.e.,

$$\left[a\frac{\partial u}{\partial \vec{n}_k}\right]_{\Sigma_k} = a_k \frac{\partial u_k}{\partial \vec{n}_k}\bigg|_{\Sigma_k} - a_{k+1}\frac{\partial u_{k+1}}{\partial \vec{n}_k}\bigg|_{\Sigma_k}.$$

Without loss of generality we assume that there exists $d > 0$ such that G_0^d is *a convex rotational cone* with the vertex at $\mathcal{O}$ and the aperture ω_0, thus

$$\Gamma_0^d = \left\{(r, \omega) \;\middle|\; x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^{n} x_i^2\,;\ r \in (0, d),\ \omega_1 = \frac{\omega_0}{2},\ \omega_0 \in (0, \pi)\right\};$$

$\Gamma_a^b = \{(r, w) \mid \ 0 \le a < r < b;\ w \in \partial\Omega\} \cap \partial G$ – the lateral surface of layer G_a^b.

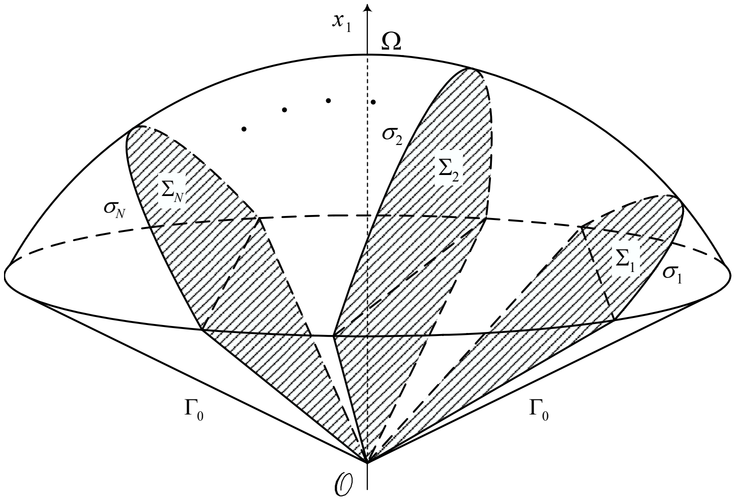

Fig. 1

We use the standard function spaces:

— $C^k(\overline{G_i})$ with the norm $|u_i|_{k,G_i}$,

— the Lebesgue space $L_p(G_i)$, $p \geq 1$ with the norm $\|u_i\|_{p,G_i}$,

— the Sobolev space $W^{k,p}(G_i)$ with the norm $\|u_i\|_{k,p;G_i}$,

— direct sum $\mathbf{C}^k(\overline{G}) = C^k(\overline{G_1}) \dotplus \ldots \dotplus C^k(\overline{G_N})$ with the norm

$$|u|_{k,G} = \sum_{i=1}^{N} |u_i|_{k,G_i} ;$$

— $\mathbf{L}_p(G) = L_p(G_1) \dotplus \ldots \dotplus L_p(G_N)$ with the norm

$$\|u\|_{\mathbf{L}_p(G)} = \sum_{i=1}^{N} \Big(\int\limits_{G_i} |u_i|^p dx \Big)^{\frac{1}{p}};$$

— $\mathbf{W}^{k,p}(G) = W^{k,p}(G_1) \dotplus \ldots \dotplus W^{k,p}(G_N)$ with the norm

$$\|u\|_{k,p;G} = \sum_{i=1}^{N} \Big(\int\limits_{G_i} \sum_{|\beta|=0}^{k} |D^\beta u_i|^p dx \Big)^{\frac{1}{p}}.$$

We define the weighted Sobolev spaces: $\mathbf{V}^k_{p,\alpha}(G) = V^k_{p,\alpha}(G_1) \dotplus \ldots \dotplus V^k_{p,\alpha}(G_N)$ for integer $k \geq 0$ and real α, where $V^k_{p,\alpha}(G_i)$ denotes the space of all distributions

$u \in \mathcal{D}'(G_i)$ satisfying $r^{\frac{\alpha}{p}+|\beta|-k}|D^\beta u_i| \in L_p(G_i)$, $i = 1, \ldots, N$. $\mathbf{V}^k_{p,\alpha}(G)$ is a Banach space with the norm

$$\|u\|_{\mathbf{V}^k_{p,\alpha}(G)} = \sum_{i=1}^{N} \Bigg(\int\limits_{G_i} \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)} |D^\beta u_i|^p \, dx \Bigg)^{\frac{1}{p}}.$$

$\mathbf{V}^{k-\frac{1}{p}}_{p,\alpha}(\partial G)$ is the space of functions φ, given on ∂G, with the norm

$$\|\varphi\|_{\mathbf{V}^{k-\frac{1}{p}}_{p,\alpha}(\partial G)} = \inf \|\Phi\|_{\mathbf{V}^k_{p,\alpha}(G)},$$

where the infimum is taken over all functions Φ such that $\Phi\big|_{\partial G} = \varphi$ in the sense of traces. We denote $\mathbf{W}^k(G) \equiv \mathbf{W}^{k,2}(G)$, $\overset{\circ}{\mathbf{W}}{}^k_\alpha(G) \equiv \mathbf{V}^k_{2,\alpha}(G)$.

Definition 1
The function $u(x)$ is called a *weak* solution of the problem (LN) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_0$ and satisfies the integral identity

$$\int\limits_G a u_{x_j} \eta_{x_j} \, dx + \int\limits_\Sigma \frac{1}{r}\beta(\omega) u(x)\eta(x)\, ds + \int\limits_{\partial G} \alpha(x)\frac{1}{r}\gamma(\omega)u(x)\eta(x)\, ds$$
$$= \int\limits_{\partial G} \alpha(x) g(x)\eta(x)\, ds + \int\limits_\Sigma h(x)\eta(x)\, ds - \int\limits_G (pu(x) + f(x))\,\eta(x)\, dx$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_0(G)$. The integrals above are sums:

$$\int\limits_G f(x)\, dx = \sum_{i=1}^{N} \int\limits_{G_i} f_i(x)\, dx, \qquad \int\limits_\Sigma h(x)\, ds = \sum_{k=1}^{N-1} \int\limits_{\Sigma_k} h_k(x)\, ds, \quad \text{etc.}$$

Remark 1
In the Dirichlet boundary condition case ($\alpha(x) \equiv 0$) we assume, without loss of generality, that

$$g\big|_{\partial G \cap \mathcal{D}} = 0 \implies u\big|_{\partial G \cap \mathcal{D}} = 0.$$

We assume that $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known. Let us define numbers

$$\begin{cases} a_* = \min\{a_1, \ldots, a_N\} > 0; \\ a^* = \max\{a_1, \ldots, a_N\} > 0; \\ p^* = \max\{p_1, \ldots, p_N\} \geq 0; \\ [a]_{\Sigma_k} = a_k - a_{k+1}, \qquad\qquad k = 1, \ldots, N-1; \\ a_0 = \max\limits_{1\leq k\leq N-1} \big|[a]_{\Sigma_k}\big|; \\ \widetilde{a} = \max(a^*, a_0). \end{cases}$$

We assume that:

(a) $f(x) \in \mathbf{L}_{\frac{q}{2}}(G) \cap \mathbf{L}_2(G)$; $q > n$;

(b) $\gamma(\phi) \geq \gamma_0 > \widetilde{a} \tan \frac{\omega_0}{2}$ *on* ∂G;
$\beta_k(\phi) \geq \beta_0 > \widetilde{a} \tan \frac{\omega_0}{2}$ *on* Σ_k, $k = 1, \ldots, N-1$;

(c) *there exist numbers* $f_0 \geq 0$, $g_0 \geq 0$, $h_0 \geq 0$, $s > 1$, $\beta \geq s-2$ *such that*

$$|f(x)| \leq f_0 |x|^{\beta}, \qquad |g(x)| \leq g_0 |x|^{s-1},$$
$$|h_k(x)| \leq h_0 |x|^{s-1}, \quad k = 1, \ldots, N-1.$$

We consider the following **eigenvalue problem (*EVP*)**.

Let $\Omega \subset S^{n-1}$ with smooth boundary $\partial\Omega$ be the intersection of the cone $\mathcal{C}$ with the unit sphere S^{n-1}. Let $\vec{\nu}$ be the exterior normal to $\partial\mathcal{C}$ at points of $\partial\Omega$ and $\vec{\tau}_k$ be the exterior with respect to Ω_k normal to Σ_k (lying in the plane tangent to Ω_k), $k = 1, \ldots, N-1$. Let $\gamma(\phi)$, $\phi \in \partial\Omega$ be a positive bounded piecewise smooth function, $\beta_k(\phi)$ be a positive continuous function on σ_k, $k = 1, \ldots, N-1$. We consider the eigenvalue problem for the Laplace-Beltrami operator $\triangle_\phi$ on the unit sphere

$$\begin{cases} a_i \left(\triangle_\phi \psi_i + \vartheta \psi_i\right) = 0, & \phi \in \Omega_i,\ a_i \text{ are positive constants; } i = 1, \ldots, N; \\ [\psi]_{\sigma_k} = 0, & k = 1, \ldots, N-1; \\ \left[a \dfrac{\partial \psi}{\partial \vec{\tau}_k}\right]_{\sigma_k} + \beta_k(\phi)\psi\big|_{\sigma_k} = 0, & k = 1, \ldots, N-1; \\ \alpha(\phi) a \dfrac{\partial \psi}{\partial \vec{\nu}} + \gamma(\phi)\psi\big|_{\partial\Omega} = 0, & \end{cases} \qquad (EVP)$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has a non-zero weak solutions (eigenfunctions).

Our main result is the following theorem. Let ϑ be the smallest positive solution of (EVP) and let

$$\lambda = \frac{2 - n + \sqrt{(n-2)^2 + 4\vartheta}}{2}. \qquad (1.1)$$

THEOREM 1
Let u be a weak solution of the problem (LN) and assumptions (a)-(c) *be satisfied. Assume that the domain G and parameters in* (a)-(c) *are such that λ defined above satisfies $\lambda > 1$. Then there are $d \in (0,1)$ and constants $C_0 > 0$, $c > 0$ depending only on n, a_*, a^*, p^*, λ, q, ω_0, f_0, h_0, g_0, β_0, γ_0, s, M_0,* meas G, diam G *such that for all $x \in G_0^d$*

$$|u(x)| \le C_0 \begin{cases} |x|^{\lambda}, & \textit{if } s > \lambda; \\ |x|^{\lambda} \ln^c \left(\dfrac{1}{|x|}\right), & \textit{if } s = \lambda; \\ |x|^{s}, & \textit{if } s < \lambda. \end{cases}$$

Suppose, in addition, that

$$\gamma(\omega) \in \mathbf{C}^1(\partial G), \qquad f(x) \in \mathbf{V}^0_{q,2q-n}(G),$$

$$h(x) \in V^{1-\frac{1}{q}}_{q,2q-n}(\Sigma), \qquad g(x) \in \mathbf{V}^{1-\frac{1}{q}}_{q,2q-n}(\partial G);$$

$q > n$ *and there is a number*

$$\tau_s =: \sup_{\varrho>0} \varrho^{-s}\Big(\|h\|_{V^{1-\frac{1}{q}}_{q,2q-n}(\Sigma^{\varrho}_{\frac{\varrho}{2}})} + \|g\|_{\mathbf{V}^{1-\frac{1}{q}}_{q,2q-n}(\Gamma^{\varrho}_{\frac{\varrho}{2}})}\Big).$$

Then for all $x \in G^d_0$

$$|\nabla u(x)| \le C_1 \begin{cases} |x|^{\lambda-1}, & \textit{if } s > \lambda; \\ |x|^{\lambda-1} \ln^c \left(\dfrac{1}{|x|}\right), & \textit{if } s = \lambda; \\ |x|^{s-1}, & \textit{if } s < \lambda. \end{cases}$$

Furthermore, the following is true

— $u \in \mathbf{V}^2_{q,2q-n}(G)$, $q > n$ *and*

$$\|u\|_{\mathbf{V}^2_{q,2q-n}(G^{\varrho}_0)} \le C_2 \begin{cases} \varrho^{\lambda}, & \textit{if } s > \lambda; \\ \varrho^{\lambda} \ln^c \left(\dfrac{1}{\varrho}\right), & \textit{if } s = \lambda; \\ \varrho^{s}, & \textit{if } s < \lambda; \end{cases}$$

— *if* $f(x) \in \overset{\circ}{\mathbf{W}}{}^0_{\alpha}(G)$, $\int_{\Sigma} r^{\alpha-1}h^2(x)\,ds + \int_{\partial G} r^{\alpha-1}g^2(x)\,ds < \infty$, *where* $4 - n - 2\lambda < \alpha \le 2$, *then* $u(x) \in \overset{\circ}{\mathbf{W}}{}^1_{\alpha-2}(G)$ *and*

$$\int\limits_G a(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)\,dx + \int\limits_{\Sigma} r^{\alpha-3}\beta(\phi)u^2(x)\,ds + \int\limits_{\partial G} \alpha(x) r^{\alpha-3}\gamma(\phi)u^2(x)\,ds \le C\Big\{\int\limits_G (u^2 + (1+r^{\alpha})f^2(x))\,dx + \int\limits_{\Sigma} r^{\alpha-1}h^2(x)\,ds + \int\limits_{\partial G} \alpha(x) r^{\alpha-1} g^2(x)\,ds\Big\},$$

where the constant $C > 0$ *depends only on* q, n, a_*, a^*, α, λ *and the domain* G.

1.1. Eigenvalue transmission problem in a composite plane domain with an angular point

Let $G \subset \mathbb{R}^2$ be bounded domain with the boundary curve ∂G smooth everywhere except at the origin $\mathcal{O} \in \partial G$. Near the point $\mathcal{O}$ it is a fan that consists of N corners with vertices at $\mathcal{O}$. Thus $G = \bigcup_{i=1}^{N} G_i$; $\partial G = \bigcup_{j=0}^{N+1} \Gamma_j$; $\Sigma = \bigcup_{k=1}^{N-1} \Sigma_k$. Here Σ_k, $k = 1, \ldots, N-1$ are rays that divide G into angular domains G_i, $i = 1, \ldots, N$. Let ω_i be apertures at the vertex $\mathcal{O}$ in domains G_i, $i = 1, \ldots, N$. We define the value $\theta_k = \omega_1 + \omega_2 + \ldots + \omega_k$. Let $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ be the curvilinear portion of the boundary ∂G. In this case we have $\lambda = \sqrt{\vartheta}$.

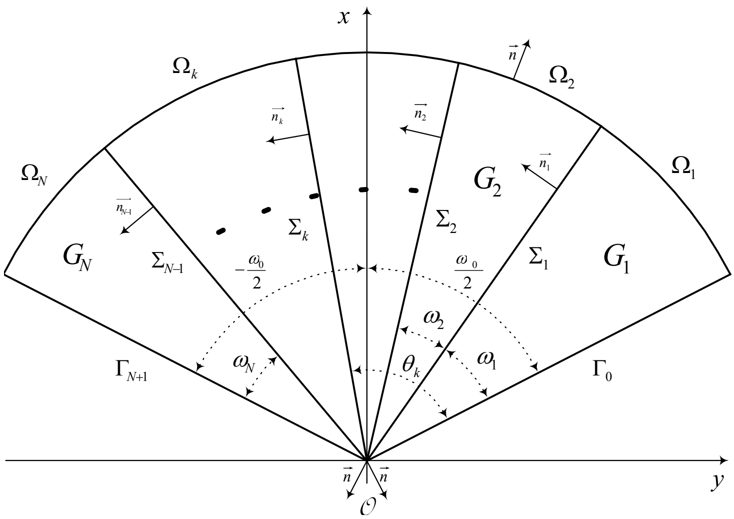

Fig. 2

We also assume that

$$\Gamma_0 = \{(r, \omega) \mid r > 0,\ \omega = 0\}, \qquad \Gamma_{N+1} = \{(r, \omega) \mid r > 0,\ \omega = \theta_N\},$$

$$\beta_k\big|_{\sigma_k} = \beta_k(\theta_k) = \beta_k = \text{const}, \quad \gamma(0) = \gamma_1 = \text{const}, \quad \gamma(\omega_0) = \gamma_N = \text{const}.$$

The eigenvalue problem in this case has the form

$$\begin{cases} \psi_i'' + \lambda^2 \psi_i(\omega) = 0, & \omega \in \Omega_i = \{\omega_{i-1} < \omega < \omega_i\}, \\ & i = 1, \dots, N; \\ \psi_i(\theta_i) = \psi_{i+1}(\theta_i), & i = 1, \dots, N-1; \\ a_i \psi_i'(\theta_i) - a_{i+1}\psi_{i+1}'(\theta_i) + \beta_i \psi_i(\theta_i) = 0, & i = 1, \dots, N-1; \\ \alpha_1 a_1 \psi_1'(0) + \gamma_1 \psi_1(0) = 0; \\ \alpha_N a_N \psi_N'(\omega_0) + \gamma_N \psi_N(\omega_0) = 0, \end{cases}$$

where $a_1, \alpha_N \in \{0, 1\}$.

By direct calculation, we get

$$\psi_i(\omega) = A_i \cos(\lambda\omega) + B_i \sin(\lambda\omega), \qquad i = 1, \dots, N$$

and constants $A_1, \dots, A_N$; $B_1, \dots, B_N$ are defined by the algebraic homogeneous system of equations:

$$\begin{cases} \lambda \alpha_1 a_1 B_1 + \gamma_1 = 0; \\ A_{i+1} = \left(\cos^2(\lambda\theta_i) + \dfrac{a_i}{a_{i+1}} \sin^2(\lambda\theta_i) - \dfrac{\beta_i}{\lambda a_{i+1}} \sin(\lambda\theta_i)\cos(\lambda\theta_i) \right) \cdot A_i \\ \quad + \left(\sin(\lambda\theta_i)\cos(\lambda\theta_i) \left(1 - \dfrac{a_i}{a_{i+1}}\right) - \dfrac{\beta_i}{\lambda a_{i+1}} \sin^2(\lambda\theta_i) \right) \cdot B_i, \\ \qquad i = 1, \dots, N-1; \\ B_{i+1} = \left(\sin(\lambda\theta_i)\cos(\lambda\theta_i) \left(1 - \dfrac{a_i}{a_{i+1}}\right) + \dfrac{\beta_i}{\lambda a_{i+1}} \cos^2(\lambda\theta_i) \right) \cdot A_i \\ \quad + \left(\sin^2(\lambda\theta_i) + \dfrac{a_i}{a_{i+1}} \cos^2(\lambda\theta_i) + \dfrac{\beta_i}{\lambda a_{i+1}} \sin(\lambda\theta_i)\cos(\lambda\theta_i) \right) \cdot B_i, \\ \qquad i = 1, \dots, N-1; \\ (\gamma_N \cos(\lambda\omega_0) - \lambda\alpha_N a_N \sin(\lambda\omega_0)) \cdot A_N + (\gamma_N \sin(\lambda\omega_0) \\ \quad + \lambda\alpha_N a_N \cos(\lambda\omega_0)) \cdot B_N = 0. \end{cases}$$

The least positive eigenvalue λ is determined by the vanishing of the determinant of this system.

1.2. Four media transmission problem

Our goal is to derive the eigenvalues equation corresponding to our transmission problem for $N = 4$. Let S^1 be the unit circle in $\mathbb{R}^2$. We denote: $\Omega_i = G_i \cap S^1$ for $i = 1, 2, 3, 4$. The eigenvalue problem is the following:

$$
\begin{cases}
\psi_i'' + \lambda^2\psi_i(\omega) = 0, & \omega \in \Omega_i\,; \text{ for } i = 1,2,3,4;\\
\psi_2(\omega_1) = \psi_1(\omega_1);\\
\psi_3(\theta_2) = \psi_2(\theta_2);\\
\psi_4(\theta_3) = \psi_3(\theta_3);\\
a_1\psi_1'(\omega_1) - a_2\psi_2'(\omega_1) + \beta_1\psi_1(\omega_1) = 0;\\
a_2\psi_2'(\theta_2) - a_3\psi_3'(\theta_2) + \beta_2\psi_2(\theta_2) = 0;\\
a_3\psi_3'(\theta_3) - a_4\psi_4'(\theta_3) + \beta_3\psi_3(\theta_3) = 0;\\
\alpha_1 a_1\psi_1'(0) + \gamma_1\psi_1(0) = 0;\\
\alpha_4 a_4\psi_4'(\theta_4) + \gamma_4\psi_4(\theta_4) = 0,
\end{cases}
\tag{1.2}
$$

where $\alpha_1 = \alpha\big|_{\Gamma_0} = \alpha\big|_{\omega=0}$, $\alpha_4 = \alpha\big|_{\Gamma_5} = \alpha\big|_{\omega=\theta_4}$, $\gamma_1 = \gamma(0)$, $\gamma_4 = \gamma(\theta_4)$; $\alpha_{1,4} \in \{0,1\}$.

A general solution of (1.2) is

$$\psi_i(\omega) = A_i\cos(\lambda\omega) + B_i\sin(\lambda\omega) \qquad \text{for } i = 1,2,3,4,$$

with arbitrary constants A_i , B_i ($i = 1,2,3,4$). Boundary condition of (1.2) force A_i , B_i to satisfy the following system of linear equations:

$$
\begin{cases}
A_2\cos\lambda\omega_1 + B_2\sin\omega_1 - A_1\cos\lambda\omega_1 - B_1\sin\lambda\omega_1 = 0,\\
A_3\cos\lambda\theta_2 + B_3\sin\theta_2 - A_2\cos\lambda\theta_2 - B_2\sin\lambda\theta_2 = 0,\\
A_4\cos\lambda\theta_3 + B_4\sin\theta_3 - A_3\cos\lambda\theta_3 - B_3\sin\lambda\theta_3 = 0,\\
\lambda a_2A_2\sin\lambda\omega_1 - \lambda a_2B_2\cos\lambda\omega_1 - \lambda a_1A_1\sin\lambda\omega_1 + \lambda a_1B_1\cos\lambda\omega_1\\
\qquad +\beta_1A_1\cos\lambda\omega_1 + \beta_1B_1\sin\lambda\omega_1 = 0,\\
\lambda a_3A_3\sin\lambda\theta_2 - \lambda a_3B_3\cos\lambda\theta_2 - \lambda a_2A_2\sin\lambda\theta_2 + \lambda a_2B_2\cos\lambda\theta_2\\
\qquad +\beta_2A_2\cos\lambda\theta_2 + \beta_2B_2\sin\lambda\theta_2 = 0,\\
\lambda a_4A_4\sin\lambda\theta_3 - \lambda a_4B_4\cos\lambda\theta_3 - \lambda a_3A_3\sin\lambda\theta_3 + \lambda a_3B_3\cos\lambda\theta_3\\
\qquad +\beta_3A_3\cos\lambda\theta_3 + \beta_3B_3\sin\lambda\theta_3 = 0,\\
\alpha_1a_1\lambda B_1 + \gamma_1A_1 = 0,\\
\alpha_4a_4\lambda A_4\sin\lambda\theta_4 - \alpha_4a_4\lambda B_4\cos\lambda\theta_4 - \gamma_4A_4\cos\lambda\theta_4 - \gamma_4B_4\sin\lambda\theta_4 = 0.
\end{cases}
$$

This system has a non-trivial solution if its determinant vanishes. This gives the eigenvalues equation, which is too complex to state here in full generality. We provide an explicite form only in special cases of boundary conditions.

1. Dirichlet problem: $\alpha_1 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0$; $\gamma_1 = \gamma_4 = 1$.

$$
\begin{aligned}
\lambda^3 a_2a_3^2 &\sin\lambda\omega_1\cos\lambda\omega_2\sin\lambda\omega_3\sin\lambda\omega_4\\
&+ a_2^2a_3\sin\lambda\omega_1\sin\lambda\omega_2\cos\lambda\omega_3\sin\lambda\omega_4\\
&+ a_2^2a_4\sin\lambda\omega_1\sin\lambda\omega_2\sin\lambda\omega_3\cos\lambda\omega_4\\
&+ a_1a_3^2\cos\lambda\omega_1\sin\lambda\omega_2\sin\lambda\omega_3\sin\lambda\omega_4
\end{aligned}
$$

$$
\begin{aligned}
&- a_2a_3a_4 \sin\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1a_3a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1a_2a_4 \cos\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1a_2a_3 \cos\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
= 0.&
\end{aligned}
$$

In the isotropic case ($a_1 = a_2 = a_3 = a_4$) we recover the well known result:

$$\sin(\lambda\theta_4) = 0 \implies \lambda_n = \frac{\pi n}{\theta_4}, \qquad n = 1, 2\dots\ .$$

Corollary: $\lambda = \frac{\pi}{\omega_0} > 1$, if $0 < \omega_0 < \pi$.

2. Neumann problem: $\alpha_1 = \alpha_4 = 1$; $\beta_1 = \beta_2 = \beta_3 = 0$; $\gamma_1 = \gamma_4 = 0$.

$$
\begin{aligned}
-a_1^2a_2a_4^2 &\sin\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2a_3a_4^2 \sin\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2a_3^2a_4 \sin\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1a_2^2a_4^2 \cos\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&+ a_1^2a_2a_3a_4 \sin\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&+ a_1a_2^2a_3a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&+ a_1a_2a_3^2a_4 \cos\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&+ a_1a_2a_3a_4^2 \cos\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
= 0.&
\end{aligned}
$$

In the isotropic case ($a_1 = a_2 = a_3 = a_4$) we recover again:

$$\sin(\lambda\theta_4) = 0 \implies \lambda_n = \frac{\pi n}{\theta_4}, \qquad n = 0, 1, 2\dots\ .$$

Corollary: $\lambda = \frac{\pi}{\omega_0} > 1$, if $0 < \omega_0 < \pi$.

3. Mixed problem: $\alpha_1 = \gamma_4 = 1$, $\alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0$; $\gamma_1 = 0$.

$$
\begin{aligned}
a_1^2a_3^2 &\sin\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2a_3a_4 \sin\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1^2a_2a_3 \sin\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1^2a_2a_4 \sin\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1a_2a_3^2 \cos\lambda\omega_1 \cos\lambda\omega_2 \sin\lambda\omega_3 \sin\lambda\omega_4 \\
&- a_1a_2^2a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \sin\lambda\omega_3 \cos\lambda\omega_4 \\
&- a_1a_2^2a_4 \cos\lambda\omega_1 \sin\lambda\omega_2 \cos\lambda\omega_3 \sin\lambda\omega_4 \\
&+ a_1a_2a_3a_4 \cos\lambda\omega_1 \cos\lambda\omega_2 \cos\lambda\omega_3 \cos\lambda\omega_4 \\
= 0.&
\end{aligned}
$$

In the isotropic case $(a_1 = a_2 = a_3 = a_4)$ we hence recover:

$$\cos(\lambda\theta_4) = 0 \implies \lambda_n = \frac{\pi(2n-1)}{2\theta_4}, \qquad n = 1, 2 \ldots .$$

COROLLARY: $\lambda = \frac{\pi}{2\omega_0} > 1$, if $0 < \omega_0 < \frac{\pi}{2}$.

4. ROBIN PROBLEM: $\alpha_1 = \alpha_4 = 1$.

   In the isotropic case $(a_1 = a_2 = a_3 = a_4 = 1;\ \beta_1 = \beta_2 = \beta_3 = 0)$ we obtain:

$$\tan(\lambda\omega_0) = \frac{\lambda(\gamma_4 - \gamma_1)}{\lambda^2 + \gamma_1\gamma_4}.$$

1.3. Three media transmission problem

Our goal is the derivation of the eigenvalues equation that corresponds to our transmission problem for the case $N = 3$. Let S^1 be the unit circle in $\mathbb{R}^2$ centered at $\mathcal{O}$. We denote: $\Omega_i = G_i \cap S^1$; $i = 1, 2, 3$. The eigenvalue problem is the following one:

$$\begin{cases} \psi_i'' + \lambda^2\psi_i(\omega) = 0, & \omega \in \Omega_i\,;\ (i = 1, 2, 3); \\ \psi_1(\omega_1) = \psi_2(\omega_1);\ \psi_3(\theta_2) = \psi_2(\theta_2); \\ a_2\psi_2'(\omega_1) - a_1\psi_1'(\omega_1) + \beta_1\psi_1(\omega_1) = 0; \\ a_3\psi_3'(\theta_2) - a_2\psi_2'(\theta_2) + \beta_2\psi_2(\theta_2) = 0; \\ \alpha_1 a_1\psi_1'(0) + \gamma_1\psi_1(0) = 0; \\ \alpha_3 a_3\psi_3'(\theta_3) + \gamma_3\psi_3(\theta_3) = 0. \end{cases} \tag{1.3}$$

We find a general solution of (1.3):

$$\psi_i(\omega) = A_i\cos(\lambda\omega) + B_i\sin(\lambda\omega) \qquad \text{for } i = 1, 2, 3,$$

where A_i, B_i $(i = 1, 2, 3)$ are arbitrary constants. From the boundary condition of (1.3) we obtain the homogenous algebraic system of six linear equations determining A_i , B_i $(i = 1, 2, 3)$. The determinant of the system must be equal to zero for a nontrivial solution of this system to exist. The latter gives the required eigenvalues λ-equation:

$$
\begin{aligned}
&[\lambda^2\alpha_3 a_3^2(\beta_1\gamma_1+\lambda^2\alpha_1 a_1^2)-\gamma_3(\beta_1\beta_2\gamma_1+\lambda^2\alpha_1\beta_2 a_1^2-\lambda^2\gamma_1 a_2^2)]\\
&\quad\times\sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\quad+\lambda a_1\cdot[\lambda^2\alpha_3 a_3^2(\gamma_1-\beta_1\alpha_1)+\gamma_3(\beta_1\beta_2\alpha_1-\gamma_1\beta_2-\lambda^2\alpha_1 a_2^2)]\\
&\quad\times\cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\quad-\lambda a_3\cdot[\gamma_3(\beta_1\gamma_1+\lambda^2\alpha_1 a_1^2)+\alpha_3(\beta_1\beta_2\gamma_1+\lambda^2\alpha_1\beta_2 a_1^2-\lambda^2\gamma_1 a_2^2)]\\
&\quad\times\sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\quad+\lambda^2 a_1 a_3\cdot[\gamma_3(\beta_1\alpha_1-\gamma_1)+\alpha_3(\beta_1\beta_2\alpha_1-\gamma_1\beta_2-\lambda^2\alpha_1 a_2^2)]\\
&\quad\times\cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\quad-\lambda a_2\cdot[\gamma_3(\beta_2\gamma_1+\lambda^2\alpha_1 a_1^2+\beta_1\gamma_1)-\lambda^2\alpha_3\gamma_1 a_3^2]\\
&\quad\times\sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\quad+\lambda^2 a_1 a_2\cdot[\gamma_3(\beta_2\alpha_1+\alpha_1\beta_1-\gamma_1)-\lambda^2\alpha_3\alpha_1 a_3^2]\\
&\quad\times\cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\quad-\lambda^2 a_2 a_3\cdot[\gamma_1\gamma_3+\alpha_3(\beta_2\gamma_1+\lambda^2\alpha_1 a_1^2+\beta_1\gamma_1)]\\
&\quad\times\sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\quad+\lambda^3 a_1 a_2 a_3\cdot[\alpha_1\gamma_3+\alpha_3(\beta_2\alpha_1+\alpha_1\beta_1-\gamma_1)]\\
&\quad\times\cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&=0.
\end{aligned}
\tag{1.4}
$$

We consider special cases of boundary conditions.

1. DIRICHLET PROBLEM: $\alpha_1=\alpha_3=\beta_1=\beta_2=0$; $\gamma_1=\gamma_3=1$.

$$
\begin{aligned}
&a_1 a_3\cdot\cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad+a_1 a_2\cdot\cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\qquad+a_2 a_3\cdot\sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad-a_2^2\cdot\sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
&=0.
\end{aligned}
$$

In the isotropic case ($a_1=a_2=a_3$) we obtain the well known result:

$$\sin(\lambda\theta_3)=0 \implies \lambda_n=\frac{\pi n}{\theta_3}, \qquad n=1,2,\dots\ .$$

COROLLARY: $\lambda=\frac{\pi}{\theta_3}>1$, if $\theta_3=\omega_1+\omega_2+\omega_3<\pi$.

2. NEUMANN PROBLEM: $\beta_1=\beta_2=\gamma_1=\gamma_3=0$; $\alpha_1=\alpha_3=1$.

$$
\begin{aligned}
&a_2^2\cdot\cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad+a_2 a_3\cdot\cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3)\\
&\qquad+a_1 a_2\cdot\sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3)\\
&\qquad-a_1 a_3\cdot\sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3)\\
&=0.
\end{aligned}
$$

In the isotropic case ($a_1=a_2=a_3$) we hence obtain:

$$\sin(\lambda\theta_3) = 0 \implies \lambda_n = \frac{\pi n}{\theta_3}, \qquad n = 0, 1, 2, \dots .$$

COROLLARY: $\lambda = \frac{\pi}{\theta_3} > 1$, if $\theta_3 = \omega_1 + \omega_2 + \omega_3 < \pi$.

3. MIXED PROBLEM: $\alpha_1 = \gamma_3 = 1$, $\alpha_3 = \beta_1 = \beta_2 = \gamma_1 = 0$.

$$\begin{aligned} a_2^2 \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) & \\ + a_1a_3 \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) & \\ + a_1a_2 \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - a_2a_3 \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) & \\ = 0. & \end{aligned}$$

In the isotropic case ($a_1 = a_2 = a_3$):

$$\cos(\lambda\theta_3) = 0 \implies \lambda_n = \frac{\pi(2n-1)}{2\theta_3}, \qquad n = 1, 2, \dots .$$

COROLLARY: $\lambda = \frac{\pi}{2\theta_3} > 1$, if $\theta_3 = \omega_1 + \omega_2 + \omega_3 < \frac{\pi}{2}$.

4. ROBIN PROBLEM: $\alpha_1 = 1$, $\alpha_3 = 1$; $\beta_1 = \beta_2 = 0$.

$$\begin{aligned} (\lambda^2 a_1^2 a_3^2 + \gamma_1\gamma_3 a_2^2) \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - \lambda \cdot (\gamma_3 a_1 a_2^2 - \gamma_1 a_1 a_3^2) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - \lambda a_3 \cdot (\gamma_3 a_1^2 - \gamma_1 a_2^2) \cdot \sin(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) & \\ - a_1 a_3(\gamma_1\gamma_3 + \lambda^2 a_2^2) \cdot \cos(\lambda\omega_1)\sin(\lambda\omega_2)\cos(\lambda\omega_3) & \\ - \lambda a_2 \cdot (\gamma_3 a_1^2 - \gamma_1 a_3^2) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - a_1 a_2 \cdot (\gamma_1\gamma_3 + \lambda^2 a_3^2) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\sin(\lambda\omega_3) & \\ - a_2 a_3 \cdot (\gamma_1\gamma_3 + \lambda^2 a_1^2) \cdot \sin(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) & \\ + \lambda a_1 a_2 a_3 \cdot (\gamma_3 - \gamma_1) \cdot \cos(\lambda\omega_1)\cos(\lambda\omega_2)\cos(\lambda\omega_3) & \\ = 0. & \end{aligned}$$

In the isotropic case ($a_1 = a_2 = a_3 = 1$) we recover (see [3], §10.1.7, Example 1):

$$\tan(\lambda\theta_3) = \frac{\lambda(\gamma_3 - \gamma_1)}{\lambda^2 + \gamma_1\gamma_3}.$$

1.4. Two media transmission problem

Here we consider in detail 2-dimensional transmission problem with two different media ($\omega_1 = \omega_2 = \frac{\omega_0}{2}$) for the Laplace operator in an angular symmetric domain and investigate the corresponding eigenvalue problem. Suppose $n = 2$, the domain G lies inside the angle

$$G_0 = \left\{(r,\omega) \,\middle|\, r > 0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}\right\}, \qquad \omega_0 \in]0, 2\pi[\,;$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_\pm = \left\{(r,\omega) \mid r > 0;\ \omega = \pm\frac{\omega_0}{2}\right\}, \qquad \Sigma_0 = \{(r,\omega) \mid r > 0;\ \omega = 0\}$$

and we put

$$\beta(\omega)\big|_{\Sigma_0} = \beta(0) = \beta = \text{const} \geq 0, \qquad \gamma(\omega)\big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = \text{const} > 0.$$

We consider the following problem:

$$\begin{cases} a_\pm \triangle u_\pm = f_\pm(x), & x \in G_\pm; \\ [u]_{\Sigma_0} = 0; & \\ \left[a\dfrac{\partial u}{\partial \vec{n}}\right]_{\Sigma_0} + \dfrac{1}{|x|}\beta u(x) = h(x), & x \in \Sigma_0; \\ \alpha_\pm a_\pm \dfrac{\partial u_\pm}{\partial \vec{n}} + \dfrac{1}{r}\gamma_\pm u_\pm(x) = g_\pm(x), & x \in \Gamma_\pm \setminus \mathcal{O}. \end{cases} \tag{1.5}$$

It is well known that the homogeneous problem ($f(x) = h(x) = g(x) = 0$) has solution of the form $u(r,\omega) = r^\lambda \psi(\omega)$, where λ is an eigenvalue and $\psi(\omega)$ is the corresponding eigenfunction of the problem

$$\begin{cases} \psi_+'' + \lambda^2 \psi_+(\omega) = 0, & \text{for } \omega \in (0, \frac{\omega_0}{2}); \\ \psi_-'' + \lambda^2 \psi_-(\omega) = 0, & \text{for } \omega \in (-\frac{\omega_0}{2}, 0); \\ \psi_+(0) = \psi_-(0); & \\ a_+\psi_+'(0) - a_-\psi_-'(0) = \beta\psi(0); & \\ \pm\alpha_\pm a_\pm \psi'\left(\pm\dfrac{\omega_0}{2}\right) + \gamma_\pm \psi\left(\pm\dfrac{\omega_0}{2}\right) = 0. & \end{cases} \tag{1.6}$$

The case $\lambda = 0$

In this case the solution of our equations has the form

$$\psi_\pm(\omega) = A_\pm \cdot \omega + B_\pm\,.$$

From the boundary conditions we obtain $B_+ = B_- = B$ and to find A_+, A_-, B, we have the system

$$\begin{cases} a_+A_+ - a_-A_- - \beta B = 0, \\ \left(\alpha_+ a_+ + \dfrac{\omega_0}{2}\gamma_+\right) A_+ + \gamma_+ B = 0, \\ -\left(\alpha_- a_- + \dfrac{\omega_0}{2}\gamma_-\right) A_- + \gamma_- B = 0. \end{cases}$$

Since $A_+^2 + A_-^2 + B^2 \neq 0$, the determinant must be equal to zero; this means

$$\beta\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right)\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right) + a_+\gamma_+\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right) \\ + a_-\gamma_-\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right) \\ = 0. \tag{1.7}$$

Thus if the equality (1.7) is satisfied, then $\lambda = 0$ and the corresponding eigenfunctions are

$$\psi(\omega) = \begin{cases} a_-\gamma_-\left\{\left(\omega - \frac{\omega_0}{2}\right)\gamma_+ - \alpha_+a_+\right\}, & \omega \in (0, \frac{\omega_0}{2}); \\ a_+\gamma_+\left\{\left(\omega + \frac{\omega_0}{2}\right)\gamma_- - \alpha_-a_-\right\}, & \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \quad \text{if } \beta = 0;$$

$$\psi(\omega) = \begin{cases} -\gamma_+\left(\alpha_-a_- + \frac{\omega_0\gamma_-}{2}\right)(\omega + \frac{a_+}{\beta}) - \frac{a_-\gamma_-}{\beta}\left(\alpha_+a_+ + \frac{\omega_0\gamma_+}{2}\right), \\ \qquad\qquad \omega \in (0, \frac{\omega_0}{2}); \\ \gamma_-\left(\alpha_+a_+ + \frac{\omega_0\gamma_+}{2}\right)(\omega - \frac{a_-}{\beta}) - \frac{a_+\gamma_+}{\beta}\left(\alpha_-a_- + \frac{\omega_0\gamma_-}{2}\right), \\ \qquad\qquad \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \quad \text{if } \beta \neq 0.$$

The case $\lambda \neq 0$

In this case the solution of our equations has the form

$$\psi_\pm(\omega) = A_\pm\cos(\lambda\omega) + B_\pm\sin(\lambda\omega).$$

From the boundary conditions we obtain $A_+ = A_- = A$ and to find A, B_+, B_- we have the system

$$\begin{cases} \beta A - \lambda a_+B_+ + \lambda a_-B_- = 0, \\ \left(\gamma_+\cos\frac{\lambda\omega_0}{2} - \lambda\alpha_+a_+\sin\frac{\lambda\omega_0}{2}\right)A + \left(\gamma_+\sin\frac{\lambda\omega_0}{2} + \lambda\alpha_+a_+\cos\frac{\lambda\omega_0}{2}\right)B_+ = 0, \\ \left(\gamma_-\cos\frac{\lambda\omega_0}{2} - \lambda\alpha_-a_-\sin\frac{\lambda\omega_0}{2}\right)A - \left(\gamma_-\sin\frac{\lambda\omega_0}{2} + \lambda\alpha_-a_-\cos\frac{\lambda\omega_0}{2}\right)B_- = 0. \end{cases}$$

Since $A^2 + B_+^2 + B_-^2 \neq 0$, the determinant must be zero; this means that λ is defined by the transcendental equation

$$\beta(\lambda^2\alpha_+\alpha_-a_+a_- + \gamma_+\gamma_-) + \lambda^2(a_+ - a_-)(\alpha_-a_-\gamma_+ - \alpha_+a_+\gamma_-) \\ + \lambda[\beta(\alpha_-a_-\gamma_+ + \alpha_+a_+\gamma_-) \\ + (a_+ + a_-)(\gamma_+\gamma_- - \lambda^2\alpha_+\alpha_-a_+a_-)]\sin(\lambda\omega_0) \\ + [\beta(\lambda^2\alpha_+\alpha_-a_+a_- - \gamma_+\gamma_-) \\ + \lambda^2(a_+ + a_-)(\alpha_-a_-\gamma_+ + \alpha_+a_+\gamma_-)]\cos(\lambda\omega_0) \\ = 0. \tag{1.8}$$

Now we investigate special cases of the boundary conditions.

1. THE DIRICHLET PROBLEM: $\alpha\pm = 0$. Equation (1.8) takes the form

$$\beta(1-\cos(\lambda\omega_0)) + \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0.$$

Hence we get

$$\lambda = \begin{cases} \dfrac{\pi}{\omega_0}, & \text{if } \beta = 0; \\ \text{the least positive root of } \tan\dfrac{\lambda\omega_0}{2} = -\dfrac{a_+ + a_-}{\beta}\cdot\lambda, & \text{if } \beta \neq 0 \end{cases}$$

and the corresponding eigenfunction is

$$\psi(\omega) = \begin{cases} \sin\lambda\left(\dfrac{\omega_0}{2} - \omega\right), & \omega \in (0, \frac{\omega_0}{2}); \\ \sin\lambda\left(\dfrac{\omega_0}{2} + \omega\right), & \omega \in (-\frac{\omega_0}{2}, 0). \end{cases}$$

2. THE NEUMANN PROBLEM: $\gamma\pm = 0$. Equation (1.8) takes the form

$$\beta(1+\cos(\lambda\omega_0)) - \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0.$$

Hence we get $\lambda = \min\{\lambda^*, \frac{\pi}{\omega_0}\}$, where λ^* is the least positive root of the transcendental equation

$$\tan\frac{\lambda\omega_0}{2} = \frac{\beta}{a_+ + a_-}\cdot\frac{1}{\lambda}.$$

We find the corresponding eigenfunctions

$$\psi(\omega) = \begin{cases} a_-\sin\dfrac{\pi\omega}{\omega_0}, & \omega \in (0, \frac{\omega_0}{2}); \\ a_+\sin\dfrac{\pi\omega}{\omega_0}, & \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \qquad \lambda = \frac{\pi}{\omega_0};$$

$$\psi(\omega) = \begin{cases} \cos\lambda^*\left(\omega - \dfrac{\omega_0}{2}\right), & \omega \in (0, \frac{\omega_0}{2}); \\ \cos\lambda^*\left(\omega + \dfrac{\omega_0}{2}\right), & \omega \in (-\frac{\omega_0}{2}, 0), \end{cases} \qquad \lambda = \lambda^*.$$

3. MIXED PROBLEM: $\alpha_+ = 1$, $\alpha_- = 0$; $\gamma_+ = 0$, $\gamma_- = 1$. Equation (1.8) takes the form

$$\beta\sin(\lambda\omega_0) + \lambda(a_+ + a_-)\cos(\lambda\omega_0) = \lambda(a_+ - a_-). \tag{1.9}$$

In particular, if $\beta = 0$, then

$$\lambda = \frac{2}{\omega_0}\arctan\sqrt{\frac{a_-}{a_+}} > 1, \qquad \text{if } \omega_0 < 2\arctan\sqrt{\frac{a_-}{a_+}}$$

as $a_+a_- > 0$; and the corresponding eigenfunction is

$$\psi(\omega) = \begin{cases} \cos(\lambda\omega) + \sqrt{\dfrac{a_-}{a_+}} \cdot \sin(\lambda\omega), & \omega \in (0, \frac{\omega_0}{2}); \\ \cos(\lambda\omega) + \sqrt{\dfrac{a_+}{a_-}} \cdot \sin(\lambda\omega), & \omega \in (-\frac{\omega_0}{2}, 0). \end{cases}$$

If λ is the least positive root of the transcendental equation (1.9), then we find the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} \sin\dfrac{\lambda\omega_0}{2} \cos\lambda\left(\omega - \dfrac{\omega_0}{2}\right), & \omega \in (0, \frac{\omega_0}{2}); \\ \cos\dfrac{\lambda\omega_0}{2} \sin\lambda\left(\omega + \dfrac{\omega_0}{2}\right), & \omega \in (-\frac{\omega_0}{2}, 0). \end{cases}$$

4. The Robin problem: $\alpha_\pm = 1$; $\gamma_\pm \neq 0$. Equation (1.8) takes the form

$$\begin{aligned} &\beta(\lambda^2 a_+ a_- + \gamma_+\gamma_-) + \lambda^2(a_+ - a_-)(a_-\gamma_+ - a_+\gamma_-) \\ &\qquad + \lambda[\beta(a_-\gamma_+ + a_+\gamma_-) + (a_+ + a_-)(\gamma_+\gamma_- - \lambda^2 a_+ a_-)]\sin(\lambda\omega_0) \\ &\qquad + [\beta(\lambda^2 a_+ a_- - \gamma_+\gamma_-) + \lambda^2(a_+ + a_-)(a_-\gamma_+ + a_+\gamma_-)]\cos(\lambda\omega_0) \\ &= 0. \end{aligned}$$

In particular, in the case of the problem without the interface ($a_+ = a_- = 1$, $\beta = 0$) we obtain the least eigenvalue as the least positive root of the transcendental equation

$$\tan(\lambda\omega_0) = \frac{\lambda(\gamma_+ + \gamma_-)}{\lambda^2 - \gamma_+\gamma_-} \tag{1.10}$$

and the corresponding eigenfunction is

$$\psi(\omega) = \lambda\cos\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right] - \gamma_+ \sin\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right]$$

(see [3], §10.1.7).

In order to have $\lambda > 1$ we show that the condition $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2}$ from the assumption (b) of our Theorem is satisfied. In fact, we rewrite the equation (1.10) in the equivalent form $\lambda = \frac{1}{\omega_0}\left(\arctan\frac{\gamma_+}{\lambda} + \arctan\frac{\gamma_-}{\lambda}\right)$. It follows that

$$1 < \lambda < \frac{1}{\omega_0}(\arctan\gamma_+ + \arctan\gamma_-) \Longrightarrow \omega_0 < \arctan\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-}, \quad \text{provided that } \gamma_+\gamma_- < 1 \tag{1.11}$$

has to be fulfilled. But our condition from the assumption (b) means that $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2}$. Hence we obtain

$$\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-} \geq \frac{2\gamma_0}{1 - \gamma_0^2} > \frac{2\tan\frac{\omega_0}{2}}{1 - \tan^2\frac{\omega_0}{2}} = \tan\omega_0, \qquad \omega_0 < \frac{\pi}{2}.$$

Thus we established (1.11). In the case $\gamma_\pm \ge \gamma_0 > \tan\frac{\omega_0}{2} \ge 1$ for $\omega_0 \in [\frac{\pi}{2}, \pi)$ the inequality $\lambda > 1$ is fulfilled a fortiori, because of the property of the monotonic increase of the eigenvalues together with the increase of $\gamma(\omega)$ (see for example [4], chapter VI, §2, Theorem 6). In fact, $\lambda = 1$ is the solution of the equation (1.10) under assumption $\gamma_\pm = \tan\frac{\omega_0}{2}$.

2. Problem (L)

We assume that $G = G_+ \cup G_- \cup \Sigma_0$ is divided into two subdomains G_+ and G_- by a hyperplane $\Sigma_0 = G \cap \{x_n = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$. We assume also that $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known and, without loss of generality, that there exists $d > 0$ such that G_0^d is *a rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0, 2\pi)$, thus

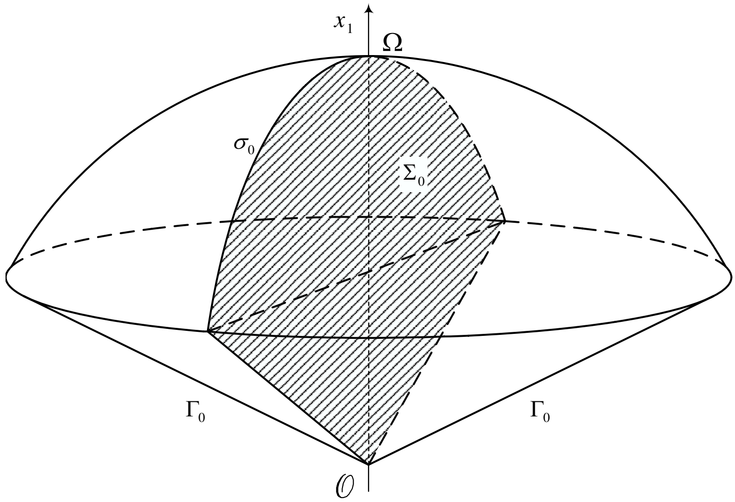

Fig. 3

$$\Gamma_0^d = \left\{ (r,\omega) \,\middle|\, x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^{n} x_i^2\,;\ r \in (0,d),\ \omega_1 = \frac{\omega_0}{2} \right\}.$$

Definition 2

A function $u(x)$ is called a *weak* solution of the problem (L) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G)$ and satisfies the integral identity

$$\int_G \left\{ a^{ij}(x)u_{x_j}\eta_{x_i} - a^i(x)u_{x_i}\eta(x) - a(x)u\eta(x) \right\} dx$$

$$+\int\limits_{\Sigma_0}\frac{\beta(\omega)}{r}u(x)\eta(x)\,ds+\int\limits_{\partial G}\frac{\gamma(\omega)}{r}u(x)\eta(x)\,ds$$
$$=\int\limits_{\partial G}g(x)\eta(x)\,ds+\int\limits_{\Sigma_0}h(x)\eta(x)\,ds-\int\limits_{G}f(x)\eta(x)\,dx$$

for all functions $\eta(x)\in \mathbf{C}^0(\overline{G})\cap \overset{\circ}{\mathbf{W}}{}_0^1$.

Regarding the equation we assume that the following conditions are satisfied:

(a) *the condition of the uniform ellipticity:*

$$\nu_\pm\xi^2\le a_\pm^{ij}(x)\xi_i\xi_j\le \mu_\pm\xi^2,\quad \forall x\in\overline{G_\pm},\quad \forall\xi\in\mathbb{R}^n;$$
$$\nu_\pm,\mu_\pm=\text{const}>0,\ \textit{and}\ a_\pm^{ij}(0)=a\delta_i^j\,,$$

where δ_i^j *is the Kronecker symbol and*

$$a=\begin{cases}a_+\,, & x\in G_+\,;\\ a_-\,, & x\in G_-\,,\end{cases}$$

with positive constants $a_\pm$*; we denote*

$$a_*=\min\{a_+,\,a_-\}>0,\qquad a^*=\max\{a_+,\,a_-\}>0;$$
$$\nu_*=\min\{\nu_-,\nu_+\};\qquad \mu^*=\max(\mu_-,\mu_+);$$

(b) $a^{ij}(x)\in \mathbf{C}^0(\overline{G})$, $a^i(x)\in \mathbf{L}_p(G)$, $a(x),f(x)\in \mathbf{L}_{\frac{p}{2}}(G)\cap\mathbf{L}_2(G)$; $p>n$. *The inequalities*

$$\Big(\sum_{i,j=1}^n|a_\pm^{ij}(x)-a_\pm^{ij}(y)|^2\Big)^{\frac12}\le a_\pm\mathcal{A}(|x-y|);$$
$$|x|\Big(\sum_{i=1}^n|a_\pm^{i}(x)|^2\Big)^{\frac12}+|x|^2|a_\pm(x)|\le a_\pm\mathcal{A}(|x|)$$

hold for $x,y\in\overline{G}$, *Here* $\mathcal{A}(r)$ *is a monotonically increasing, nonnegative function, continuous at* 0 *with* $\mathcal{A}(0)=0$*;*

(c) $a(x)\le 0$ *in* G*;* $\beta(\omega)\ge\nu_0>0$ *on* σ_0*;* $\gamma(\omega)\ge\nu_0>0$ *on* ∂G*;*

(d) *there exist numbers* $f_1\ge0$, $g_1\ge0$, $h_1\ge0$, $s>1$, $\beta\ge s-2$ *such that*

$$|f(x)|\le f_1|x|^\beta,\qquad |g(x)|\le g_1|x|^{s-1},\qquad |h(x)|\le h_1|x|^{s-1},$$

$\gamma(\omega)$ *is a positive bounded piecewise smooth function on* $\partial\Omega$, $\sigma(\omega)$ *is a positive continuous function on* σ_0*;*

(aa) $\left|\sum\limits_{i=1}^{n} \frac{\partial a^{ij}(x)}{\partial x_i}\right| \le K$ *for all* $j = 1, \dots, n$.

Our main result is the following theorem.

THEOREM 2
Let u be a weak solution of the problem (L), the assumptions (a)-(d), (aa) *are satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Let λ be as in* (1.1)*; $N = 2$. Then there are $d \in (0,1)$ and constants $C > 0$, $c > 0$ depending only on n, ν_*, μ^*, p, λ, $\left\|\sum_{i=1}^{n} |a^i(x)|^2\right\|_{\mathbf{L}_{\frac{p}{2}}(G)}$, K, ω_0, f_1, h_1, g_1, ν_0, s, M_0,* $\operatorname{meas} G$, $\operatorname{diam} G$ *and on the quantity $\int_0^1 \frac{\mathcal{A}(r)}{r}dr$ such that for all $x \in G_0^d$*

$$|u(x)| \le C_0 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1 + \frac{1}{\sqrt{\sigma_0}} h_1 \right) \cdot \begin{cases} |x|^{\lambda}, & \text{if } s > \lambda; \\ |x|^{\lambda} \ln^c \left(\frac{1}{|x|} \right), & \text{if } s = \lambda; \\ |x|^{s}, & \text{if } s < \lambda. \end{cases}$$

Suppose, in addition, that

$$a^{ij}(x) \in \mathbf{C}^1(G), \qquad \sigma(\omega) \in C^1(\sigma_0), \qquad \gamma(\omega) \in \mathbf{C}^1(\partial G),$$
$$f(x) \in \mathbf{V}^0_{p,2p-n}(G), \qquad h(x) \in V^{1-\frac{1}{p}}_{p,2p-n}(\sigma_0), \qquad g(x) \in \mathbf{V}^{1-\frac{1}{p}}_{p,2p-n}(\partial G);$$

$p > n$ and there is a number

$$\tau_s =: \sup_{\varrho > 0} \varrho^{-s} \Big(\|h\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Sigma^{\varrho}_{\varrho/2})} + \|g\|_{\mathbf{V}^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^{\varrho}_{\varrho/2})} \Big).$$

Then for all $x \in G_0^d$

$$|\nabla u(x)| \le C_1 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1 + \frac{1}{\sqrt{\sigma_0}} h_1 + \tau_s \right)$$
$$\times \begin{cases} |x|^{\lambda-1}, & \text{if } s > \lambda; \\ |x|^{\lambda-1} \ln^c \left(\frac{1}{|x|} \right), & \text{if } s = \lambda; \\ |x|^{s-1}, & \text{if } s < \lambda. \end{cases}$$

Furthermore, $u \in \mathbf{V}^2_{p,2p-n}(G)$ and

$$\|u\|_{\mathbf{V}^2_{p,2p-n}(G_0^{\varrho})} \le C_2 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1 + \frac{1}{\sqrt{\sigma_0}} h_1 + \tau_s \right)$$
$$\times \begin{cases} \varrho^{\lambda}, & \text{if } s > \lambda; \\ \varrho^{\lambda} \ln^c \left(\frac{1}{\varrho} \right), & \text{if } s = \lambda; \\ \varrho^{s}, & \text{if } s < \lambda. \end{cases}$$

3. Problem $(\boldsymbol{WL})$

We consider problem (WL) that is the transmission problem for a quasi-linear equation with semi-linear principal part.

DEFINITION 3
The function $u(x)$ is called a *weak* solution of the problem (WL) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^1(G)$ and satisfies the integral identity

$$\int_G \left\{|u|^q a^{ij}(x) u_{x_j} \eta_{x_i} + b(x,u,u_x)\eta(x)\right\} dx$$
$$+ \int_{\Sigma_0} \frac{\beta(\omega)}{r} u|u|^q \eta(x)\, ds + \int_{\partial G} \frac{\gamma(\omega)}{r} u|u|^q \eta(x)\, ds$$
$$= \int_{\partial G} g(x,u)\eta(x)\, ds + \int_{\Sigma_0} h(x,u)\eta(x)\, ds$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^1(G)$.

Regarding the equation we assume that the following conditions are satisfied.

Let $q \geq 0$, $0 \leq \mu < q+1$, $s > 1$, $f_1 \geq 0$, $g_1 \geq 0$, $h_1 \geq 0$, $\beta \geq s-2$ be given numbers.

(a) *The condition of the uniform ellipticity:*

$$a_\pm \xi^2 \leq a_\pm^{ij}(x)\xi_i\xi_j \leq A_\pm \xi^2, \quad \forall x \in \overline{G_\pm}, \quad \forall \xi \in \mathbb{R}^n;$$
$$a_\pm, A_\pm = const > 0,\ a_\pm^{ij}(0) = a\delta_i^j\,,$$

where δ_i^j is the Kronecker symbol;

$$a = \begin{cases} a_+\,, & x \in G_+\,; \\ a_-\,, & x \in G_-\,; \end{cases}$$

$$\begin{cases} a_* = \min\{a_+,\, a_-\} > 0; \\ a^* = \max\{a_+,\, a_-\} > 0; \\ A^* = \max(A_-, A_+). \end{cases}$$

(b) $a_\pm^{ij}(x) \in \mathbf{C}^0(\overline{G})$ *and the inequality*

$$\left(\sum_{i,j=1}^n |a_\pm^{ij}(x) - a_\pm^{ij}(y)|^2\right)^{\frac{1}{2}} \leq \mathcal{A}(|x-y|)$$

holds for $x, y \in \overline{G}$, *where* $\mathcal{A}(r)$ *is a monotonically increasing, nonnegative function, continuous at* 0 *with* $\mathcal{A}(0) = 0$.

(c) $|b(x, u, u_x)| \le a\mu|u|^{q-1}|\nabla u|^2 + b_0(x)$; $0 \le \mu < 1 + q$, $b_0(x) \in L_{p/2}(G)$, $n < p < 2n$.

(d) $\beta(\omega) \ge \nu_0 > 0$ *on* σ_0; $\gamma(\omega) \ge \nu_0 > 0$ *on* ∂G.

(e) $\frac{\partial h(x,u)}{\partial u} \le 0$, $\frac{\partial g(x,u)}{\partial u} \le 0$.

(f) $|b_0(x)| \le f_1|x|^\beta$, $|g(x, 0)| \le g_1|x|^{s-1}$, $|h(x, 0)| \le h_1|x|^{s-1}$.

We assume without loss of generality that there exists $d > 0$ such that G_0^d is *a rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0, 2\pi)$.

Our main result is the following statement.

Theorem 3

Let u *be a weak solution of the problem* (WL), *the assumptions* (a)-(f) *are satisfied with* $\mathcal{A}(r)$ *Dini-continuous at zero. Let us assume that* $M_0 = \max_{x\in\overline{G}} |u(x)|$ *is known. Let* λ *be as in* (1.1) *for* $N = 2$. *Then there are* $d \in (0, 1)$ *and constants* $C_0 > 0$, $c > 0$ *depending only on* n, a_*, A^*, p, q, λ, μ, f_1, h_1, g_1, ν_0, s, M_0, $\operatorname{meas} G$, $\operatorname{diam} G$ *and on the quantity* $\int_0^1 \frac{\mathcal{A}(r)}{r}\,dr$ *such that for all* $x \in G_0^d$

$$|u(x)| \le C_0 \begin{cases} |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}}, & \text{if } s > \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}} \ln^c\left(\dfrac{1}{|x|}\right), & \text{if } s = \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{s}{q+1}}, & \text{if } s < \lambda\dfrac{1+q-\mu}{1+q}. \end{cases}$$

Suppose, in addition, that coefficients of the problem (WL) *satisfy such conditions, which guarantee the local estimate* $|\nabla u|_{0,G'} \le M_1$ *for any smooth* $G' \subset \overline{G} \setminus \{\mathcal{O}\}$ *(see for example [1], §4). Then for all* $x \in G_0^d$

$$|\nabla u(x)| \le C_1 \begin{cases} |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}-1}, & \text{if } s > \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}-1} \ln^c\left(\dfrac{1}{|x|}\right), & \text{if } s = \lambda\dfrac{1+q-\mu}{1+q}; \\ |x|^{\frac{s}{q+1}-1}, & \text{if } s < \lambda\dfrac{1+q-\mu}{1+q} \end{cases}$$

with $C_1 = c_1\big(\|u\|_{2(q+1),G} + f_1 + g_1 + h_1\big)$, *where* c_1 *depends on* M_0, M_1 *and* C_0 *from above.*

The idea of the proofs of Theorems 1-3 is based on the deduction of a new inequality of the Friedrichs–Wirtinger type with the *exact* constant as well as other integral-differential inequalities adapted to the transmission problem. The precise exponent of the solution decrease rate depends on this exact constant. We obtain *the Friedrichs–Wirtinger type inequality* by the variational principle:

LEMMA 1
Let ϑ be the smallest positive eigenvalue of the problem (EVP). Let $\Omega \subset S^{n-1}$ be a bounded domain. Let $\psi \in \mathbf{W}^1(\Omega)$ and satisfy the boundary and conjunction conditions from (EVP) in the weak sense. Let $\gamma(\omega)$ be a positive bounded piecewise smooth function on $\partial\Omega$, $\beta(\omega)$ be a positive continuous function on σ_0. Then

$$\vartheta \int_\Omega a\psi^2(\omega)\,d\Omega \le \int_\Omega a|\nabla_\omega \psi(\omega)|^2\,d\Omega + \int_{\sigma_0} \beta(\omega)\psi^2(\omega)\,d\sigma + \int_{\partial\Omega} \alpha(x)\gamma(\omega)\psi^2(\omega)\,d\sigma.$$

LEMMA 2
Let G_0^d be the conical domain and $\nabla v(\varrho,\cdot) \in \mathbf{L}_2(\Omega)$ for almost all $\varrho \in (0,d)$. Assume that for almost all $\varrho \in (0,d)$

$$V(\rho) = \int_{G_0^\varrho} ar^{2-n}|\nabla v|^2\,dx + \int_{\Sigma_0^\varrho} r^{1-n}\beta(\omega)v^2(x)\,ds + \int_{\Gamma_0^\varrho} r^{1-n}\gamma(\omega)v^2(x)\,ds < \infty.$$

Then

$$\int_\Omega a\left(\varrho v\frac{\partial v}{\partial r} + \frac{n-2}{2}v^2\right)\bigg|_{r=\varrho} d\Omega \le \frac{\varrho}{2\lambda}V'(\varrho).$$

At last we derive a result that asserts *the local estimate at the boundary* (near the conical point) of the weak solution of problem (WL).

THEOREM 4
Let $u(x)$ be a weak solution of the problem (WL). Suppose that assumptions (a), (c)-(e) are satisfied. Suppose, in addition, that $h(x) \in L_\infty(\Sigma_0)$, $g(x) \in L_\infty(\partial G)$. Then the inequality

$$\sup_{G_0^{\varkappa\varrho}} |u(x)| \le \frac{C}{(1-\varkappa)^{\frac{n}{t}(q+1)}}\Big\{\varrho^{-\frac{n}{t}(q+1)}\|u\|_{t(q+1),G_0^\varrho} + \varrho^{\frac{2}{q+1}(1-\frac{n}{p})}\|b_0\|^{\frac{1}{q+1}}_{p/2,G_0^\varrho}$$
$$+ \varrho^{\frac{1}{q+1}}\left(\|g(x,0)\|^{\frac{1}{q+1}}_{\infty,\Gamma_0^\varrho} + \|h(x,0)\|^{\frac{1}{q+1}}_{\infty,\Sigma_0^\varrho}\right)\Big\}$$

holds for any $t > 0$, $\varkappa \in (0,1)$ and $\varrho \in (0,d)$, where $C = \mathrm{const}(n, a_*, A^*, t, p, \mu, G)$ and $d \in (0,1)$.

Examples

Here we consider two dimensional transmission problem for the Laplace operator with absorbtion term in an angular domain and investigate the corresponding eigenvalue problem. Suppose $n = 2$, and the domain G lies inside the angle

$$G_0 = \Big\{(r,\omega) \,\Big|\, r > 0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}\Big\}, \quad \omega_0 \in]0, 2\pi[\,;$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_\pm = \Big\{(r,\omega) \,\Big|\, r > 0;\ \omega = \pm\frac{\omega_0}{2}\Big\}, \qquad \Sigma_0 = \{(r,\omega) \mid r > 0;\ \omega = 0\}$$

and we put

$$\beta(\omega)\big|_{\Sigma_0} = \beta = \text{const} \geq 0, \qquad \gamma(\omega)\big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = \text{const} > 0.$$

We consider the following problem:

$$\begin{cases} \dfrac{d}{dx_i}\left(|u|^q u_{x_i}\right) = a_0 r^{-2} u|u|^q - \mu u|u|^{q-2}|\nabla u|^2, & x \in G_0 \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0; & \\ \left[a|u|^q \dfrac{\partial u}{\partial n}\right]_{\Sigma_0} + \dfrac{\beta}{|x|} u|u|^q = 0, & x \in \Sigma_0; \\ \alpha_\pm a_\pm |u_\pm|^q \dfrac{\partial u_\pm}{\partial n} + \dfrac{\gamma_\pm}{|x|} u_\pm |u_\pm|^q = 0, & x \in \Gamma_\pm \setminus \mathcal{O}, \end{cases}$$

where

$$a = \begin{cases} a_+\,, & x \in G_+\,; \\ a_-\,, & x \in G_-\,, \end{cases}$$

$a_\pm$ are positive constants; $a_0 \geq 0$, $0 \leq \mu < 1 + q$, $q \geq 0$; $\alpha_\pm \in \{0; 1\}$. We make the function change $u = v|v|^{\varsigma-1}$ with $\varsigma = \frac{1}{q+1}$ and consider our problem for the function $v(x)$:

$$\begin{cases} \triangle v + \mu\varsigma v^{-1}|\nabla v|^2 = a_0(1+q) r^{-2} v, \quad \varsigma = \dfrac{1}{1+q}, & x \in G_0 \setminus \Sigma_0; \\ [v]_{\Sigma_0} = 0; & \\ \left[a\dfrac{\partial v}{\partial n}\right]_{\Sigma_0} + (1+q)\beta \dfrac{v(x)}{|x|} = 0, & x \in \Sigma_0; \\ \alpha_\pm a_\pm \dfrac{\partial v_\pm}{\partial n} + (1+q)\gamma_\pm \dfrac{v_\pm(x)}{|x|} = 0, & x \in \Gamma_\pm \setminus \mathcal{O}. \end{cases}$$

We want to find the exact solution of this problem in the form $v(r,\omega) = r^{\varkappa}\psi(\omega)$. For $\psi(\omega)$ we obtain the problem

$$\begin{cases} \psi''(\omega) + \dfrac{\mu\varsigma}{\psi(\omega)}\psi'^2(\omega) + \left\{(1+\mu\varsigma)\varkappa^2 - a_0(1+q)\right\}\cdot\psi(\omega) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \omega \in (-\frac{\omega_0}{2}, 0) \cup (0, \frac{\omega_0}{2}); \\ [\psi]_{\omega=0} = 0; \\ [a\psi'(0)] = (1+q)\beta\psi(0); \\ \pm\alpha_\pm a_\pm \psi'_\pm\left(\pm\dfrac{\omega_0}{2}\right) + (1+q)\gamma_\pm\psi_\pm\left(\pm\dfrac{\omega_0}{2}\right) = 0. \end{cases}$$

We assume that $\varkappa^2 > a_0\frac{(1+q)^2}{1+q+\mu}$ and define the value $\Upsilon = \sqrt{\varkappa^2 - a_0\frac{(1+q)^2}{1+q+\mu}}$. We consider separately two cases: $\mu = 0$ and $\mu \neq 0$.

The case $\mu = 0$

In this case we get

$$\psi_\pm(\omega) = A\cos(\Upsilon\omega) + B_\pm\sin(\Upsilon\omega),$$

where constants A, $B_\pm$ should be determined from the conjunction and boundary conditions.

1. The Dirichlet problem: $\alpha_\pm = 0$, $\gamma_\pm \neq 0$. Direct calculations will give

$$\psi_\pm(\omega) = \cos(\Upsilon\omega) \mp \cot\left(\Upsilon\frac{\omega_0}{2}\right)\cdot\sin(\Upsilon\omega), \qquad \Upsilon = \begin{cases} \dfrac{\pi}{\omega_0}, & \text{if } \beta = 0; \\ \Upsilon^*, & \text{if } \beta \neq 0, \end{cases}$$

where Υ^* is the least positive root of the transcendental equation

$$\Upsilon\cdot\cot\left(\Upsilon\frac{\omega_0}{2}\right) = -\frac{1+q}{a_+ + a_-}\beta$$

and from the graphic solution we obtain $\frac{\pi}{\omega_0} < \Upsilon^* < \frac{2\pi}{\omega_0}$. The corresponding eigenfunctions are

$$\psi_\pm(\omega) = \begin{cases} \cos\left(\dfrac{\pi\omega}{\omega_0}\right), & \text{if } \beta = 0; \\ \cos(\Upsilon^*\omega) \mp \cot\left(\Upsilon^*\frac{\omega_0}{2}\right)\cdot\sin(\Upsilon^*\omega), & \text{if } \beta \neq 0. \end{cases}$$

2. The Neumann problem: $\alpha_\pm = 1$, $\gamma_\pm = 0$. Direct calculations give

$$\Upsilon = \begin{cases} \dfrac{\pi}{\omega_0}, & \text{if } \beta = 0; \\ \Upsilon^*, & \text{if } \beta \neq 0, \end{cases}$$

where Υ^* is the least positive root of the transcendental equation

$$\Upsilon \cdot \tan\left(\Upsilon \frac{\omega_0}{2}\right) = \frac{1+q}{a_+ + a_-}\beta$$

and from the graphic solution we obtain $0 < \Upsilon^* < \frac{\pi}{\omega_0}$. The corresponding eigenfunctions are

$$\psi_\pm(\omega) = \begin{cases} a_\mp \sin\left(\dfrac{\pi\omega}{\omega_0}\right), & \text{if } \beta = 0; \\ \cos(\Upsilon^*\omega) \pm \tan\left(\Upsilon^* \frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), & \text{if } \beta \neq 0. \end{cases}$$

3. Mixed problem: $\alpha_+ = 1$, $\alpha_- = 0$; $\gamma_+ = 0$, $\gamma_- = 1$. Direct calculations give: $\Upsilon = \Upsilon^*$, where Υ^* is the least positive root of the transcendental equation

$$a_+ \tan\left(\Upsilon \frac{\omega_0}{2}\right) - a_- \cot\left(\Upsilon \frac{\omega_0}{2}\right) = \frac{1+q}{\Upsilon}\beta.$$

The corresponding eigenfunctions are

$$\psi_+(\omega) = \cos(\Upsilon^*\omega) + \tan\left(\Upsilon^* \frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), \qquad \omega \in \left[0, \tfrac{\omega_0}{2}\right];$$
$$\psi_-(\omega) = \cos(\Upsilon^*\omega) + \cot\left(\Upsilon^* \frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), \qquad \omega \in \left[-\tfrac{\omega_0}{2}, 0\right].$$

4. The Robin problem: $\alpha_\pm = 1$, $\gamma_\pm \neq 0$. Direct calculations give:
   1) $\frac{\gamma_+}{\gamma_-} = \frac{a_+}{a_-} \Longrightarrow \psi_\pm(\omega) = a_\mp \sin(\Upsilon^*\omega)$, where Υ^* is the least positive root of the transcendental equation

   $$\Upsilon \cdot \cot\left(\Upsilon \frac{\omega_0}{2}\right) = -(1+q)\frac{\gamma_+}{a_+}$$

   and from the graphic solution we obtain $\frac{\pi}{\omega_0} < \Upsilon^* < \frac{2\pi}{\omega_0}$.
   2) $\frac{\gamma_+}{\gamma_-} \neq \frac{a_+}{a_-} \Longrightarrow A \neq 0$ and $\psi_\pm(0) \neq 0$;

further see below the general case $\mu \neq 0$.

The case $\mu \neq 0$

It is obvious that in this case $\psi(0) \neq 0$. By setting $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$, we arrive at the problem for $y(\omega)$

$$\begin{cases} y' + (1+\mu\varsigma)y^2(\omega) + (1+\mu\varsigma)\varkappa^2 - a_0(1+q) = 0, \quad \omega \in \left(-\frac{\omega_0}{2}, 0\right) \cup \left(0, \frac{\omega_0}{2}\right); \\ a_+ y_+(0) - a_- y_-(0) = (1+q)\beta; \\ \pm\alpha_\pm a_\pm y_\pm\left(\pm\frac{\omega_0}{2}\right) + (1+q)\gamma_\pm = 0. \end{cases}$$

Integrating the equation of our problem we find

$$y_\pm(\omega) = \varUpsilon \tan\left\{\varUpsilon\left(C_\pm - (1+\mu\varsigma)\omega\right)\right\}, \qquad \forall C_\pm.$$

From the boundary conditions we have

$$C_\pm = \pm(1+\mu\varsigma)\frac{\omega_0}{2} \mp \frac{1}{\varUpsilon}\arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \varUpsilon}.$$

Finally, in virtue of the conjunction condition, we get the equation for $\varkappa$:

$$\begin{aligned} a_+ \cdot \frac{\alpha_+ a_+ \varUpsilon \tan\left\{(1+\mu\varsigma)\varUpsilon\frac{\omega_0}{2}\right\} - (1+q)\gamma_+}{\alpha_+ a_+ \varUpsilon + (1+q)\gamma_+ \tan\left\{(1+\mu\varsigma)\varUpsilon\frac{\omega_0}{2}\right\}} & \\ + a_- \cdot \frac{\alpha_- a_- \varUpsilon \tan\left\{(1+\mu\varsigma)\varUpsilon\frac{\omega_0}{2}\right\} - (1+q)\gamma_-}{\alpha_- a_- \varUpsilon + (1+q)\gamma_- \tan\left\{(1+\mu\varsigma)\varUpsilon\frac{\omega_0}{2}\right\}} & \\ = \frac{1+q}{\varUpsilon}\beta, \qquad \text{where } 1+\mu\varsigma = \frac{1+q+\mu}{1+q}. & \end{aligned}$$

Thus we obtain

$$y_\pm(\omega) = \varUpsilon \tan\left\{\varUpsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \varUpsilon}\right\}$$

and, because of $(\ln\psi(\omega))' = y(\omega)$, it follows that

$$\psi_\pm(\omega) = \cos^{\frac{1+q}{1+q+\mu}}\left\{\varUpsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \varUpsilon}\right\}.$$

At last, returning to the function u we establish a solution of our problem

$$u_\pm(r,\omega) = r^{\frac{\varkappa}{1+q}}\cos^{\frac{1}{1+q+\mu}}\left\{\varUpsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \varUpsilon}\right\}.$$

If we consider **the Dirichlet problem without the interface**: $\alpha_\pm = 0$, $a_\pm = 1$, $\beta = 0$, then we can calculate

$$u(r,\omega) = r^{\widetilde{\lambda}}\cos^{\frac{1}{1+q+\mu}}\left(\frac{\pi\omega}{\omega_0}\right); \qquad \widetilde{\lambda} = \frac{\sqrt{(\pi/\omega_0)^2 + a_0(1+q+\mu)}}{1+q+\mu}.$$

It recovers a well known result (see [2], p. 374, Example 4.6). Now we can verify that the derived exact solution satisfies the estimate of Theorem 3. In fact, in our case we have: the value λ is equal $\vartheta = \frac{\pi}{\omega_0}$ and therefore

$$|u(r,\omega)| \le r^{\widetilde{\lambda}} \le r^{\frac{\pi}{\omega_0}\cdot\frac{1}{1+q+\mu}} \le r^{\frac{\pi}{\omega_0}\cdot\frac{1+q-\mu}{(1+q)^2}},$$

since $a_0 \ge 0$ and $\frac{1}{1+q+\mu} \ge \frac{1+q-\mu}{(1+q)^2}$.

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Received: 7 May 2008; final version: 31 May 2008;
available online: 29 September 2008.