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Lichawski-Matkowski-Miś theorem on locally defined operators for functions of several variables

Abstract. Let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$ and $C^m(D)$ be the space of functions $f|_D$ such that $f \in C^m(G)$. The representation formulas for locally defined operators mapping $C^m(D)$ into $C^0(D)$ and into $C^1(D)$ are given.

1. Introduction

For a real interval $I \subset \mathbb{R}$ and a nonnegative integer m, we denote by $C^m(I)$ the set of all m-times continuously differentiable functions $\varphi: I \longrightarrow \mathbb{R}$. An operator $K: C^m(I) \longrightarrow C^0(I)$ or $C^m(I) \longrightarrow C^1(I)$ is said to be locally defined if for every two functions $\varphi, \psi \in C^m(I)$ and for every open subinterval $J \subset I$ the relation $\varphi|_J = \psi|_J$ implies that $K(\varphi)|_J = K(\psi)|_J$. Answering a question posed by F. Neuman, the authors of [1] gave a representation formula for locally defined operators $K: C^m(I) \longrightarrow C^0(I)$. Namely, they proved that: *every locally defined operator* $K: C^m(I) \to C^0(I)$ *must be of the form*

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(m)}(x))$$

for a certain function $h: I \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$. Moreover, they proved that *every locally defined operator* $K: C^m(I) \longrightarrow C^1(I)$ *must be of the form*

$$K(\varphi)(x) = h(x, \varphi(x), \ldots, \varphi^{(m-1)}(x)).$$

In this paper we generalize this result showing that analogous representation theorems hold true for locally defined operators $K: C^m(D) \longrightarrow C^0(D)$ and $C^m(D)$ into $C^1(D)$, where D is a regular closed set in the open subspace $G \subset \mathbb{R}^n$ and $C^m(D)$ is the space of functions $f|_D$ such that $f \in C^m(G)$. The proofs of our theorems are similar in spirit to the proofs of Theorems 2 and 3 in [1].

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2. Preliminaries

Let $\mathbb{N}_0$ be a set of nonnegative integers and let $\mathbb{N}_0^n := \prod_{i=1}^n \mathbb{N}_0$ for $n \in \mathbb{N}$. In this paper, for $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and $i = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, we put

$$\begin{aligned} |k| &:= k_1 + \ldots + k_n \,, \\ k! &:= (k_1!) \cdot \ldots \cdot (k_n!), \\ k + i &:= (k_1 + i_1, \ldots, k_n + i_n), \\ k - i &:= (k_1 - i_1, \ldots, k_n - i_n) \quad \text{for all } i \leq k, \end{aligned}$$

where the notation $i \leq k$ means that $i_s \leq k_s$ for every $s \in \{1, \ldots, n\}$.

Moreover, for $i = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we put

$$x^i := x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \qquad \text{and} \qquad \|x\| := \sqrt{\sum_{i=1}^n x_i^2}.$$

As a consequence of the Whitney Extension Theorem (cf. [2]) we get the following lemma.

Lemma 1

Let $B \subset \mathbb{R}^n$ be a compact set with only one cluster point $z \in \mathbb{R}^n$. Suppose that $m \in \mathbb{N}_0$ and

$$\{f^k \mid \ f^k : B \longmapsto \mathbb{R}, \ k \in \mathbb{N}_0^n, \ |k| \leq m\} \qquad \textit{where } f^{(0,\ldots,0)} = f$$

is a family of functions satisfying the condition

$$f^k(x) - \sum_{|i| \leq m-|k|} \frac{f^{k+i}(z)}{i!}(x-z)^i = o(\|x-z\|^{m-|k|}) \qquad \textit{as } x \to z \tag{1}$$

for all $x \in B$, $|k| \leq m$, $k \in \mathbb{N}_0^n$. If for some $\alpha > 0$,

$$x \neq y \implies \|x - y\| \geq \alpha \max\{\|x - z\|, \|y - z\|\}, \qquad x, y \in B,$$

then there exists a function g of the class C^m on $\mathbb{R}^n$ satisfying the condition

$$\frac{\partial^{|k|} g}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}(x) = f^k(x) \qquad \textit{for all } x \in B, \ k \in \mathbb{N}_0^n \ \textit{and } |k| \leq m. \tag{2}$$

3. Locally defined operators mapping $C^m(D)$ into $C^0(D)$ and into $C^1(D)$

Let G be a nonempty and open set in the Euclidean space $\mathbb{R}^n$. By $C^m(G)$ we denote the space of m-times continuously differentiable functions on G.

Definition 1
Let G be an open set in the Euclidean space $\mathbb{R}^n$ and let $D \subset G$ be a regular closed set in the subspace G, i.e., $D = G \cap \operatorname{cl} \operatorname{int} D$. A function $f\colon D \longrightarrow \mathbb{R}$ is said to be of the *class* C^m *on* D if there exists a function $g \in C^m(G)$ such that $g|_D = f$, i.e.,

$$C^m(D) = \{f|_D :\ f \in C^m(G)\}.$$

Let $J_i \subset \mathbb{R}$, $i = 1, \ldots, n$, be open (closed) intervals. A set $J \subset \mathbb{R}^n$,

$$J = \prod_{i=1}^{n} J_i\,,$$

the Cartesian product of the intervals J_i, will be called an *open* (*closed*) *interval* in $\mathbb{R}^n$.

Now, we introduce the definition of locally defined operators of the type $K\colon C^m(D) \longrightarrow C^k(D)$.

Definition 2
Let $m, k \in \mathbb{N}_0$ and let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$. An operator $K\colon C^m(D) \longrightarrow C^k(D)$ is said to be *locally defined* if for every two functions $\varphi, \psi \in C^m(D)$ and for every open interval $J \subset \mathbb{R}^n$

$$\varphi|_{D\cap J} = \psi|_{D\cap J} \implies K(\varphi)|_{D\cap J} = K(\psi)|_{D\cap J}\,.$$

We shall need the following lemma.

Lemma 2 (cf. [3], Theorem)
Let $m, k \in \mathbb{N}_0$ and a closed interval $D \subset \mathbb{R}^n$ be fixed and let $K\colon C^m(D) \longrightarrow C^k(D)$ be a locally defined operator. Then for every $x_o \in D$, $\varphi, \psi \in C^m(D)$, if

$$\frac{\partial^{|j|}\varphi}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}(x_o) = \frac{\partial^{|j|}\psi}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}(x_o) \qquad \text{for all } j \in \mathbb{N}_0^n\,,\ |j| \le m,$$

then

$$\frac{\partial^{|i|}K(\varphi)}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}(x_o) = \frac{\partial^{|i|}K(\psi)}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}(x_o) \qquad \text{for all } i \in \mathbb{N}_0^n\,,\ |i| \le k.$$

Before formulating the main theorems we have to introduce the following notation. Let $m \in \mathbb{N}_0$ be fixed. Then

$$S(k) := \sum_{s=0}^{m-k} \binom{n+s-1}{s}$$

denotes the cardinality of the set of all partial derivatives of $m-k$ times continuously differentiable function $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$.

Theorem 1
Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$. If an operator $K\colon C^m(D) \longrightarrow C^0(D)$ is locally defined, then there exists a unique function $h\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x), \frac{\partial \phi}{\partial x_1}(x), \dots, \frac{\partial \phi}{\partial x_n}(x), \dots, \frac{\partial^m \phi}{\partial x_1^m}(x), \dots, \frac{\partial^m \phi}{\partial x_n^m}(x)\right)$$

for all $\phi \in C^m(D)$ and $x \in D$.

Proof. The proof is based on the concept of Theorem 2 in [1]. In order to define a function $h\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ let us fix arbitrarily $z = (z_1, \dots, z_n) \in D$ and $y_{(j_1,\dots,j_n)} \in \mathbb{R}$ such that $j_1, \dots, j_n \in \{0, \dots, m\}$, $|j| \le m$. Let us take a polynomial

$$P_{z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}}(x_1, \dots, x_n)$$
$$:= \sum_{\substack{j_1,\dots,j_n=0 \\ j_1+\dots+j_n \le m}}^{m} \frac{y_{(j_1,\dots,j_n)}}{j_1! \dots j_n!}(x_1 - z_1)^{j_1} \cdot \ldots \cdot (x_n - z_n)^{j_n}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n$$

and put

$$h(z_1, \dots, z_n, y_{(0,\dots,0)}, \dots, y_{(0,\dots,m)}) := K(P_{z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}})(z_1, \dots, z_n).$$

For any $\phi \in C^m(D)$, $j \in \mathbb{N}_0^n$ and $|j| \le m$

$$\frac{\partial^{|j|}\phi}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(z_1, \dots, z_n) = \frac{\partial^{|j|} P_{z_1,\dots,z_n,\phi(z),\frac{\partial\phi}{\partial x_1}(z),\dots,\frac{\partial^m\phi}{\partial x_n^m}(z)}}{\partial x_1^{j_1} \dots \partial x_n^{j_m}}(z_1, \dots, z_n).$$

Hence, by Lemma 2 for $|i| = 0$, we obtain

$$K(\phi)(z_1, \dots, z_n) = K\big(P_{z_1,\dots,z_n,\phi(z),\frac{\partial\phi}{\partial x_1}(z),\dots,\frac{\partial^m\phi}{\partial x_n^m}(z)}\big)(z_1, \dots, z_n)$$

and therefore

$$K(\phi)(z_1, \dots, z_n) = h\left(z_1, \dots, z_n, \phi(z), \frac{\partial \phi}{\partial x_1}(z), \dots, \frac{\partial^m \phi}{\partial x_n^m}(z)\right).$$

Now, we prove the uniqueness of h. Let $h_1\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ be a function such that

$$K(\phi)(z_1, \dots, z_n) = h_1\left(z_1, \dots, z_n, \phi(z), \frac{\partial \phi}{\partial x_1}(z), \dots, \frac{\partial^m \phi}{\partial x_n^m}(z)\right)$$

for all $\phi \in C^m(D)$ and $z = (z_1, \dots, z_n) \in D$. In order to show that $h = h_1$, let us fix an arbitrary $(z_1, \dots, z_n) \in D$ and $y_{(j_1,\dots,j_n)} \in \mathbb{R}$, $j_1, \dots, j_n \in \{0, \dots, m\}$, $|j| \le m$.

According to the definitions of h_1 and h, we have

$$\begin{aligned} h_1(z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}) &= K(P_{z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}})(z_1,\dots z_n) \\ &= h(z_1,\dots,z_n,y_{(0,\dots,0)},\dots,y_{(0,\dots,m)}), \end{aligned}$$

which completes the proof.

Corollary 1

Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and an open set $G \subset \mathbb{R}^n$ be fixed. If an operator $K\colon C^m(G) \longrightarrow C^0(G)$ is locally defined, then there exists a unique function $h\colon G \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x), \frac{\partial\phi}{\partial x_1}(x),\dots,\frac{\partial\phi}{\partial x_n}(x),\dots,\frac{\partial^m\phi}{\partial x_1^m}(x),\dots,\frac{\partial^m\phi}{\partial x_n^m}(x)\right)$$

for all $\phi \in C^m(G)$ and $x \in G$.

The following result may be proved in much the same way as Theorem 3 in [1].

Theorem 2

Let $m,n \in \mathbb{N}$ and let D be a regular closed set in the open subspace $G \subset \mathbb{R}^n$. If an operator $K\colon C^m(D) \longrightarrow C^1(D)$ is locally defined, then there exists a unique function $h\colon D \times \mathbb{R}^{S(1)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\left(x, \phi(x),\dots,\frac{\partial^{m-1}\phi}{\partial x_1^{m-1}}(x),\dots,\frac{\partial^{m-1}\phi}{\partial x_n^{m-1}}(x)\right)$$

for all $\phi \in C^m(D)$ and $x = (x_1,\dots,x_n) \in D$.

Proof. By Theorem 1 there exists a unique function $h\colon D \times \mathbb{R}^{S(0)} \longrightarrow \mathbb{R}$ such that for all $\phi \in C^m(D)$ and $(x_1,\dots,x_n) \in D$

$$\begin{aligned} &K(\phi)(x_1,\dots,x_n) \\ &= h\left(x_1,\dots,x_n,\phi(x_1,\dots,x_n),\dots,\frac{\partial^m\phi}{\partial x_1^m}(x_1,\dots,x_n),\dots,\frac{\partial^m\phi}{\partial x_n^m}(x_1,\dots,x_m)\right). \end{aligned}$$

In order to prove this theorem it is enough to show that for all $i \in \mathbb{N}_0^n$ such that $|i| = m$ we have

$$\frac{\partial h}{\partial y_i}(x_1,\dots,x_n,y_{(0,\dots,0)},\dots,y_{(m,0,\dots,0)},\dots,y_{(0,\dots,0,m)}) = 0. \tag{3}$$

Let us fix $x_o \in D$ and $y_i \in \mathbb{R}$ where $i \in \mathbb{N}_0^n$, $|i| \le m$ and let us choose an arbitrary i_0, $|i_0| = m$, and a real sequence $(y_{i_0,N})_{N=0}^{\infty}$ such that

$$y_{i_0,0} = y_{i_0}; \qquad y_{i_0,N} \neq y_{i_0}, \quad N \in \mathbb{N}; \qquad \lim_{N\to\infty} y_{i_0,N} = y_{i_0,0}.$$

Let ϕ_N, for every $N \in \mathbb{N}_0$, denotes the polynomial

$$\phi_N(x) := \sum_{\substack{|r|\le m \\ r\ne i_0}} \frac{y_r}{r!}(x-x_o)^r + \frac{y_{i_0,N}}{i_0!}(x-x_o)^{i_0}, \qquad x \in D.$$

Fix an $\varepsilon > 0$. Since all functions $K(\phi_N)$ are continuous, for all $N \in \mathbb{N}$ there exists $\delta_N > 0$ such that

$$\|x - x_o\| < \delta_N \;\Rightarrow\; |K(\phi_N)(x) - K(\phi_N)(x_o)| < \varepsilon |y_{i_0,N} - y_{i_0,0}|, \quad x \in D. \tag{4}$$

Take an arbitrary $\alpha > 0$ and choose a set $B = \{x_N : \ N \in \mathbb{N}_0\} \subset D$ satisfying all the conditions listed in Lemma 1 with $z = x_o$ and such that

$$\|x_N - x_o\| < \delta_N\,, \qquad N \in \mathbb{N} \tag{5}$$

and

$$\lim_{N\to\infty} \frac{y_{i_0,N} - y_{i_0,0}}{\|x_N - x_o\|} = \infty. \tag{6}$$

Now define functions $f^i\colon \mathbb{B} \longrightarrow \mathbb{R}$, $i \in \mathbb{N}_0^n$, $|i| \le m$, by the formula

$$f^i(x_N) := \phi_N^i(x_N), \qquad N \in \mathbb{N}_0\,.$$

First we show that the family $\{f^i \mid f^i\colon \mathbb{B} \longrightarrow \mathbb{R},\ i \in \mathbb{N}_0^n\,,\ |i| \le m\}$ fulfills (1) for all $i \in \mathbb{N}_0^n$ such that $i \le i_0$.

Since for all $N \in \mathbb{N}_0$

$$f^i(x_N) = \sum_{\substack{|r|\le m-|i| \\ r\ne i_0-i}} \frac{y_{i+r}}{r!}(x_N - x_o)^r + \frac{y_{i_0,N}}{(i_0-i)!}(x_N-x_o)^{i_0-i},$$

and

$$\sum_{|r|\le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N - x_o)^r = \sum_{\substack{|r|\le m-|i| \\ r\ne i_0-i}} \frac{y_{i+r}}{r!}(x_N-x_o)^r + \frac{y_{i_0,0}}{(i_0-i)!}(x_N-x_o)^{i_0-i},$$

we infer that

$$\begin{aligned}
\left| f^i(x_N) - \sum_{|r|\le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N-x_o)^r \right| &= \left| \frac{y_{i_0,N} - y_{i_0,0}}{(i_0-i)!}(x_N-x_o)^{i_0-i} \right| \\
&= \frac{|y_{i_0,N} - y_{i_0,0}|}{(i_0-i)!}\; |(x_N-x_o)^{i_0-i}| \\
&\le \frac{|y_{i_0,N} - y_{i_0,0}|}{(i_0-i)!} \|x_N - x_o\|^{|i_0-i|} \\
&= \frac{|y_{i_0,N} - y_{i_0,0}|}{(i_0-i)!} \|x_N - x_o\|^{m-|i|} \\
&= o(\|x_N - x_o\|^{m-|i|}).
\end{aligned}$$

In the second case, when $i \in \mathbb{N}_0^n$ is such that $|i| \le m$ does not satisfy the inequality $i \le i_0$, we have

$$f^i(x_N) = \sum_{|r|\le m-|i|} \frac{y_{i+r}}{r!}(x_N - x_o)^r = \sum_{|r|\le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N - x_o)^r$$

and therefore

$$f^i(x_N) - \sum_{|r|\le m-|i|} \frac{f^{i+r}(x_o)}{r!}(x_N - x_o)^r = 0.$$

Thus the family $\{f^i \mid f^i\colon \mathbb{B} \longrightarrow \mathbb{R},\ i \in \mathbb{N}_0^n,\ |i| \le m\}$ fulfills (1) and according to Lemma 1 there exists a function $g \in C^m(\mathbb{R}^n)$ such that

$$\frac{\partial^{|i|} g}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(x_N) = \frac{\partial^{|i|} \phi_N}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(x_N), \qquad N \in \mathbb{N}_0,\ i \in \mathbb{N}_0^n,\ |i| \le m. \tag{7}$$

Hence and by (4), (5), (7) and Lemma 2 we have

$$\begin{aligned}
&\left|\frac{h(x_o, y_{(0,\dots,0)}, \dots, y_{i_0,N}, \dots, y_{(0,\dots,m)}) - h(x_o, y_{(0,\dots,0)}, \dots, y_{i_0,0}, \dots, y_{(0,\dots,m)})}{y_{i_0,N} - y_{i_0,0}}\right| \\
&\quad = \left|\frac{K(\phi_N)(x_o) - K(\phi_0)(x_o)}{y_{i_0,N} - y_{i_0,0}}\right| \\
&\quad \le \left|\frac{K(\phi_N)(x_N) - K(\phi_N)(x_o)}{y_{i_0,N} - y_{i_0,0}}\right| + \left|\frac{K(\phi_N)(x_N) - K(\phi_0)(x_o)}{y_{i_0,N} - y_{i_0,0}}\right| \\
&\quad \le \varepsilon + \left|\frac{K(g)(x_N) - K(g)(x_o)}{y_{i_0,N} - y_{i_0,0}}\right| \\
&\quad = \varepsilon + \frac{|K(g)(x_N) - K(g)(x_o)|}{\|x_N - x_o\|} \cdot \frac{\| x_N - x_o\|}{|y_{i_0,N} - y_{i_0,0}|}.
\end{aligned}$$

Since $K(g) \in C^1(D)$, we conclude that

$$\lim_{N\to\infty} \frac{|K(g)(x_N) - K(g)(x_o)|}{\|x_N - x_o\|} < \infty.$$

Hence and by (6) we obtain (3) for $i = i_0 \in \mathbb{N}_0^n$ such that $|i_0| = m$ and the proof is completed.

Corollary 2

Let $m, n \in \mathbb{N}$ and an open set $G \subset \mathbb{R}^n$ be fixed. If an operator $K\colon C^m(G) \longrightarrow C^1(G)$ is locally defined, then there exists a unique function $h\colon G \times \mathbb{R}^{S(1)} \longrightarrow \mathbb{R}$ such that

$$K(\phi)(x) = h\Big(x, \phi(x), \dots, \frac{\partial^{m-1}\phi}{\partial x_1^{m-1}}(x), \dots, \frac{\partial^{m-1}\phi}{\partial x_n^{m-1}}(x)\Big)$$

for all $\phi \in C^m(G)$ and $x \in G$.

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References

[1] K. Lichawski, J. Matkowski, J. Miś, *Locally defined operators in the space of differentiable functions*, Bull. Polish Acad. Sci. Math. **37** (1989), no. 1-6, 315-325.

[2] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63-89.

[3] M. Wróbel, *Some consequences of locally defined operators*, in: Proceedings of the XII$^{\text{th}}$ Czech–Polish–Slovak Mathematical School, Faculty of Education of University J.E. Purkyně in Ústi nad Labem, 2005, 82-86.

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