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Gauss's definition of the gamma function

Abstract. The present paper gives a historical account on extending the factorial function to complex numbers by Gauss.

A short excursion into history of complex numbers

Complex numbers (more exactly, square roots of negative numbers) have been used since the mid of the 16th century in mathematics. These numbers appeared when Italian mathematicians tried to solve polynomial equations of higher degrees. The use of complex numbers sometimes led to certain obscurities. A change of the perception of the complex numbers came around the turn of the 18th and 19th century when Karl Friedrich Gauss (1777-1855) published several papers where he used an idea of complex numbers.

Further progress came with representation of complex numbers by points or vectors in the plane. This idea occurred for the first time in the works of Caspar Wessel (1745-1818) and Jean Robert Argand (1768-1822); Argand introduced the term textitle module for an absolute value of the complex numbers. Both Wessel's and Argand's articles were written probably independently and these works have never received general awareness. The geometric interpretation of complex numbers was completely accepted when Gauss wrote his treatise *Theoria residuorum biquadraticorum* (Theory of the biquadratic residues) (1831), see [1].

In 1837 William Rowan Hamilton (1805-1865) introduced complex numbers as ordered pairs of real numbers. In 1847 Louis Augustin Cauchy (1789-1857) presented an algebraic definition of complex numbers.

A Gauss's letter to Bessel

On 21st November 1811 Karl Friedrich Gauss wrote a letter to Friedrich Wilhelm Bessel (1784-1846) about his development on general factorials, see [3]. Gauss wrote the following text in that letter (authors translation).

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Thus the product

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot x = \prod x$$

is the function that, in my opinion, must be introduced into Calculus, (...). But if one wants to avoid countless Kramp's paradigms and paradoxes and contradictions, $1 \cdot 2 \cdot 3 \cdot \ldots \cdot x$ must not be used as the definition of $\prod x$, because such a definition has a precise meaning only when x is an integer; rather, one must start with a more general definition, which is applicable even to imaginary values of x, of which that one is a special case. I have chosen the following one

$$\prod x = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \cdot k^x}{(x+1)(x+2)(x+3) \cdot \ldots \cdot (x+k)},$$

where k tends to infinity.

Let us remark that Christian Kramp (1760-1826) was one of the mathematicians who sought a general rule for non-integer values of factorials. He introduced so-called "numerical factorial" by the form

$$a^{\frac{b}{c}} = a(a+c)(a+2c) \cdot \ldots \cdot (a+(b-1)c) \tag{1}$$

in the book [4]. The product on the right-hand side of (1) was studied in the first half of the 19th century under the name "analytic factorial". In 1856 Karl Weierstrass (1815-1897) finished these activities, when he demonstrated nonsense resulting from that definition, see [8]. Moreover, Kramp was the first mathematician who used the notation $n!$ for n-factorials in the book [5].

Gauss acquired his knowledge about the function $\prod x$ during an investigation of properties of the hypergeometric series. The generalized hypergeometric function

$${}_pF_q(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p, x)$$

is defined by the sum of a hypergeometric series, i.e., series $\sum_{k=0}^{\infty} a_k x^k$ for which $a_0 = 1$ and the ratio of consecutive terms $\frac{a_{k+1}}{a_k}$ can be expressed as the fraction

$$\frac{a_{k+1}}{a_k} = \frac{(\alpha_1 + k)(\alpha_2 + k) \cdot \ldots \cdot (\alpha_p + k)}{(k+1)(\beta_1 + k)(\beta_2 + k) \cdot \ldots \cdot (\beta_q + k)} x.$$

If $p = 2$ and $q = 1$, we get *Gauss's hypergeometric function* ${}_2F_1(\alpha, \beta, \gamma, x)$, which is the sum of the series

$$1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \ldots \tag{2}$$

Gauss dealt with that series in [2]. He stated a condition for the convergence of (2) in the terms of the coefficients α, β, γ, x, which is described in the following theorem.

Theorem

Let $x \in \mathbb{C}$ and $\gamma \in \mathbb{C}\backslash\{0, -1, -2, -3, \ldots\}$. If $|x| < 1$, then the series (2) is convergent. If $|x| = 1$, then the series (2) is convergent if and only if $|\gamma - \alpha - \beta| > 0$ holds. In case of $|x| > 1$, the series (2) is divergent.

The introduction to the theory of the gamma function

Gauss derived a following formula for hypergeometric series

$$F(\alpha, \beta, \gamma, 1) = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} F(\alpha, \beta, \gamma + 1, 1). \tag{3}$$

Then he generalized equation (3) into the form

$$\begin{aligned} &F(\alpha, \beta, \gamma, 1) \\ &= \frac{(\gamma - \alpha)(\gamma + 1 - \alpha) \cdot \ldots \cdot (\gamma + k - 1 - \alpha)}{\gamma(\gamma + 1) \cdot \ldots \cdot (\gamma + k - 1)} \\ &\quad \times \frac{(\gamma - \beta)(\gamma + 1 - \beta) \cdot \ldots \cdot (\gamma + k - 1 - \beta)}{(\gamma - \alpha - \beta)(\gamma + 1 - \alpha - \beta) \cdot \ldots \cdot (\gamma + k - 1 - \alpha - \beta)} F(\alpha, \beta, \gamma + k, 1), \end{aligned} \tag{4}$$

where $k \in \mathbb{N}$. This recurrence formula became a starting point for his next investigation about the gama function.

For $k \in \mathbb{N}$, $z \in \mathbb{C}$, Gauss introduced the function $\prod(k, z)$ by

$$\prod(k, z) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}{(z + 1)(z + 2)(z + 3) \cdot \ldots \cdot (z + k)} k^z.$$

A simple calculation shows that the expression

$$\frac{\prod(k, \gamma - 1) \cdot \prod(k, \gamma - \alpha - \beta - 1)}{\prod(k, \gamma - \alpha - 1) \cdot \prod(k, \gamma - \beta - 1)}$$

can be reduced to the fraction

$$\begin{aligned} &\frac{(\gamma - \alpha)(\gamma + 1 - \alpha) \cdot \ldots \cdot (\gamma + k - 1 - \alpha)}{\gamma(\gamma + 1) \cdot \ldots \cdot (\gamma + k - 1)} \\ &\qquad \times \frac{(\gamma - \beta)(\gamma + 1 - \beta) \cdot \ldots \cdot (\gamma + k - 1 - \beta)}{(\gamma - \alpha - \beta)(\gamma + 1 - \alpha - \beta) \cdot \ldots \cdot (\gamma + k - 1 - \alpha - \beta)}. \end{aligned}$$

It means that the equation (4) can be transformed to the form

$$F(\alpha, \beta, \gamma, 1) = \frac{\prod(k, \gamma - 1) \cdot \prod(k, \gamma - \alpha - \beta - 1)}{\prod(k, \gamma - \alpha - 1) \cdot \prod(k, \gamma - \beta - 1)} \cdot F(\alpha, \beta, \gamma + k, 1).$$

It is easily seen that $\prod(k,z)$ is defined for all $z \in \mathbb{C}$ except the negative integers. If z is a nonnegative integer, we get (for all $k \in \mathbb{N}$)

$$\begin{aligned}
\prod(k,0) &= 1,\\
\prod(k,1) &= \frac{1\cdot k}{k+1} = \frac{1}{1+\frac{1}{k}},\\
\prod(k,2) &= \frac{1\cdot 2\cdot k^2}{(k+1)(k+2)} = \frac{1\cdot 2}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)},\\
&\vdots\\
\prod(k,z) &= \frac{1\cdot 2\cdot 3\cdot\ldots\cdot z}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)\cdot\ldots\cdot\left(1+\frac{z}{k}\right)}.
\end{aligned}$$

Gauss determined the values of the function $\prod(k,z+1)$ for given k and z by the recurrence formula

$$\prod(k,z+1) = \prod(k,z)\cdot\frac{z+1}{1+\frac{z+1}{k}}.$$

In a similar way, Gauss found a recurrence formula with respect to k

$$\prod(k+1,z) = \prod(k,z)\cdot\frac{\left(1+\frac{1}{k}\right)^{z+1}}{1+\frac{1+z}{k}}. \tag{5}$$

The formula (5) yields the following equalities

$$\begin{aligned}
\prod(1,z) &= \frac{1}{1+z},\\
\prod(2,z) &= \frac{1}{1+z}\cdot\frac{\left(\frac{2}{1}\right)^{z+1}}{\frac{2+z}{1}} = \frac{1}{1+z}\cdot\frac{2^{z+1}}{2+z},\\
&\vdots\\
\prod(k,z) &= \frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z\cdot(2+z)}\cdot\frac{3^{z+1}}{2^z\cdot(3+z)}\cdot\ldots\cdot\frac{k^{z+1}}{(k-1)^z\cdot(k+z)}.
\end{aligned}$$

Then Gauss supposed $k\to\infty$, z as fixed point and set $k=h$, $z<h$. If h increase to $h+1$, then the value of $\log\prod(k,z)$ increase too. It holds by (5)

$$\log\prod(h+1,z) - \log\prod(h,z) = \log\frac{\left(1+\frac{1}{h}\right)^{z+1}}{1+\frac{1+z}{h}} = \log\frac{\left(1+\frac{1}{h}\right)^{z}}{1+\frac{z}{h+1}}. \tag{6}$$

According to the equality $\frac{h+1}{h} = \frac{1}{\frac{h}{h+1}}$ and the equation (6), Gauss got

$$\log\prod(h+1,z) - \log\prod(h,z) = -z\log\left(1-\frac{1}{h+1}\right) - \log\left(1+\frac{z}{h+1}\right). \tag{7}$$

Both logarithms on the right-hand side of (7) can be expressed in the form of power series

$$z\left(\frac{1}{h+1}+\frac{1}{2(h+1)^2}+\dots\right)-\left(\frac{z}{h+1}-\frac{z^2}{2(h+1)^2}+\dots\right).$$

Assuming the absolute convergence of both series, Gauss got for the increase of $\log\prod(h,z)$ the convergent series

$$\frac{z(1+z)}{2(h+1)^2}+\frac{z(1-z^2)}{3(h+1)^3}+\frac{z(1+z^3)}{4(h+1)^4}+\frac{z(1-z^4)}{5(h+1)^5}+\dots\,.$$

If the value k increases from $h+1$ to $h+2$, then one has

$$\log\prod(h+2,z)-\log\prod(h+1,z)=\frac{z(1+z)}{2(h+2)^2}+\frac{z(1-z^2)}{3(h+2)^3}+\frac{z(1+z^3)}{4(h+2)^4}+\dots\,.$$

Generally, if the value k increases from h to $h+n$, then the difference $\log\prod(h+n,z)-\log\prod(h,z)$ equals

$$\begin{aligned}&\frac{1}{2}z(1+z)\left(\frac{1}{(h+1)^2}+\frac{1}{(h+2)^2}+\frac{1}{(h+3)^2}+\dots+\frac{1}{(h+n)^2}\right)\\&\quad+\frac{1}{3}z(1-z^2)\left(\frac{1}{(h+1)^3}+\frac{1}{(h+2)^3}+\frac{1}{(h+3)^3}+\dots+\frac{1}{(h+n)^3}\right)\\&\quad+\frac{1}{4}z(1+z^3)\left(\frac{1}{(h+1)^4}+\frac{1}{(h+2)^4}+\frac{1}{(h+3)^4}+\dots+\frac{1}{(h+n)^4}\right)+\dots\,.\end{aligned}$$

For $n\to\infty$ Gauss obtained an absolute convergent double series

$$\sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\left(\frac{z+z^{2i}}{2i(h+j)^{2i}}+\frac{z-z^{2i+1}}{(2i+1)(h+j)^{2i+1}}\right)\right].$$

Gauss proved the finitness of $\lim_{k\to\infty}\prod(k,z)$ for every $\mathbb{C}\backslash\{-1,-2,-3,\dots\}$. Value of this limit depends on z only, hence the function $\lim_{k\to\infty}\prod(k,z)$ depends also only on z. Thus Gauss introduced the function $\prod z$ by equation

$$\prod z=\lim_{k\to\infty}\frac{1\cdot2\cdot3\cdot\ldots\cdot k\cdot k^z}{(z+1)(z+2)(z+3)\cdot\ldots\cdot(z+k)},$$

or by an infinite product

$$\prod z=\frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z(2+z)}\cdot\frac{3^{z+1}}{2^z(3+z)}\cdot\frac{4^{z+1}}{3^z(4+z)}\cdot\dots,$$

which represents an analog of the Euler's definition, see [7]. But there is an important difference in the domain of definition in comparison to Euler's definition. Euler considered reals without the negative integers, while Gauss took the set $\mathbb{C}\backslash\{-1,-2,-3,\dots\}$ as the domain of the function $\prod z$.

This definition of Gauss played an important role in later development of the theory of functions. For example, it was the impuls which led Karl Weierstrass to the idea about elementary factors used in his factorization theorem. The notation $\prod z$ comes from Gauss. Later, in 1809, Adrien-Marie Legendre (1752-1833) introduced a standard notation $\Gamma(z)$ instead of $\prod(z-1)$, see [6].

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