Annales Academiae Paedagogicae Cracoviensis
Folia 45 Studia Mathematica VI

Report of Meeting

11th International Conference on Functional Equations and Inequalities, Będlewo, September 17–23, 2006

Contents

The *Eleventh International Conference on Functional Equations and Inequalities* was held from September 17 to September 23, 2006 in Będlewo, Poland. The series of ICFEI meetings has been organized by the *Institute of Mathematics of the Pedagogical University of Cracow* since 1984. For the second time, the conference was organized jointly with the *Stefan Banach International Mathematical Center* and hosted by the *Mathematical Research and Conference Center* in Będlewo.

The Organizing Committee consisted of Professor Janusz Brzdęk (Chairman), Dr. Paweł Solarz, Miss Janina Wiercioch and Mr. Władyław Wilk.

The Scientific Committee consisted of Professors Nicole Brillouët-Belluot, Dobiesław Brydak (Honorary Chairman), Janusz Brzdęk (Chairman), Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Marek Cezary Zdun and Dr. Jacek Chmieliński (Scientific Secretary).

The 73 participants came from 11 countries: Austria, Canada, China, Croatia, France, Germany, Hungary, India, Israel, Poland and Russia.

Professor J. Brzdęk welcomed participants in the name of the Organizing Committee and then an opening address was given by Professor Eugeniusz Wachnicki, the Vice-Rector of the Pedagogical University of Cracow. The opening ceremony was followed by the first scientific session chaired by Professor János Aczél with the first lecture given by Professor Walter Benz.

During 20 regular sessions 65 talk were delivered. They focused on functional equations in single and several variables, functional inequalities, stability theory, convexity, multifunctions, theory of iteration, means, differential and difference equations, dynamical systems, applications of functional equations in physics and other topics. Several contributions have been made during special *Problems and Remarks*' sessions. Additional session devoted to stability problems was organized by Professor Boris Paneah on Friday evening, September 22.

On Tuesday, September 19, a picnic was organized in the park surrounding the Center. On the next day afternoon participants visited the castle and arboretum in Kórnik, as well as the park with ancient oaks in Rogalin. In the evening the piano recital was performed by Professor Hans-Heinrich Kairies.

On Thursday, September 21, a banquet was held in the Palace in Będlewo, and another piano recital was performed by Dr. Marek Czerni. On the following day a guitar concert *An Introduction to Flamenco* was given by Dr. Grzegorz Guzik.

The final session on Saturday, September 23 was chaired by Professor Bogdan Choczewski who also closed the conference. In the closing address, he gave some concluding information about the meeting and conveyed best regards for the participants from the Honorary Chairman of the ICFEI, Professor Dobiesław Brydak. It was announced that the 12th ICFEI will be organized in 2008.

The following part of the report contains abstracts of talks (in alphabetical order of the authors), problems and remarks (in chronological order of presentation) and the list of participants (with addresses). It has been compiled by Dr. Jacek Chmieliński.

Abstracts of Talks

Mirosław Adamek *On generalized affine functions*

Let $\mathcal{F}$ be a family of real functions defined on a nonempty interval $I \subset \mathbb{R}$. We say that $\mathcal{F}$ is a *two-parameter family* on I if, for any two different points $x_1, x_2 \in I$ and for any $t_1, t_2 \in \mathbb{R}$, there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = t_i \qquad \text{for } i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $x_1, x_2 \in I$ and values $t_1, t_2 \in \mathbb{R}$ will be denoted by $\varphi_{(x_1,t_1)(x_2,t_2)}$.

A function $f\colon I \longrightarrow \mathbb{R}$ is called $\mathcal{F}$-*affine* if for any different points $x_1, x_2 \in I$ and $t \in [0, 1]$ the following equality holds

$$f(tx_1 + (1-t)x_2) = \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(tx_1 + (1-t)x_2).$$

If a function f satisfies the above equality only with a fixed $t \in (0,1)$, then such function we call $(\mathcal{F}, t)$-*affine*.

In the talk we present some general properties about them and also Kuchn-type theorem for $(\mathcal{F}, t)$-affine functions.

Roman Badora *On the stability of the nonlinear functional equation*

In the talk we present an elementary proof of the following generalization of Baker's theorem (J.A. Baker, *The stability of certain functional equations*, Proc. Amer. Math. Soc. **112** (1991), 729-732) on the stability of the nonlinear functional equation.

THEOREM
Let S be a nonempty set and let (X,d) be a complete metric space. Assume that $f\colon S \longrightarrow S$ and the function $F\colon S \times X \longrightarrow X$ satisfies

$$d(F(t,x), F(t,y)) \le \lambda(t) d(x,y), \qquad t \in S,\ x,y \in X,$$

where $\lambda\colon S \longrightarrow \mathbb{R}$. Suppose that $\phi\colon S \longrightarrow X$ satisfies

$$d(\phi(t), F(t, \phi(f(t)))) \le \varepsilon(t), \qquad t \in S,$$

where $\varepsilon\colon S \longrightarrow \mathbb{R}$ and

$$\sum_{n=2}^{\infty} \varepsilon(f^{n-1}(t)) \prod_{i=0}^{n-2} \lambda(f^i(t)) < +\infty, \qquad t \in S.$$

Then there exists a unique function $\Phi\colon S \longrightarrow X$ such that

$$\Phi(t) = F(t, \Phi(f(t))), \qquad t \in S$$

and

$$d(\Phi(t), \phi(t)) \le \varepsilon(t) + \sum_{n=2}^{\infty} \varepsilon(f^{n-1}(t)) \prod_{i=0}^{n-2} \lambda(f^i(t)), \qquad t \in S.$$

Anna Bahyrycz *Forti's example of an unstable homomorphism equation*

We present a proof of some property of the function introduced by G.L. Forti, which is used to show that the homomorphism equation for some group is not stable.

[1] G.L. Forti, *Remark*, Aequationes Math. **29** (1985), 90-91.

[2] G.L. Forti, *The stability of homomorphisms and amenability, with applications to functional equations*, Abh. Math. Sem. Univ. Hamburg **57** (1987), 215-226.

Karol Baron *On linear iterative equations for distribution functions*

Given a probability space $(\Omega, \mathcal{A}, P)$ and a function $\tau\colon \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ we consider the equation

$$F(x) = \int_{\Omega} F(\tau(x,\omega))P(d\omega)$$

in some classes of distribution functions. We get results on existence, uniqueness and convergence of successive approximations.

Bogdan Batko *Superstability of some alternative Cauchy functional equations*

We are going to discuss the superstability of some alternative functional equations. Consider, for instance, Mikusinski's functional equation

$$f(x+y) \cdot (f(x+y) - f(x) - f(y)) = 0, \qquad x, y \in G \tag{1}$$

in the class of complex functions defined on an abelian group G. We ask if every unbounded solution of the inequality

$$|f(x+y) \cdot (f(x+y) - f(x) - f(y))| \le \varepsilon, \qquad x, y \in G,$$

with given $\varepsilon > 0$, must be an exact solution of Mikusinski's equation (1).

Walter Benz *A conditional functional inequality*

Let X be a real inner product space of infinite or finite dimension ≥ 3, and t be a fixed element of X with $t^2 = 1$. Put $H = \{h \in X | \;\; ht = 0\}$ and observe that to every $x \in X$ there exists uniquely determined $\bar{x} \in H$ and $x_0 \in \mathbb{R}$ such that $x = \bar{x} + x_0 t$. If $x, y \in X$, $x \le y$ is defined by $\|\bar{y} - \bar{x}\| \le y_0 - x_0$. We are interested in special solutions $f\colon X \longrightarrow X$, so-called casual automorphisms, of the conditional functional inequality

$$\forall\, x, y \in X \quad x \le y \implies f(x) \le f(y),$$

generalizing a theorem of A.D. Alexandrov, V.V. Ovchinnikova and E.C. Zeeman. (See my book "Classical Geometries in Modern Contexts", Birkhäuser, Basel – Boston – Berlin, 2005.)

Elena V. Blinova *On Ω-explosions in smooth skew products of interval maps*

Joint work with L.S. Efremova.

This work is continuation of [1]. it is devoted to the research of C^0- and C^1-Ω-explosions in C^1-smooth skew products of interval maps, i.e., maps of the type

$$F(x,y) = (f(x), g_x(y)), \tag{1}$$

where $(x, y) \in I$, I is the closed rectangle in the plane, in the additional assumption on the closure of the set $\mathrm{Per}(F)$ of F-periodic points.

Definition
Let $z_1 = (x, y_1)$, $z_2 = (x, y_2)$ be periodic points of F. The point z_1 is called the accessible from z_2 ($z_2 \to^a z_1$) if for every $\varepsilon > 0$ there is an ε-chain from z_2 to z_1 by the map F restricted to $\bigcup_{y \in \mathrm{Orb}(x)} \{y\} \times I$, where $\mathrm{Orb}(x)$ is f-periodic orbit of x.

The criterion of C^0-Ω-explosion for a map (1) is given in the terms of the properties of the set $\mathrm{Per}(F)$.

Theorem A
C^1-smooth map F with the closed set of periodic points permits a C^0-Ω-explosion if and only if F satisfies one of the following conditions:

1. *the set* $\mathrm{Per}(F)$ *is connected, and there exists at least one point* $x \in \mathrm{Per}(f)$ *such that the set* $\mathrm{Per}(\tilde{g_x})$ *is not connected, here* $\tilde{g_x} = g_{f^{n-1}(x)} \circ \ldots \circ g_{f(x)} \circ g_x$*,* n *is the least period of* x*;*
2. *the set* $\mathrm{Per}(F)$ *is not connected (let* K_i *be the connected components of* $\mathrm{Per}(F)$*), and either one of the connected components satisfies the condition* (1) *or there exists a finite number of connected components* K_i*,* $i = 1, 2, \ldots, m$ *of* $\mathrm{Per}(F)$ *such that for all* $i = 1, 2, \ldots, m$*,* $K_i \to^a K_{i+1}$*, where* $K_{m+1} = K_1$*.*

Theorem B
If the set $\mathrm{Per}(F)$ *of C^1-smooth map* (1) *is closed, then there is $\varepsilon > 0$ such that for every map $F^* \in B^1_\varepsilon(F)$* $\mathrm{Per}(F^*)$ *is a closed set, here $B^1_\varepsilon(F)$ is ε-neighborhood of F in the space of C^1-smooth skew products of interval maps with C^1-norm.*

The next theorem is the main result in the research of C^1-Ω-explosions in C^1-smooth maps (1).

Theorem C
C^1-smooth map (1) *with a closed set of periodic points doesn't permit C^1-Ω-explosions.*

Finally, we present the example of the one-parameter family of C^1-smooth maps of type (1) with the closed set of periodic points, which depend continuously (but not smoothly) on the parameter, where one can observe the appearing of the periodic orbits with periods $2, 4, \ldots, 2^n$ ($n \geq 1$) at once from the fixed point.

This research is partially supported by RFBR, grant No 04-01-00457.

[1] L.S. Efremova, *On the nonwandering set and the center of triangular maps with closed set of periodic points in the base* (in Russian), Dynamical Syst. and Nonlinear Phenomena. Kiev., 1990, 15-25.

[2] J. Kupka, *Triangular maps with the chain recurrent points periodic*, Acta Math. Univ. Comenianae, **72.2** (2003), 245-251.

Nicole Brillouët-Belluot *On applications of functional equations in physics*

Nowadays, problems in Physics are generally modelled by partial differential equations. Before the development of the differential calculus, the physical processes were often described by functional equations.

Functional equations represent an alternative way of modelling problems in Physics.

In this talk we present some applications of functional equations in Physics. Through these examples, I will explain how the functional equations appear in the physical process and what can be the interest of modelling physical problems by functional equations.

Bruno Brive *Differential equations of infinite order*

Joint work with Prof. Atzmon (Tel Aviv University, Israel).

We consider the following functional equation

$$\sum_{n\geq 0} a_n \frac{d^n}{dz^n} f(z) = g(z) \tag{$*$}$$

where f and g are functions of a complex variable z and a_n are complex numbers. This is an inhomogeneous linear differential equation of (possibly) infinite order with constant coefficients. Examples of such equations are given by linear difference-differential equations with constant coefficients. Many authors have studied equations $(*)$ in various contexts: Nörlund, Valiron, Malgrange, Martineau, Guelfond, … Many results are given in Berenstein and Gay, *Complex Analysis and Special Topics in Harmonic Analysis*, Springer, 1995.

We look at $(*)$ from a functional analysis viewpoint. We consider the equation $(*)$ when f and g belong to a weighted L^p space of entire functions. Under general assumptions, the differentiation operator $D = \frac{d}{dz}$ is bounded on this space. We define the left-hand side of $(*)$ by Riesz holomorphic functional calculus.

We give a necessary and sufficient condition for the operator $\sum_{n\geq 0} a_n D^n$ to be surjective. In this case, it admits moreover a bounded linear right-inverse.

We also give an application to the particular equation

$$f(z+1) - f(z) = g(z)$$

which was first investigated by Guichard and Hurwitz.

Janusz Brzdęk *On stability of the linear functional equation*

Joint work with Dorian Popa and Bing Xu.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, X be a Banach space over $\mathbb{K}$, S be a nonempty set, $f\colon S \longrightarrow S$, $F\colon S \longrightarrow X$, m be a positive integer, and $a_j\colon S \longrightarrow \mathbb{K}$ for $j =$

$1, \ldots, m$. We present some results concerning stability of the general linear functional equation (in single variable)

$$\varphi(f^m(x)) = a_1(x)\varphi(f^{m-1}(x)) + \ldots + a_{m-1}(x)\varphi(f(x)) + a_m(x)\varphi(x) + F(x), \quad (1)$$

where $\varphi \colon S \longrightarrow X$ is the unknown function and f^p denotes the p-th iterate of f, i.e., $f^0(x) = x$ and $f^{p+1}(x) = f(f^p(x))$ for $p = 0, 1, 2, \ldots$.

For instance, in the special case where all the functions $a_1, \ldots, a_p$ are constant we have the following result.

Theorem
Suppose that $r_1, \ldots, r_m \in \mathbb{K}$ are the roots of the characteristic equation

$$r^m - a_1 r^{m-1} - \ldots - a_{m-1} r - a_m = 0,$$

$\delta > 0$, and one of the following two conditions holds.

(i) $|r_j| > 1$ *for every* $j = 1, \ldots, m$.

(ii) f *is bijective and* $|r_j| \neq 1$ *for every* $j = 1, \ldots, m$.

If a function $\varphi_s \colon S \longrightarrow X$ satisfies

$$\left\| \varphi_s(f^m(x)) - \sum_{i=1}^{m} a_i \varphi_s(f^{m-i}(x)) - F(x) \right\| \leq \delta, \qquad \forall x \in S,$$

then the equation

$$\varphi(f^m(x)) = a_1 \varphi(f^{m-1}(x)) + \ldots + a_{m-1}\varphi(f(x)) + a_m \varphi(x) + F(x)$$

has a unique solution $\varphi \colon S \longrightarrow X$ such that

$$\|\varphi_s(x) - \varphi(x)\| \leq \frac{\delta}{||r_1| - 1| \cdot \ldots \cdot ||r_m| - 1|}, \qquad \forall x \in S.$$

Pál Burai *On the equivalence of equations involving means and the solution to a problem of Daróczy*

In this work we prove the equivalence of the following two functional equations:

$$f(\mathcal{A}(x, y; p)) + f(\mathcal{H}(x, y; 1-p)) = f(x) + f(y) \qquad x, y \in I,$$

and

$$2f(\mathcal{G}(x, y)) = f(x) + f(y) \qquad x, y \in I.$$

Here I is a nonempty open interval on the positive real line, and $\mathcal{A}(x, y; p)$, $\mathcal{H}(x, y; 1-p)$, $\mathcal{G}(x, y)$ the weighted arithmetic mean with the weight p, the weighted harmonic mean with the weight $1-p$, and the geometric mean respectively.

[1] Z. Daróczy, B. Ebanks, Zs. Páles, *Jelentés a 2001. évi Schweitzer Miklós Matematikai emlékversenyröl*, Matematikai Lapok, 2001-2002/2, 58-60.

[2] Z. Daróczy, Zs. Páles, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen, **61** (2002), 157-218.

[3] Z. Daróczy, K. Lajkó, R. Lovas, Gy. Maksa, Zs. Páles, *Functional equations involving means*, submitted.

[4] Z. Daróczy, Gy. Maksa, Zs. Páles, *Functional equations involving means and their Gauss composition*, Proc. Amer. Math. Soc. **134** (2006), 521-530.

[5] B. Ebanks, *Solution of some functional equations involving symmetric means*, Publ. Math. Debrecen **61** (2002), 579-588.

[6] A. Járai, *Regularity Properties of Functional Equations in Several Variables*, Springer, Advances in Mathematics, Vol. **8**, 2005.

Jacek Chmieliński *Orthogonality preserving property, Wigner equation and stability*

We deal with the stability of the orthogonality preserving property in the class of mappings phase-equivalent to linear or conjugate-linear ones. We give a characterization of approximately orthogonality preserving mappings in this class and we show some connections between the considered stability and the stability of the Wigner equation.

Jacek Chudziak *Stability of the Gołąb–Schinzel functional equation*

Let X be a linear space over a field K of real or complex numbers. At the 38th International Symposium on Functional Equations (2000, Noszvaj, Hungary) Professor Roman Ger raised, among others, the problem of Hyers–Ulam stability of the Gołąb–Schinzel functional equation

$$f(x+f(x)y)=f(x)f(y) \tag{1}$$

for $x, y \in X$. In [1] it has been proved that in the class of functions $f\colon X \longrightarrow K$ satisfying some weak regularity assumptions, the equation (1) is superstable, i.e., every solution of the inequality

$$|f(x+f(x)y)-f(x)f(y)| \le \varepsilon \tag{2}$$

for $x, y \in X$, where ε is a fixed nonnegative real number, either is bounded or satisfies (1). However, it is known (cf. [2, 3]) that the phenomenon of superstability is caused by the fact that we mix two operations. Namely, on the right-hand side of (1) we have the product, but in (2) we measure the distance between the two sides of (1) using the difference. Therefore, it is more natural, to measure the difference between 1 and the quotients of the sides of the equation (1). In the present talk we deal with this problem.

[1] J. Chudziak, J. Tabor, *On the stability of the Gołąb–Schinzel functional equation*, J. Math. Anal. Appl. **302** (2005), 196-200.

[2] R. Ger, *Superstability is not natural*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **13** (1993), 109-123.

[3] R. Ger, P. Šemrl, *The stability of the exponential equation*, Proc. Amer. Math. Soc. **124** (1996), 779-787.

Marek Czerni *Comparison theorems for nonlinear functional inequalities*

We present some comparison theorems for the solutions ψ of nonlinear functional inequalities

$$\begin{cases} \psi[f(x)] \le G(x, \psi(x)), \\ (-1)^p \psi[f^{2M}(x)] \le (-1)^p g_{2M}(x, \psi(x)) \end{cases}$$

where $p \in \{0,1\}$, M is a fixed positive integer and functional sequence g_n is defined by the recurrent formula

$$\begin{cases} g_0(x,y) = y, \\ g_{n+1}(x,y) = G(f^n(x), g_n(x,y)), \qquad n = 0,1,2,\dots \,. \end{cases}$$

In the talk we shall consider the case when the given function G is strictly decreasing with respect to the second variable. Moreover, we shall obtain a characterization of this results in terms of lattice theory.

Thomas M.K. Davison *On the functional equation* $g(xy) + g(x\tau(y)) = 2g(x)g(y)$

A function g on G is *basic* if $\{u \in G : \; g(xuy) = g(xy), \text{ for all } x, y \text{ in } G\} = \{e\}$. Using Stetkær's results [1], we prove that if g is basic, satisfies our equation, and the domain G of g is non-abelian then the centre of G is $\mathrm{Fix}(\tau) := \{z \in G : \; \tau(z) = z\}$.

[1] H. Stetkær, *D'Alembert's functional equations on metabelian groups*, Aequationes Math. **59** (2000), 306-320.

Joachim Domsta *Regular iteration of homeomorphisms of intervals*

According to the known results, the regular iteration of a self mapping f of $\mathbb{R}_+ := (0, \infty)$ without fixed points is closely related to regularly varying solutions Ψ of the corresponding Schröder equation

$$g(\Psi(x)) = \Psi(f(x)) \qquad \text{for } x \in \mathbb{R}_+ \,,$$

where g is a multiplication by a positive constant, not equal to 1. In more general cases, g is also a selfmapping of $\mathbb{R}_+$ and the equation states a (weak) conjugacy of g to f. For regular Ψ joining f and g the relation between the regular iteration groups of f and g is analysed.

Lyudmila S. Efremova *On homoclinic points of C^1-smooth skew products of interval maps*

The structure of a neighborhood of the transverse homoclinic trajectory to the saddle periodic orbit of a C^1-smooth skew product of interval maps is investigated.

In the set of C^1-smooth Ω-stable skew products of interval maps the criterion of the existence of a homoclinic trajectory is proved.

In the space of C^1-smooth skew products of interval maps the subset is distinguished in which the maps with transverse homoclinic trajectories to saddle periodic orbits are everywhere dense.

The examples of C^1-smooth skew products belonging to the boundary of the Ω-stability domain and having the "exotic" properties are given.

The author is partially supported by RFBR, grant No 04-01-00457.

[1] L.S. Efremova, *On the fundamental property of quasiminimal sets of skew products of interval maps*, Intern. Conference "Tikhonov and Contemporary Mathematics", Moscow, Russia, June 19-25, 2006; Abstracts of session "Functional Analysis and Differential Equations", 2006, 64-65.

Danièle Fournier-Prunaret *Attractors bifurcations and basins in two-dimensional and three-dimensional biological models based on logistic maps*

Joint work with Ricardo Lopez-Ruiz.

We consider 2-D and 3-D biological models given by coupling between logistic maps. The considered maps are noninvertible. We study the evolution of attractors (periodic orbits and chaotic attractors) and their basins when parameters change. An important tool is that of critical manifolds, which are specific to noninvertible maps and separate the state space in areas where points have a different number of preimages.

Roman Ger *On a functional congruence related to Gołąb–Schinzel equation*

Anna Mureńko, in her doctoral dissertation [1] devoted to the functional equation

$$f\,(x + M(f(x))y) = f(x) \circ f(y),$$

has come across a functional congruence

$$F(x \circ y) - M(x)F(y) - F(x) \in T.$$

We are looking for a readable description of the solutions of that congruence in the case where given a linear space X over a field K, a subgroup $(T, +)$ of the additive group $(X, +)$ and a groupoid $W \subset K \setminus \{0\}$, F maps W into X and M stands for a selfmapping of K.

[1] Anna Mureńko, *O rozwiązaniach pewnego równania funkcyjnego typu Gołąba–Schinzla*, Doctoral dissertation, Kraków 2006.

Attila Gilányi *Bernstein–Doetsch and Sierpiński theorems for (M, N)-convex functions*

Joint work with Zsolt Páles.

One of the classical results of the theory of convex functions is the theorem of F. Bernstein and G. Doetsch [1] which states that if a real valued Jensen-convex function defined on an open interval I is locally bounded above at one point in I then it is continuous. According to a related result by W. Sierpiński [4], the Lebesgue measurability of a Jensen-convex function implies its continuity, too.

In this talk we generalize the theorems above for (M, N)-convex functions, calling a function $f: I \to J$, (M, N)-convex (c.f., e.g.: [4]) if it satisfies the inequality

$$f(M(x, y)) \leq N(f(x), f(y))$$

for all $x, y \in I$, where I and J are open intervals, M and N are suitable means on I and J, respectively. Our statements also generalize T. Zgraja's recent results on (M, M)-convex functions (c.f.: [5]). A special case of our theorems was presented in [2].

[1] F. Bernstein, G. Doetsch, *Zür Theorie der konvexen Funktionen*, Math. Ann. **76** (1915), 514-526.

[2] A. Gilányi, Zs. Páles, *Bernstein–Doetsch theorem for (M, N)-convex functions*, Talk, 42nd International Symposium on Functional Equations, Hradec nad Moravici, Czech Republic, June 20-27, 2004.

[3] C. Niculescu, L.E. Persson, Convex Functions and Their Applications, CMS Books in Mathematics, Springer, 2006.

[4] W. Sierpiński, *Sur les fonctions convexes mesurables*, Fund. Math. **1** (1920), 125-128.

[5] T. Zgraja, *Continuity of functions which are convex with respect to means*, Publ. Math. Debrecen **63** (2003), 401-411.

Alina Gleska *Oscillatory properties of solutions of nonhomogeneous difference equations*

Joint work with Jarosław Werbowski.

Our goal in this talk is to investigate the monotonic and oscillatory properties of solutions of the nonhomogeneous difference equation

$$(-1)^z \Delta^m y(n) = f(n, y(r_1(n)), y(r_2(n)), \ldots, y(r_k(n))), \tag{E$_\text{z}$}$$

where $z, k \in \mathbb{N}$, $m \geq 2$, $r_i \colon \mathbb{N}_{n_0} \longrightarrow \mathbb{N}_{n_0}$, $\lim_{n\to\infty} r_i(n) = \infty$ and the function $f \colon \mathbb{N}_{n_0} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ satisfies the condition

$$f(n, x_1, x_2, \ldots, x_k) \operatorname{sgn} x_1 \geq \sum_{i=1}^{k} p_i(n)|x_i|, \tag{C}$$

where $p_i \colon \mathbb{N}_{n_0} \longrightarrow \mathbb{R}_+ \cup \{0\}$ $(i = 1, 2, \ldots, k)$.

[1] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1991.

[2] G. Ladas, Ch.G. Philos, Y.G. Sficas, *Sharp conditions for the oscillation of delay difference equations*, J. Appl. Math. Simulation **2** (1989), 101-111.

Dorota Głazowska *An invariance of geometric mean with respect to Lagrangean means*

Joint work with Janusz Matkowski.

The invariance of the geometric mean G with respect to the Lagrangean mean-type mapping (L^f, L^g), i.e., the equation $G \circ (L^f, L^g) = G$, is considered. We show that the functions f and g must be of high class regularity. This fact allows to reduce the problem to a differential equation and determine the second derivatives of the generators f and g.

Dijana Ilišević *Quadratic functionals and sesquilinear forms*

The problem of the representability of quadratic functionals by sesquilinear forms arises from the well-known Jordan–von Neumann characterization of inner product spaces among normed spaces via the parallelogram identity. The aim of this talk is to present some recent algebraic Jordan–von Neumann type theorems in the setting of modules over involutive rings and algebras.

Eliza Jabłońska *Solutions of a generalized Gołąb–Schinzel functional equation*

Let X be a linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We consider solutions $f \colon X \longrightarrow \mathbb{K}$ and $M \colon \mathbb{K} \longrightarrow \mathbb{K}$ of the functional equation

$$f(x + M(f(x))y) = f(x)f(y) \qquad \text{for } x, y \in X \tag{1}$$

such that f is bounded on a set "big" in some sense. As a consequence we obtain measurable in Lebesgue and Baire sense solutions of (1). Our results refer to results of C.G. Popa and J. Brzdęk.

Justyna Jarczyk *Invariance in the class of quasi-arithmetic means with function weights*

Let $I \subset \mathbb{R}$ be an open interval. Given a function $\mu \colon I \times I \longrightarrow (0, 1)$ and a strictly monotonic function $\varphi \colon I \longrightarrow \mathbb{R}$ we consider the mean $M_\mu^\varphi \colon I \times I \longrightarrow \mathbb{R}$ defined by

$$M_{\mu}^{\varphi}(x,y) = \varphi^{-1}\left(\mu(x,y)\varphi(x) + (1-\mu(x,y))\varphi(y)\right).$$

We study the invariance of $M_{\lambda}^{\mathrm{id}}$ in such a class of means, that is the functional equation

$$\lambda(x,y)M_{\mu}^{\varphi}(x,y) + (1-\lambda(x,y))M_{\nu}^{\psi}(x,y) = \lambda(x,y)x + (1-\lambda(x,y))y.$$

In particular, we are interested in the case when

$$\lambda(x,y) = \tfrac{r(x)}{r(x)+r(y)}, \quad \mu(x,y) = \tfrac{s(x)}{s(x)+s(y)} \quad \text{and} \quad \nu(x,y) = \tfrac{t(x)}{t(x)+t(y)}$$

for every $x, y \in I$, where r, s, t are given positive functions on I. As a special case we obtain a recent result of J. Domsta and J. Matkowski [Aequationes Math. **71** (2006), 70-85; Theorem 2].

In particular, we come also to the Bajraktarević means. They satisfy

$$M_{\mu}^{\varphi}(x,y) + M_{1-\mu}^{\psi}(x,y) = x + y, \qquad x, y \in I,$$

i.e., the arithmetic mean is invariant with respect to $(M_{\mu}^{\varphi}, M_{1-\mu}^{\psi})$.

Witold Jarczyk *Almost convex functions on Abelian groups*

Joint work with Miklós Laczkovich.

A σ-ideal $\mathcal{I}$ in $\mathbb{R}^n$ is called linearly invariant if

$$A \in \mathcal{I} \text{ and } x \in \mathbb{R}^n \text{ imply } x - A \in \mathcal{I}.$$

We say that σ-ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, respectively, are conjugate if they fulfil the following Fubini condition:

for every $A \in \mathcal{I}_2$ the sections $\{y \in \mathbb{R}^n : (x,y) \in A\}$ are in $\mathcal{I}_1$ for $\mathcal{I}_1$-a.a. $x \in \mathbb{R}^n$.

In 1970 Marek Kuczma published the following result [Colloq. Math. **21** (1970), 279-284]. Here ϕ stands for the transformation of $\mathbb{R}^n \times \mathbb{R}^n$ defined by $\phi(x,y) = (x+y, x-y)$.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be conjugate proper linearly invariant σ-ideals in $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively, fulfilling the conditions

if $A \in \mathcal{I}_1$ then $aA \in \mathcal{I}_1$ for every $a \in \mathbb{R}$,

if $A \in \mathcal{I}_2$ then $\phi^{-1}(A) \in \mathcal{I}_2$.

If $D \subset \mathbb{R}^n$ is an open convex set and $f\colon D \longrightarrow \mathbb{R}$ is an $\mathcal{I}_2$-almost convex function, i.e.,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for $\mathcal{I}_2$-a.a. $(x,y) \in D$, then there exists a unique Jensen convex function $g\colon D \longrightarrow \mathbb{R}$ such that $g(x) = f(x)$ for $\mathcal{I}_1$-a.a. $x \in D$.

We present a generalization of Kuczma's result for functions defined on a subset of an Abelian group G. Convexity [almost convexity] of $f\colon A \longrightarrow \mathbb{R}$ means here that the inequality

$$2f(x) \leq f(x+h) + f(x-h)$$

holds for all [$\mathcal{I}_2$-a.a.] $(x,y) \in G \times G$ such that $x, x+h, x-h \in A$.

Hans-Heinrich Kairies *On some problems concerning a sum type operator*

The sum type operator F, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x),$$

has been thoroughly discussed in the last years. Nevertheless, there remained some open problems. We state some of them which are connected with

1. Images and pre- images of F,
2. Spectral properties of F,
3. The maximal domain of F,
4. Characterizations of $F[\varphi]$,
5. A two parameter extension of F.

Zoltán Kaiser *On stability of the Fréchet equation*

Let X be a linear space and Y be a normed space over the field of rational numbers. The stability problem concerning the Fréchet equation is the following:

Let the n-th differences of the function $f\colon X \longrightarrow Y$ are bounded, i.e.,

$$\|\Delta_{y_1,\ldots,y_n} f(x)\| \leq \varepsilon \qquad (y_1, \ldots, y_n, x \in X) \tag{1}$$

for some $\varepsilon > 0$. Is there any generalized polynomial g of degree at most $n-1$, for which $f - g$ is bounded?

Without any regularity condition of f, the first positive answer of this problem was given by D.H. Hyers [3]. M. Albert and J.A. Baker [1] gave a result in a more general form, with a shorter proof.

C. Borelli and C. Invernizzi [2] dealt with the stability of the Fréchet equation in the case that the right hand-side of (1) is an α-homogeneous function, but there was a mistake in the proof of the main theorem. Motivated by these results, we prove a stability theorem of the Fréchet equation in Banach spaces over fields with valuation.

[1] M. Albert, J.A. Baker, *Functions with bounded nth differences*, Ann. Polon. Math. **43** (1983), 93-103.

[2] C. Borelli, C. Invernizzi, *Sulla stabilitá dell'equazione funzionale dei polinomi*, Rend. Sem. Mat. Univ. Politec. Torino **57** (1999), 197-208.

[3] D.H. Hyers, *Transformations with bounded mth differences*, Pacific J. Math. **11** (1961), 591-602.

Barbara Koclęga-Kulpa *On some equation connected with Hadamard inequalities*

Joint work with Tomasz Szostok.

We consider some equations connected with Hadamard inequalities. Namely, we observe that the function $f(x) = x^2$ satisfies the condition

$$\int_x^y f(t)\,dt = (y-x)\left[\frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(x) + \frac{1}{6}f(y)\right]. \tag{1}$$

We ask about functions having properties of this kind. Moreover, we present some generalization of the equation (1), i.e.,

$$f(y) - g(y) = (y-x)[h(x+y) + \phi(x) + \psi(y)],$$

which was considered in [1] for functions acting on $\mathbb{R}$. We determine all solutions of this equation in more general case — for integral domains.

[1] T. Riedel, P.K. Sahoo, *Mean value theorems and functional equations*, World Scientific, Singapore – New Jersey – London – Hong Kong, 1998.

Imre Kocsis *A bisymmetry equation on restricted domain*

Let $X \subset \mathbb{R}$ be an interval of positive length and define the set $\Delta = \{(x,y) \in X \times X \mid x \geq y\}$. In this note we give the solution of the equation

$$F(G_1(x,y), G_2(u,v)) = G(F(x,u), F(y,v)), \qquad (x,y) \in \Delta,\ (u,v) \in \Delta,$$

where the functions $F\colon X \times X \longrightarrow X$, $G_1\colon \Delta \longrightarrow X$, $G_2\colon \Delta \longrightarrow X$, and $G\colon F(X,X) \times F(X,X) \longrightarrow X$ are continuous and strictly monotonic (in the same sense) in each variable. The result is a generalization of a previous one investigated by the author (under publication in Aequationes Mathematicae). The original problem was published by R. Duncan Luce and J.A. Marley in The Journal of Risk and Uncertainty (**30:1** (2005), 21-62).

Dorota Krassowska *On a nonlinear simultaneous system of functional inequalities*

Under some conditions on given real functions f, F, g, G we determine all the continuous at least at one point solutions φ of the simultaneous system of functional inequalities

$$\begin{cases} \varphi\left(f(x)\right) \leq F\left(\varphi(x)\right) \\ \varphi\left(g(x)\right) \leq G\left(\varphi(x)\right) \end{cases}, \qquad x \in I,$$

where $I \subset \mathbb{R}$ is an arbitrary interval.

Xiaopei Li *An iterative equation on the unit circle*

Joint work with Shengfu Deng.

A functional equation of nonlinear iterates is discussed on the circle S^1 for its continuous solutions and differentiable solutions. By lifting to $\mathbb{R}$, the existence, uniqueness and stability of those solutions are obtained. Techniques of continuation are used to guarantee the preservation of continuity and differentiability in lifting.

Arkadiusz Lisak *A characterization of some operators by functional equations*

Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and let r_n and q_n be two sequences of real numbers for every $n \in \mathbb{N}$. We define for fixed $t \neq 0$, $y, z \in \mathbb{R}$, $y \neq z$ sequence of operators ${}_t\phi^{(n)}_{y,z}$ in the following way

$$\begin{aligned} {}_t\phi^{(1)}_{y,z} f(x) &= t\frac{f(\frac{x+y}{t}) - f(\frac{x+z}{t})}{y-z}, \\ {}_t\phi^{(n+1)}_{y,z} f &= t[r_n\, {}_t\phi^{(n)}_{y,z} f + q_n\, {}_t\phi^{(n)}_{2y,2z} f] \end{aligned}$$

for $x \in \mathbb{R}$. For every $n \in \mathbb{N}$ we consider functional equations

$${}_t\phi^{(n)}_{y,z} f(x) = g(x),$$

where $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ are unknown functions, and we solve them in special cases. One of the special cases has been dealt with in [1].

Next we consider and solve the equations on the Abelian groups. We show that the equations characterize polynomials (or generalized polynomials) and their derivatives (or homomorphisms of a special form).

[1] T. Riedel, M. Sablik, A. Sklar, *Polynomials and divided differences*, Publ. Math. Debrecen **66** (2005), 313-326.

Lászlo Losonczi *Polynomials with all zeros on the unit circle*

Joint work with P. Lakatos.

We summarize recent results on polynomials all of whose zeros are on the unit circle. We give sufficient conditions for this and also necessary conditions (in terms of the coefficients), describe the methods used. Finally we mention a very general new sufficient condition for self-inversive polynomials.

Grażyna Łydzińska *On some set-valued iteration semigroups*

Let X be an arbitrary set. We present the necessary and sufficient conditions for a set-valued function $A\colon X \longrightarrow 2^{\mathbb{R}}$ under which a family of multifunctions of the form

$$A^{-1}\left(A(x) + \min\{t, q - \inf A(x)\}\right)$$

where $q := \sup A(X)$, naturally occurring in the iteration theory, is an iteration semigroup.

Andrzej Mach *On some functional equations involving involutions*

Joint work with Zenon Moszner.

We present some theorems characterizing solutions of the equation

$$f(x) = f(\varphi(x)) + g(x),$$

where φ is a given involution, and particularly differentiable solutions of the equation $f(x) = f(1-x) + 2x - 1$. The stability of this equation and non-existence of the extremal points of the set of solutions are proved.

[1] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222-224.

[2] A. Mach, Z. Moszner, *On some functional equations involving involutions*, Sitzungsberichte of the Austrian Academy of Sciences (ÖAW), in print.

[3] P. Volkmann, *Caractérisation de la fonction $f(x) = x$ par un système de deux équations fonctionnelles*, C. R. Math. Rep. Acad. Sci. Canada, **5** (1983), 27-28.

Elena Makhrova *Dendrites with the periodic points property*

Dendrite is a locally connected continuum without subsets homeomorphic to a circle.

A dendrite X is said to have the periodic points property provided that for any continuous map $f\colon X \longrightarrow X$ and for an arbitrary subcontinuum $Y \subset X$ such that $Y \subseteq f(Y)$ the last inclusion implies $Y \cap \mathrm{Per}(f) \neq \emptyset$, $\mathrm{Per}(f)$ is the periodic points set of f.

In [1] it is shown that a finite tree (a dendrite with a finite ramification points set and points of finite order) has the periodic points property, the example of a dendrite which has no the periodic points property is constructed.

In the report the structure of dendrites having the periodic points property is investigated. The next result is true.

Theorem A
Let X be a dendrite such that the derivative of the ramification points set of X is at most countable. Then X has the periodic points property.

Theorem A does not occur if the derivative of the ramification points set of a dendrite X is uncountable. The example of such dendrite is constructed here.

Necessary conditions of structure of dendrites having the periodic points property are presented.

This research is partially supported by RFBR, grant No 04-01-00457.

[1] E.N. Makhrova, *On the existence of periodic points of continuous maps of dendrites*, Some Problems of Fundamental and Applied Mathematics, Moscow 2006 (to appear).

Janusz Matkowski *On extension of solutions of simultaneous systems of functional equations*

Some sufficient conditions which allow to extend every local solution of a simultaneous system of equations to a global one are presented.

Fruzsina Mészáros *Functional equations on group*

Joint work with Zs. Ádám, K. Lajkó and Gy. Maksa.

Let G be an arbitrary group written additively. We give the general solution of the functional equation

$$f(x)\, f(x+y) = f(y)^2\, f(x-y)^2\, g(y) \qquad (x, y \in G)$$

and all the solutions of

$$f(x)\, f(x+y) = f(y)^2\, f(x-y)^2\, g(x) \qquad (x, y \in G)$$

with the additional supposition $g(x) \neq 0$ for all $x \in G$. In both cases $f, g\colon G \longrightarrow \mathbb{R}$ are unknown functions.

Janusz Morawiec *On a refinement type equation*

Joint work with Rafał Kapica.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. We show that the trivial function is the unique L^1-solution of the following refinement type equation

$$f(x) = \int_\Omega \left| \frac{\partial A}{\partial x}(x,\omega) \right| f(A(x,\omega))\, dP(\omega)$$

in a wide class of the given functions A. This class contains functions of the form $A(x,\omega) = \alpha(\omega)x - \beta(\omega)$ with $-\infty < \int_\Omega \log|\alpha(\omega)|\, dP(\omega) < 0$.

Jacek Mrowiec *Generalized convex functions in linear spaces*

The notion of generalized convex functions has been introduced by E.F. Beckenbach in the following way:
Let $\mathcal{F}$ be a two-parameter family of continuous real-valued functions defined on an open interval (a,b) such that for any two distinct points $x_1, x_2 \in (a,b)$ and any $t_1, t_2 \in \mathbb{R}$ there exists exactly one $\varphi = \varphi_{(x_1,t_1)\,(x_2,t_2)} \in \mathcal{F}$ satisfying

$$\varphi(x_i) = t_i, \qquad i = 1, 2.$$

We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is $\mathcal{F}$-convex if for any distinct $x_1, x_2 \in (a,b)$

$$f(x) \leq \varphi_{(x_1,f(x_1))\,(x_2,f(x_2))}(x) \qquad \text{for every } x \in [x_1, x_2].$$

We present a method for generating two-parameter families not only on the real line. This method allows us to extend the notion of generalized convex functions to linear spaces. Such functions have many properties of functions, which are convex in the usual sense, and proofs of their properties are easy.

Anna Mureńko *On solutions of some conditional generalizations of the Gołąb-Schinzel equation*

We deal with the conditional functional equations

$$\text{if } x, y, x + M(g(x))y > 0, \text{ then } g(x + M(g(x))y) = g(x) \circ g(y),$$

$$\text{if } x, y, x + M(g(x))y > 0, \text{ then } g(x + M(g(x))y) = g(x)g(y),$$

where $M\colon \mathbb{R} \longrightarrow \mathbb{R}$, $\circ\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g\colon (0,\infty) \longrightarrow \mathbb{R}$ is Lebesgue measurable or Baire measurable. We consider the above equations under some additional (different for each equation) assumptions.

Veerapazham Murugan *Smooth solutions for a functional equation involving series of iterates*

Joint work with P.V. Subrahmanyam.

In this talk we give sufficient conditions for the existence and uniqueness of C^2 solution for the functional equation

$$\sum_{i=1}^{\infty} \lambda_i f^i(x) = F(x), \qquad x \in [a,b] \subset \mathbb{R},$$

where λ_i's are nonnegative real numbers with $\sum_{i=1}^{\infty} \lambda_i = 1$ and F is a given C^2 function on $[a,b]$ satisfying some additional conditions. Such functional equations have been studied earlier by Kulczycki, Shengfu, Tabor, Xiaopei, Żołdak and the present authors.

Adam Najdecki *On the stability of some generalization of Cauchy, d'Alembert and quadratic functional equations*

Let $X \neq \emptyset$ be a set, $(Y,+)$ be a commutative semigroup with a complete invariant metric, $k \in \mathbb{N}$ and let $A, B\colon Y \longrightarrow Y$, $G_i\colon X \times X \longrightarrow X$ for $i \in \{1, 2, \ldots, k\}$. We consider stability of the functional equation

$$\sum_{i=1}^{k} f(G_i(x,y)) = A(f(x)) + B(f(y))$$

in the class of functions $f\colon X \longrightarrow Y$, as well as of the equation

$$\sum_{i=1}^{k} f(G_i(x,y)) = kf(x)f(y)$$

in the class of functions mapping X into a real or complex normed algebra with a multiplicative norm.

Wiesława Nowakowska *Sufficient conditions for the oscillation of solutions of iterative functional equations*

Joint work with Jarosław Werbowski.

Sufficient conditions for the oscillation of all solutions of iterative functional equations will be presented. Oscillation criteria for difference equations will be obtained.

Andrzej Olbryś *A characterization of* $(t_1, \ldots, t_n)$*-Wright affine functions*

Let $t_1, \ldots, t_n$, $n \geq 2$, be fixed positive numbers, let X be a linear space over the field $L(t_1, \ldots, t_n)$ generated by $t_1, \ldots, t_n$ (i.e., the smallest field containing the set $\{t_1, \ldots, t_n\}$) and let Y be a commutative group.

Following [1] where the definition of $(t_1, \ldots, t_n)$-Wright convex function was given we introduce the definition of $(t_1, \ldots, t_n)$-Wright affine function as a function $f\colon D \longrightarrow Y$ satisfying the following functional equation:

$$\Delta_{t_1 z, \ldots, t_n z} f(x) = 0, \qquad (x,z) \in D \times X : x + (t_1 z + \ldots + t_n z) \in D,$$

where $D \subset X$ is a $L(t_1, \ldots, t_n)$-convex set.

In the paper [2] K. Lajkó has given a characterization of $(t, 1-t)$-Wright affine functions. We extend this result to $(t_1, \ldots, t_n)$-Wright affine functions of an arbitrary order.

[1] A. Gilányi, Zs. Páles, *On Dinghas-type derivatives and convex functions of higher order*, Real Anal. Exchange **27** (2001/2002), 485-494.

[2] K. Lajkó, *On a functional equation of Alsina and Garcia-Roig*, Publ. Math. Debrecen **52** (1998), 507-515.

[3] A.Olbryś, *A characterization of* $(t_1, \ldots, t_n)$*-Wright affine functions*, submitted.

Ágota Orosz *Difference equations on discrete polynomial hypergroups*

In the classical theory of difference equations the translate of a function by n and the translation of the function n-times by 1 give the same result for all n in $\mathbb{N}$. But in the hypergroup case there are two different ways to define difference equations along these two interpretations. In this talk we give the solutions of a homogenous linear difference equation of order N on a polynomial hypergroup (which is actually a difference equation with nonconstant coefficients in the classical sense) in both cases.

Boris Paneah *Strong stability of functional equations in several variables*

We deal with compact supported Banach-valued functions F satisfying

$$(\mathcal{P}F)(x,y) := F(a(x,y)) - \sum_{j=1}^{n} \alpha_j(x,y)F(a_j(x,y)) = H(x,y)$$

for all (x,y) in a bounded domain $D \subset \mathbb{R}^2$. The prototype is the Jensen equation with $a = \alpha_1 x + \alpha_2 y,\ a_1 = x,\ a_2 = y$, and real positive numbers α_1, α_2 satisfying $\alpha_1 + \alpha_2 = 1$. The problem of its stability goes back to Ulam (1940). Our approach is novel in two essential ways.

The first is that if $a = \sum \alpha_j a_j$ on a one-dimensional submanifold $\Gamma \subset \overline{D}$ (weak Jensen operator $\mathcal{P}$), then under quite general conditions the stability problem for $\mathcal{P}$ is overdetermined: the smallness of H **only** on Γ implies the nearness of F to a linear function (strong stability).

The second is a functional analytic point of view. We consider the linear operator $\mathcal{P}_\Gamma$ – the restriction of $\mathcal{P}$ to Γ – between appropriate function spaces and give conditions of its surjectivity. The stability then follows from functional analytic considerations.

Iwona Pawlikowska *Flett-type means II*

We continue the investigation of properties of means obtained from Flett's mean value theorem. We take into account two generalizations of Flett mean value theorem and we show that means obtained from these theorems are unique. There is also discussed problem of equivalence Flett-type means to the well known means.

Zsolt Páles *A regularity problem concerning the equality of generalized quasi-arithmetic means*

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Given a continuous strictly monotone function $\varphi\colon I \longrightarrow \mathbb{R}$ and a Borel probability measure μ on $[0,1]$, the mean $M_{\varphi,\mu}\colon I^2 \longrightarrow I$ is defined by

$$M_{\varphi,\mu}(x,y) := \varphi^{-1}\Bigg(\int\limits_0^1 \varphi(tx+(1-t)y)\,d\mu(t)\Bigg) \qquad (x,y\in I).$$

The equality problem of these means is to describe all pairs (φ,μ) and (ψ,ν) such that

$$M_{\varphi,\mu}(x,y) = M_{\psi,\nu}(x,y) \qquad (x,y\in I).$$

By a recent result obtained jointly with Z. Makó, if there exists a point $p\in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p)\neq 0$, then a necessary condition for the above equality problem is that the first moments of the measures μ and ν be equal, i.e.,

$$\mu_1 := \int\limits_0^1 t\,d\mu(t) = \int\limits_0^1 t\,d\nu(t) =: \nu_1\,.$$

Introducing the notion of quasi-differentiability, we deduce this (and more general conditions) under much weaker assumptions.

Magdalena Piszczek *On a multivalued second order differential problem*

Let K be a closed convex cone with the nonempty interior in a real Banach space and let $\mathrm{cc}(K)$ denote the family of all nonempty convex compact subsets of K. Assume that continuous linear multifunctions $H,\Psi\colon K\longrightarrow \mathrm{cc}(K)$ are given. We consider the following problem

$$\begin{aligned} D^2\,\Phi(t,x) &= \Phi(t,H(x)),\\ D\,\Phi(t,x)|_{t=0} &= \{0\},\\ \Phi(0,x) &= \Psi(x) \end{aligned}$$

for $t\geq 0$ and $x\in K$, where $D\,\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t and $D^2\,\Phi(t,x) = D(D\,\Phi(t,x))$.

Barbara Przebieracz *Near iterability*

Inspired by Problem (3.1.12) posed by E. Jen in [T] we present various approaches to the concept of near-iterability. We deal with selfmappings of a real compact interval, characterize and compare a few classes of near-iterable functions in a sense. That includes

- almost iterable functions, that is continuous $f\colon X\longrightarrow X$, for which there exists an iterable $g\colon X\longrightarrow X$ such that

  $$f^n - g^n \text{ converges to } 0 \text{ everywhere in } X \tag{1}$$

  and the convergence is uniform on every interval with endpoints being two consecutive fixed points of f (cf. [J]);

- functions satisfying (1);

and some weaker condition than (1), that is

- functions f for which there exists an iterable g such that

$$f^n(x) - g^n(x) \text{ converges to 0 for every } x \in X \setminus M, \tag{2}$$

where the set M has empty interior;

- approximately iterable functions, that is continuous $f\colon X \longrightarrow X$, such that for every $\varepsilon > 0$ there exists an iterable function $g\colon X \longrightarrow X$ and a positive integer n_0 satisfying the inequality $|f^n(x) - g^n(x)| < \varepsilon$, $n \geq n_0$, $x \in X$;

- closure of the set of all iterable functions.

(This is the continuation of my talk at 10th ICFEI).

[J] W. Jarczyk, *Almost iterable functions*, Aequationes Math. **42** (1991), 202-219.

[T] Gy. Targonski, *New directions and open problems in iteration theory*, Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, No. **229**, Forschungszentrum, Graz, 1984.

Maciej Sablik *A generalization of generalized bisymmetry*

Many authors (cf. e.g. the references below) have been concerned with functional equations of generalized bisymmetry. In particular, under some assumptions the form of solutions has been determined in the case where unknown functions map finite products of intervals into reals. In the present talk we determine (again, under some regularity assumptions) the form of operators M defined in some function spaces of measurable functions and satisfying a "Fubini type" equality

$$M_{[s]}M_{[t]}x(s,t) = M_{[t]}M_{[s]}x(s,t)$$

for every real function x such that $x(s,\cdot)$, $x(\cdot,t)$ belong to a given function space. The proofs use the results obtained earlier for the generalized bisymmetry.

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York – London, 1966.

[2] J. Aczél, Gy. Maksa, M. Taylor, *Equations of generalized bisymmetry and of consistent aggregation: Weakly surjective solutions which may be discontinuous at places*, J. Math. Anal. Appl. **214** (1997), 22-35.

[3] Gy. Maksa, *Solution of generalized bisymmetry type equations without surjectivity assumptions*, Aequationes Math. **57** (1999), 50-74.

[4] Gy. Maksa, *Quasisums and generalized associativity*, Aequationes Math. **69** (2005), 6-27.

[5] A. Münnich, Gy. Maksa, R.J. Mokken, *n-variable bisection*, J. Math. Psych. **44** (2000), 569-581.

Vsevolod Sakbaev *On the spaces of functions integrable with respect to finite additive measure and the generalized convergence*

The investigation of qualitative properties of the dynamical systems and the ill-posed boundary-value problems by regularization methods lead to considering of divergent sequences. To describe the behavior of the divergent sequence by the regular methods of generalized summation and to obtain the distributions on the set of its limit points we consider the measures on the set of natural numbers $\mathbb{N}$ which concentrated on the infinity. Then any nonnegative normalized measure which concentrated on the infinity defines the regular method of generalized summation of sequences such that any bounded sequence is summable. We investigate the procedure of weak integration of vector-valued functions on the set with bounded additive measure and the properties of the spaces of integrable functions. The Hilbert space of square integrable function is applied to description of the regularization of ill-posed Cauchy problem ([1]).

This work is partially supported by RFBR, grant No 04-01-00457.

[1] V.Zh. Sakbaev, *Set-valued mappings specified by regularization of the Schrödinger equation with degeneration*, Comp. Math. Math. Phys. **46** (2006), 651-665.

Adolf Schleiermacher *On real places*

Let K be a formally real field and $\varphi\colon K \longrightarrow \mathbb{R} \cup \{\infty\}$ a mapping which satisfies

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \tag{$*$}$$

whenever the righthand sides in these equations are defined. Here the operations are extended as usual to the symbol ∞ so that $x + \infty = \infty$, $y \cdot \infty = \infty$ for $x, y \in \mathbb{R}$, $y \neq 0$ and $\infty \cdot \infty = \infty$ while $\infty + \infty$ and $0 \cdot \infty$ remain undefined. A mapping φ satisfying $(*)$ is called a real place if in addition $\varphi(1) = 1$. This last requirement serves to exclude trivial cases for which $\varphi(K) \subseteq \{0, \infty\}$. In K we introduce a partial ordering $\leq_P$ defined as usual by its set P of non-negative elements. A real place φ will be called order preserving if for $a \leq_P b$ and $\varphi(a), \varphi(b) \in \mathbb{R}$ we have $\varphi(a) \leq \varphi(b)$. If $\varphi(K) \subseteq \mathbb{R}$ then a real place φ is simply an isomorphic embedding of K in $\mathbb{R}$. The object of this talk is to investigate the fields K with partial ordering $\leq_P$ for which all order preserving real places are embeddings.

It is known that this occurs when all orderings of K compatible with $\leq_P$ are Archimedean. Other necessary and sufficient conditions will be obtained by studying various properties of the partial ordering $\leq_P$. For instance, we consider the topology $\mathcal{T}$ defined by $\leq_P$ or we study the unit interval $T = \{x : -1 \leq_P x \leq_P 1\}$ as a convex set in the vector space $K_{\mathbb{Q}}$ over $\mathbb{Q}$.

For a subset S of $K_{\mathbb{Q}}$ the core of S consists of all $x \in S$ such that each line through x contains an open segment (a, b) with $x \in (a, b)$ and $(a, b) \subseteq S$. Here of course, $(a, b) = \{\rho a + (1 - \rho)b : \rho \in \mathbb{Q} \text{ and } 0 < \rho < 1\}$. Note that in infinite dimensional vector spaces there exist convex sets whose affine hull is the whole space but whose core is nevertheless empty. Conditions characterizing the fields for which all order preserving real places are embeddings are for instance:

1. *The sequence $\frac{1}{2^n}$ converges to zero with respect to topology $\mathcal{T}$.*
2. *In the vector space $K_{\mathbb{Q}}$ the (convex) set T has non-empty core.*
3. *The topology $\mathcal{T}$ can be defined by a spectral norm μ satisfying $\mu(1) = 1$, $\mu(\lambda x) = |\lambda|\mu(x)$ for all $x \in K$, $\lambda \in \mathbb{Q}$, and $\mu(x) \leq 1$ if and only if $x \in T$.*

Stanisław Siudut *Stability of the Cauchy equation for convolutions*

Let S be a real or complex normed algebra with multiplication $*$ and let F be a S-valued function defined on S.

The equation

$$F(x * y) - F(x) * F(y) = 0 \qquad (x, y \in S)$$

will be called superstable if for each F satisfying

$$\|F(x * y) - F(x) * F(y)\| \leq \delta, \qquad x, y \in S,$$

where $\delta > 0$, either F is a bounded function or $F(x * y) = F(x) * F(y)$ for all $x, y \in S$.

The above equation is not superstable for some functions algebras with convolution multiplication $*$. However, under some additional assumptions on the range $F(S)$ of F, if the set $\{F(x*y) - F(x)*F(y) : x, y \in S\}$ is bounded in S then F is a bounded function or $\|F(x*y) - F(x)*F(y)\| = 0$ for all $x, y \in S$.

Dariusz Sokołowski *Solutions with exponential character to a linear functional equation and roots of its characteristic equation*

We deal with the functional equation

$$\varphi(x) = \int_S \varphi\left(x + M(s)\right) \sigma(ds) \tag{1}$$

and its characteristic equation

$$\int_S e^{\lambda M(s)} \sigma(ds) = 1 \tag{2}$$

assuming that (S, Σ, σ) is a finite measure space, $M \colon S \longrightarrow \mathbb{R}$ is a Σ-measurable bounded function and $\sigma(M \neq 0) > 0$. By a *solution* of (1) we mean a real function φ defined on a set of the form $(a, +\infty) \cap W$ with $W + \langle M(S) \rangle \subset W$, and such that for every

$$x \in (a + \sup\{|M(s)| : \ s \in S\}, +\infty) \cap W$$

the integral $\int_S \varphi(x + M(s))\sigma(ds)$ exists and (1) holds. Our main result reads.

THEOREM
Assume λ is a real number. If for some solution φ of (1) *either a finite and nonzero limit*

$$\lim_{x \to +\infty} \frac{\varphi(x)}{xe^{\lambda x}}$$

exists, or

$$0 < \liminf_{x \to +\infty} \frac{\varphi(x)}{e^{\lambda x}} \leqslant \limsup_{x \to +\infty} \frac{\varphi(x)}{e^{\lambda x}} < +\infty,$$

then λ is a root of (2).

Paweł Solarz *Iterative roots for homeomorphisms with a rational rotation number*

Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism such that its rotation number is rational, i.e., $\alpha(F) = \frac{q}{n}$, where $q, n \in \mathbb{N}$, $\gcd(q, n) = 1$ and $q < n$. Denote by $\operatorname{Per} F$ the set of all periodic points of F and let

$$\begin{aligned} \mathcal{M}_F^+ := \Big\{ & u \in \operatorname{Per} F : \exists\, w \in \operatorname{Per} F \setminus \{u\} : \\ & \big(\, \overrightarrow{(u,w)} \cap \operatorname{Per} F = \emptyset \wedge \forall\, z \in \overrightarrow{(u,w)}\ z \in \overrightarrow{(u, F^n(z))}\,\big) \Big\}. \end{aligned}$$

The continuous and orientation-preserving solutions of the equation

$$G^m(z) = F(z), \qquad z \in S^1,$$

where $m \geq 2$ is an integer, exist if and only if some orientation-preserving continuous solution $\Phi\colon \operatorname{Per} F \to S^1$ of the equation

$$\Phi(F(z)) = e^{2\pi \mathrm{i} \frac{q}{n}} \Phi(z), \qquad z \in \operatorname{Per} F$$

satisfies

$$e^{2\pi \mathrm{i} \frac{q+jn}{mn}} \Phi[\operatorname{Per} F] = \Phi[\operatorname{Per} F] \quad \text{and} \quad e^{2\pi \mathrm{i} \frac{q+jn}{mn}} \Phi[\mathcal{M}_F^+] = \Phi[\mathcal{M}_F^+]$$

for some $j \in \{0, \dots, m-1\}$.

Tomasz Szostok *On some conditional Cauchy equation*

Let X be a real inner product space. The Cauchy equation with the right-hand side multiplied by some constant is considered. This equation is assumed for all $x, y \in X$ satisfying the equality $\frac{\|x-y\|}{\|x+y\|} = \alpha$ where $\alpha \in (0, \infty)$ is given. Solutions of this conditional equation under some assumptions are determined.

Jacek Tabor *Shadowing with multidimensional time in Banach spaces*

As it is well-known, many stability problems can be reduced to the following one concerning difference equations:

PROBLEM
Let $(A_k)_{k\in\mathbb{Z}}$ be a sequence of bounded linear operators in a Banach space X and let $(y_k)_{k\in\mathbb{Z}} \subset X$ be such that

$$\|y_{k+1} - A_k y_k\| \le \delta \qquad \text{for } k \in \mathbb{Z}.$$

Does there exist a solution to $x_{k+1} = Ax_k$ such that

$$\|x_k - y_k\| \le \varepsilon\,?$$

The aim of the talk is to deal with the generalization of the above problem to the case of multidimensional time. Applying the Taylor functional calculus [3] we generalize the results from [2] to the case of Banach spaces [1].

[1] Z. Mączyńska, Jacek Tabor, *Shadowing with multidimensional time in Banach spaces,* J. Math. Anal. Appl., to appear.

[2] S.Yu. Pilyugin, Sergei B. Tikhomirov, *Shadowing in actions of some Abelian groups*, Fund. Math. **179** (2003), 83-96.

[3] J.L. Taylor, *The analytic-functional calculus for several commuting operators*, Acta Math. **125** (1970), 1-38.

Józef Tabor *Restricted stability of the Cauchy equation in metric semigroups*

Joint work with Jacek Tabor.
Let $f\colon [0,\infty) \longrightarrow [1,\infty)$ be defined by

$$f(x) := \frac{1}{3}x + 1.$$

Then

$$f(x+y) - f(x) - f(y)| \le 1 \qquad \text{for } x, y \in [0,\infty),$$

but there is no additive function $a\colon [0,\infty) \longrightarrow [1,\infty)$ satisfying the condition

$$\sup_{x\in[0,\infty)} |f(x) - a(x)| < \infty.$$

However, function f can be approximated by an additive one on $[3,\infty)$. Namely, we have

$$\left|f(x) - \frac{1}{3}x\right| \le 1 \qquad \text{for } x \in [3,\infty).$$

Such an effect is caused by the fact that the division by 2 is not globally performable in the set $[1,\infty)$.

Our aim is to deal with the stability of the Cauchy equation in such a frame.

Peter Volkmann *Characterization of the absolute value of complex linear functionals by functional equations*

Joint work with Karol Baron.

Let V be a complex vector space. The functions $f(x) = |\varphi(x)|$ $(x \in V)$ with some linear $\varphi\colon V \longrightarrow \mathbb{C}$ are characterized by each of the two equations

$$\sup_{\alpha\in\mathbb{R}} f(x+e^{\alpha i}y) = f(x)+f(y) \qquad \text{and} \qquad \inf_{\alpha\in\mathbb{R}} f(x+e^{\alpha i}y) = |f(x)-f(y)|.$$

The paper will appear in Sem. LV (http://www.mathematik.uni-karlsruhe.de/~semlv).

Janusz Walorski *On some solutions of the Schröder equation in Banach spaces*

Let X be a Banach space. We consider the problem of existence and uniqueness of solutions of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

where the function $f\colon X \longrightarrow X$ and the bounded linear operator $A\colon X \longrightarrow X$ are given.

Szymon Wąsowicz *On error bounds for Gauss–Legendre and Lobatto quadrature rules*

For a function $f\colon[-1,1] \longrightarrow \mathbb{R}$ let

$$\begin{aligned}
\mathcal{G}_2(f) &:= \frac{1}{2}\left(f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)\right),\\
\mathcal{G}_3(f) &:= \frac{5}{18}f\left(-\frac{\sqrt{15}}{5}\right)+\frac{4}{9}f(0)+\frac{5}{18}f\left(\frac{\sqrt{15}}{5}\right),\\
\mathcal{L}(f) &:= \frac{1}{12}f(-1)+\frac{5}{12}f\left(-\frac{\sqrt{5}}{5}\right)+\frac{5}{12}f\left(\frac{\sqrt{5}}{5}\right)+\frac{1}{12}f(1),\\
\mathcal{S}(f) &:= \frac{1}{6}\left(f(-1)+4f(0)+f(1)\right).
\end{aligned}$$

The operators $\mathcal{G}_2$ and $\mathcal{G}_3$ are connected with Gauss–Legendre quadrature rules. The operators $\mathcal{L}$ and $\mathcal{S}$ are connected with Lobatto and Simpson's quadrature rules, respectively. We establish the following inequalities:

Proposition
If $f\colon[-1,1] \longrightarrow \mathbb{R}$ is 3-convex, then $\mathcal{G}_2(f) \le \mathcal{G}_3(f) \le \mathcal{S}(f)$ and $\mathcal{L}(f) \le \mathcal{S}(f)$.

We apply this result to give the error bounds for quadrature rules $\mathcal{G}_3$ and $\mathcal{L}$ for four times differentiable functions (instead of six times differentiable functions as in the classical results known from numerical analysis).

[1] Sz. Wąsowicz, *On error bounds for Gauss–Legendre and Lobatto quadrature rules*, J. Inequal. Pure Appl. Math. **7**(3) (2006), Art. 84, 1-7.

Marek Żołdak *Asymptotic stability of isometries in p-homogeneous F-spaces*

Joint work with Józef Tabor.

The equation of isometry in Banach spaces is stable in Hyers–Ulam sense. It happens that in complete Frechet spaces this equation is not stable. We prove that if $p \in (0,1]$, $r > 0$, $\varepsilon > 0$; X, Y are complete p-homogeneous spaces, $f\colon X \longrightarrow Y$ is a surjective mapping such that $f(0) = 0$ and

$$\left| \|f(x) - f(y)\|^r - \|x - y\|^r \right| \le \varepsilon \qquad \text{for } x, y \in X,$$

then there exist a linear surjective isometry U and a constant $L > 0$ such that

$$\|f(x) - U(x)\| \le L(\varepsilon^p \|x\|^{1-pr} + \varepsilon^{\frac{1}{r}}) \qquad \text{for } x \in X,$$

when $pr < 1$, and

$$\|f(x) - U(x)\| \le L\varepsilon^{\frac{1}{r}} \qquad \text{for } x \in X,$$

when $pr > 1$.

Problems and Remarks

1. Remark. *Dynamical systems generated by two maps*

Let $\delta_j\colon I \longrightarrow I$, $j = 1, 2$, be continuous maps of the interval $I = [-1, 1]$ into itself satisfying:

1° all δ_j do not decrease; $\qquad 2^\circ$ $\mathcal{R}(\delta_1) \cap \mathcal{R}(\delta_2) = \{0\}$; $\qquad 3^\circ$ $\mathcal{R}(\delta_1) \cup \mathcal{R}(\delta_2) = I$

with $\mathcal{R}(\delta_j)$ a range of δ_j. The semigroup Φ_δ generated by δ_1, δ_2 consists of al maps $\delta_J\colon I \longrightarrow I$ of the form $\delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1}$, where $J = (j_1, \ldots, j_n)$ is an arbitrary multi-index with $j_k = 1$ or $j_k = 2$. A sequence $(t_1, \ldots, t_n, \ldots)$ of points $t_k \in I$ is called *orbit* if for all $k = 1, 2, \ldots$ we have

$$t_{k+1} = \delta_{j_k}(t_k), \qquad j_k \in \{1, 2\}. \tag{$*$}$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be arbitrary disjoint closed subsets in I and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. An orbit $(t_1, t_2, \ldots)$ is called *$\mathcal{T}$-guiding* if in $(*)$ $j_k = 1$ as $t_k \in \mathcal{T}_2$ and $j_k = 2$ as $t_k \in \mathcal{T}_1$. This notion plays a crucial role when studying various forms of the solvability of general linear functional equations. For example, when describing the kernel of the Cauchy type operator $CF := F(\delta_1 + \delta_2) - F(\delta_1) - F(\delta_2)$ with the above

δ_1, δ_2 the result follows immediately if we note that the maximal value of any element $F \in \ker C$ spreads along $\mathcal{T}$-guiding orbits, where $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{T}_j = \{t \mid \delta_j'(t) = 0\}$, see [B. Paneah, Funct. Anal. Appl., **37** (2003), 46-60]. Finally, *attractor* $\mathcal{A}$ in Φ_δ is a collection of points $x \in I$ such that for any point $t \in I$ there is a $\mathcal{T}$-guiding orbit $(t, \delta_{j_1}(t), \ldots)$ converging to x. The main problem (solution of which finds immediately many applications) is as follows: given maps δ_1, δ_2 and sets $\mathcal{T}_1$, $\mathcal{T}_2$ to describe all attractors of the dynamical system Φ_δ . A particular solution of the problem is given in the above mentioned paper.

Boris Paneah

2. Remark.

During the 44th International Symposium on Functional Equations held in Louisville in May, 2006, Janusz Brzdęk asked on all self-mappings of a given semigroup satisfying the equation

$$f(x) + f(y + f(y)) = f(y) + f(x + f(y)). \tag{1}$$

Recently, Marcin Balcerowski from Katowice proved some results on (1) as well as on the more general equation

$$f(x) + f(y + g(y)) = f(y) + f(x + g(y)). \tag{2}$$

Among them the following can be proved.

Theorem

Let G be a group and let $g\colon G \longrightarrow G$. Assume that the group $\langle g(G)\rangle$ generated by $g(G)$ is G. Let H be an Abelian group. Then $f\colon G \longrightarrow H$ satisfies (2) *if and only if it is affine, that is*

$$f(x) = a(x) + b, \qquad x \in G$$

with an additive $a\colon G \longrightarrow H$ and a $b \in H$.

Corollaries

1. *Let G be an abelian group and let $f\colon G \longrightarrow G$. Assume that $\langle f(G)\rangle = G$. Then f is a solution of* (1) *if and only if it is affine.*
2. *A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous solution of* (1) *if and only if*
$$f(x) = ax + b, \qquad x \in \mathbb{R},$$
with some $a, b \in \mathbb{R}$.
3. *A function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic solution of* (1) *if and only if*
$$f(z) = az + b, \qquad z \in \mathbb{C},$$
with some $a, b \in \mathbb{C}$.

Witold Jarczyk

3. Problem.

Let $D \subset \mathbb{R}^2$ be an open region. *Determine the general solution of*

$$k(x+y) = f(x)g(y) + h(y) \qquad ((x,y) \in D). \tag{1}$$

More exactly, determine all $f\colon D_1 \longrightarrow \mathbb{R},\ \ g,h\colon D_2 \longrightarrow \mathbb{R},\ k\colon D_+ \longrightarrow \mathbb{R}$ satisfying (1), where

$$\begin{aligned} D_1 &:= \{x \mid \exists y:\ (x,y) \in D\}, \\ D_2 &:= \{y \mid \exists x:\ (x,y) \in D\}, \\ D_+ &:= \{x+y \mid (x,y) \in D\}. \end{aligned} \tag{2}$$

BACKGROUND

I solved equation (1) (Proc. Amer. Math. Soc. **133** (2005), 3227-3233) when k is *locally nonconstant* (not constant on neighbourhood of any point in D_+; called *philandering* by Lundberg, Sablik et al.)

No other assumption. The *problem is to eliminate this one assumption.*

Why is the equation (1) *interesting?*

$$f(x+y) = f(x)g(y) + h(y) \qquad (k = f) \tag{3}$$

is fundamental to characterising power means among quasiarithmetic means.

For $k = h$ the equation

$$k(x+y) = f(x)g(y) + h(y) \qquad ((x,y) \in D) \tag{1}$$

i.e.

$$h(x+y) = f(x)g(y) + h(y) \qquad ((x,y) \in D) \tag{4}$$

played an important role in comparison of utility representations (Gilányi-Ng-Aczél, J. Math. Anal. Appl. **304** (2005), 572-583).

Of course, also *the Pexider equations*

$$k(x+y) = f(x)g(y) \qquad ((x,y) \in D) \tag{5}$$

and

$$k(x+y) = f(x) + h(y) \qquad ((x,y) \in D) \tag{6}$$

are particular cases of (1).

As is known, (6) *can be solved by extension*, that is there exist $F, H, K\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$F = f \text{ on } D_1, \qquad H = h \text{ on } D_2, \qquad K = k \text{ on} D_+$$

and

$$K(u+v) = F(u) + H(v) \qquad \text{for } (u,v) \in \mathbb{R}^2.$$

Surprisingly, *for* (5) *such an extension is in general, not possible* (possible only if k is nowhere zero on D_+), as Fulvia Skoff showed by *counterexample.*

By a constructive method, Baker, Aczél and Skoff *found the general solutions of* (5). Similarly, *if k is not locally nonconstant, extension would not work* in general for (1), *another (constructive?) method* would be needed to find the general solution of (1).

János Aczél

4. Remark and Problem. *On the stability of the Hermite–Hadamard inequality*

The convexity of a continuous real function $f\colon I \longrightarrow \mathbb{R}$ defined on an open interval $I \subseteq \mathbb{R}$ is characterized by both sides of the well-known Hermite–Hadamard inequality, i.e., we have the following

Fact 1
The following three assertions are equivalent:

(i) f *is convex;*

$$\text{(ii)} \qquad \frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2} \qquad (x, y \in I,\ x<y);$$

$$\text{(iii)} \qquad f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt \qquad (x, y \in I,\ x<y).$$

For a proof and further generalizations see the book of Niculescu and Persson [4] and the paper [1].

Related to ε-convexity, we have the next (easy to verify)

Fact 2
Assume that f is ε-convex in the following sense

$$\text{(i)}^* \qquad f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon \qquad (x, y \in I,\ t \in [0,1]).$$

Then

$$\text{(ii)}^* \qquad \frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2}+\varepsilon \qquad (x, y \in I,\ x<y);$$

$$\text{(iii)}^* \qquad f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt+\varepsilon \qquad (x, y \in I,\ x<y).$$

Conversely, if (ii)* *and* (iii)* *hold then f is 4ε-convex.*

Proof. Assume that f is ε-convex. Then, integrating (i)* with respect to t over $[0,1]$, one obtains (ii)*. To deduce (iii)*, observe that (i)* implies

$$f\left(\frac{x+y}{2}\right) \le \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}+\varepsilon \qquad (x,y\in I,\ t\in[0,1]).$$

Integrating this inequality with respect to t over $[0,1]$, one arrives at (iii)*.

If (ii)* and (iii)* hold then

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}+2\varepsilon \qquad (x,y\in I).$$

Now, using the result of Ng and Nikodem [3], the 4ε-convexity of f follows.

A problem presented at the 5th Katowice–Debrecen Winter Seminar in Będlewo was if any of the inequalities (ii)* or (iii)* implies the $c\varepsilon$-convexity of f for some positive constant c. By a recent paper of Nikodem, Riedel and Sahoo [5], the answers to both of these questions are negative, i.e., neither (ii)* nor (iii)* imply the $c\varepsilon$-convexity of f for any $c>0$.

Briefly, in [5] the following result was proved:

1. The function $fx) := \ln x$, $(x>0)$ satisfies (ii)* with $\varepsilon = 1$ but it is not c-convex for any $c>0$.

2. For all $n\in\mathbb{N}$ there exists a function f_n which satisfies (iii)* with $\varepsilon=1$ but not c-convex for any $0<c<n$.

Related to another version of approximate convexity that was studied in [2] we have

Fact 3

Assume that f is $(\varepsilon,1)$-Jensen-convex in the following sense

$$\text{(i)}^{**}\qquad f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}+\varepsilon|x-y| \qquad (x,y\in I,\ t\in[0,1]).$$

Then

$$\text{(ii)}^{**}\qquad \frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2}+\varepsilon|x-y| \qquad (x,y\in I,\ x<y);$$

$$\text{(iii)}^{**}\qquad f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt+\frac{\varepsilon}{2}|x-y| \qquad (x,y\in I,\ x<y).$$

Conversely, if (ii)** *and* (iii)** *hold then f is $\left(\frac{3}{2}\varepsilon,1\right)$-Jensen-convex.*

Proof. Assume that f is $(\varepsilon,1)$-convex. Then, by the main result of [2],

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+2\varepsilon T(t)|x-y| \qquad (x,y\in I,\ t\in[0,1]),$$

where $T\colon\mathbb{R}\longrightarrow\mathbb{R}$ denotes the Takagi-function defined by

$$T(t) := \sum_{n=0}^{\infty} \frac{\mathrm{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R}).$$

Now, integrating this inequality with respect to t over $[0,1]$, using that $\int_0^1 T(t)\,dt = \frac{1}{2}$, one obtains (ii)**. To deduce (iii)**, observe that (i)** implies, for $x, y \in I$, $t \in [0,1]$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2} + \varepsilon\,|1-2t|\,|x-y|.$$

Integrating this inequality with respect to t over $[0,1]$, one gets (iii)*.

If (ii)** and (iii)** hold then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \frac{3}{2}\varepsilon|x-y| \qquad (x, y \in I),$$

which means the $(\frac{3}{2}\varepsilon, 1)$-Jensen-convexity of f.

Motivated by the above fact, we can raise the following

PROBLEM
Does either (ii)** or (iii)** imply the $(c\varepsilon, 1)$-convexity of f for some positive constant c?

[1] M. Bessenyei, Zs. Páles, *Characterizations of convexity via Hadamard's inequality*, Math. Inequal. Appl. **9** (2006), 53-62.

[2] A. Házy, Zs. Páles, *On approximately midconvex functions*, Bull. London Math. Soc. **36** (2004), 339-350.

[3] C.T. Ng, K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), 103-108.

[4] C.P. Niculescu, L.E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, **23**, Springer, New York, 2006.

[5] K. Nikodem, T. Riedel, P. Sahoo, *The stability problem of the Hermite–Hadamard inequality*, submitted.

Zsolt Páles

5. Remark. *Functional equations involving weighted quasi-arithmetic means and their Gauss composition* (presented by Zs. Páles)

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Let $M_i \colon I^2 \longrightarrow I$ $(i = 1, 2, 3)$ be weighted quasi-arithmetic means with the property

$$M_3 = M_1 \otimes M_2\,,$$

where $\otimes$ denote the Gauss composition of M_1 and M_2. We consider the following two functional equations for the unknown $f \colon I \longrightarrow \mathbb{R}$:

(1) $f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \qquad (x, y \in I)$,

(2) $2f(M_3(x,y)) = f(x) + f(y) \quad (x, y \in I)$.

It is known, that all solutions of (2) are solutions of (1), too. We give a complete characterization for the means M_i $(i = 1, 2, 3)$ so that arbitrary solution of (1) also satisfy (2).

Zoltán Daróczy

6. Remark.

In 1960 the following system of equalities was solved by Aczél and Gołąb (see [1], also [2])

$$H(s,t,x) = H(u,t,H(s,u,x)), \tag{1}$$

$$H(s,s,x) = x. \tag{2}$$

One can observe that equations (1) and (2) themselves do not need any algebraic structures in the domain of H so we could assume that the function H acts as follows $H\colon S \times S \times X \longrightarrow X$ where S and X are sets.

Moreover, it is known that if $(S, +)$ is a group and $F\colon S \times X \longrightarrow X$ satisfies the translation equation

$$F(s+t, x) = F(t, F(s,x))$$

with natural initial condition

$$F(0,x) = x, \tag{3}$$

then the function $H\colon S \times S \times X \longrightarrow X$ defined by

$$H(s,t,x) := F(s-t, x)$$

satisfies the system of (1) and (2).

Nevertheless, condition (3) is common but in some situations is not fulfilled by solution of the translation equation. It leads to the idea of solving equation (1) without equality (2). In this direction we have proved the following proposition.

Proposition

Let S, X be sets and let $H\colon S \times S \times X \longrightarrow X$ be a solution of equation (1). *Therefore there are functions $\Phi, \Psi\colon S \times X \longrightarrow X$ such that*

$$H(s,t,x) := \Psi(t, \Phi(s,x)) \qquad \text{for every } s, t \in S,\ x \in X. \tag{4}$$

Moreover, if $\Phi, \Psi\colon S \times X \longrightarrow X$ are functions such that for every $u \in S$: $\Psi(u,\cdot)^{-1} = \Phi(u,\cdot)$ on the set $\Phi(S \times X)$, then the function $H\colon S \times S \times X \longrightarrow X$ given by the formula (4) *is a solution of equation* (1).

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[2] Z. Moszner, *Les equations et les inégalités liées à l'équation de translation*, Opuscula Math. **19** (1999), 19-43.

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7. Problem.
Is the following conjecture true?

CONJECTURE
Let the diffeomorphism $\Psi\colon(0,\infty)\longrightarrow(0,\infty)$ *have no fixed point. If for every increasing self-diffeomorphism* g *of the closed interval* $[0,\infty)$ *the function*

$$g_\Psi(x) := \Psi^{-1}\left(g(\Psi(x))\right), \qquad x>0,$$

(with value 0 *at zero) is again a self-diffeomorphism of* $[0,\infty)$*, then the derivative* $D\Psi$ *of* Ψ *is slowly varying at zero.*

For making the problem more readable, let us sketch a proof of the inverse claim. For, let the diffeomorphism Ψ have slowly varying derivative, i.e., let

$$\lim_{x\to 0}\frac{D\Psi(\lambda\cdot x)}{D\Psi(x)} = 1 \qquad \text{for all } \lambda>0.$$

Then both, Ψ and Ψ^{-1} are regularly varying with exponent 1 (we are omitting the details). Moreover for the derivative of g_Ψ we have

$$Dg_\Psi(x) = \frac{D\Psi(x)}{D\Psi(\Psi^{-1}\circ g\circ\Psi(x))}\cdot Dg(\Psi(x)).$$

With the use of $Dg(0)>0$, by the regular variability of Ψ^{-1} we obtain that the ratio of the arguments of Ψ has a finite and positive limit as follows,

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\Psi^{-1}\circ g\circ\Psi(x)} &= \lim_{x\to 0}\frac{\Psi^{-1}\circ\Psi(x)}{\Psi^{-1}\circ g\circ\Psi(x)} = \lim_{x\to 0}\frac{\Psi(x)}{g\circ\Psi(x)}\\ &= \lim_{y\to 0}\frac{y}{g(y)} = (Dg(0))^{-1}\in(0,\infty).\end{aligned}$$

By the slow variability of $D\Psi$ and by continuity of Dg, the limit $Dg_\Psi(0^+)$ equals $1\cdot Dg(0)$. By similar arguments from $Dg_\Psi(0)=\lim_{x\to 0}\frac{g_\Psi(x)}{x}$ one can get that $Dg_\Psi(0)=Dg(0)$, too. Thus, Dg_Ψ is continuous at zero, which closes the most important step for the inverse claim.

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8. Remark.

The Theorem formulated on p. 159 of the report on the 10th ICFEI (Ann. Acad. Paed. Cracov. Studia Math., **5** (2006)) is not true. A counterexample: $F(x,y) = x$, $(x,y) \in \mathbb{R}^2$, was communicated to the speaker by Professor Karol Baron.

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9. Remark. *Regular variability in functional equations*

This remark is related to the lecture presented by professor Zsolt Páles (see Abstracts of Talks, page 157). Some of the results use the regular variability almost everywhere for obtaining uniqueness of the generating function from the mean, dependent additionally on some generating measure (mean of a mixed type). We want to point at the fact that the regular variability has been used already in the following (obviously much simpler) problem of restoring f from the mean defined as follows

$$M_f(x,y) := f^{-1}\left(\frac{xf(x)+yf(y)}{x+y}\right), \qquad x,y \in I \tag{1}$$

where f is a continuous and strictly monotonic function defined on an interval I of positive reals. For a point $x_0 \in I$ let the auxiliary function

$$\delta_0(u) := f(x_0+u) - f(x_0) \qquad \text{whenever } x_0 + u \in I \tag{2}$$

be regularly varying at 0 with non-zero exponent, i.e., let

$$\lim_{u\to 0} \frac{\delta_0(\lambda\cdot u)}{\delta_0(u)} = \lambda^\rho \qquad \text{for } \lambda > 0, \text{ where } \rho \in (0,\infty). \tag{3}$$

(For measurable functions the definition is equivalent to the notion introduced by J. Karamata in [5]; for review of the regular variability see [1], [4] or [6], and for the facts suitable for the functional equations, see [1].) In terms of

$$\mu_0(u) := M_f(x_0, x_0+u) - x_0, \qquad w_0(u) := \frac{x_0+u}{2x_0+u}, \tag{4}$$

definition (1) implies the following homogeneous equation

$$\delta_0(\mu_0(u)) \;=\; w_0(u)\cdot\delta_0(u) \qquad \text{whenever } u \in I - x_0\,, \tag{5}$$

where $I - x_0 := \{x - x_0 :\ x \in I\}$. It is shown in [3] that

$$f(x) = f(x_0) + (f(x_1) - f(x_0))\cdot \lim_{n\to\infty}\left(\frac{{\mu_0}^n(x-x_0)}{{\mu_0}^n(x_1-x_0)}\right)^\rho \cdot \frac{W_{0;n}(x_1-x_0)}{W_{0;n}(x-x_0)}$$

where μ_0 is given by the x_0-cut of M_f according to (4), and

$$W_{0;n}(u) := \prod_{j=0}^{n-1} w_0({\mu_0}^j(u)) \qquad \text{for } u \in I - x_0\,. \tag{6}$$

whenever $(x - x_0) \cdot (x_1 - x_0) > 0$, $x_1, x \in I$.

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10. Remark. *Embedding commuting functions into a regular iteration group*

An increasing continuous self-mapping $f\colon (0,\infty) \to (0,\infty)$ is said to be Szekeresian if $f(x) < x$, for all $x > 0$, possesses the derivative at zero $D_0 f := \lim_{x\to 0} \frac{f(x)}{x}$ in $(0,1)$ and if the Szekeres principal function

$$\varphi_f(x\|y) := \lim_{n\to\infty} \frac{f^n(x)}{f^n(y)}, \qquad x > 0,$$

is continuous for some $y > 0$. If additionally f is homeomorphic onto $(0,\infty)$ then $\varphi_f(\cdot\,\|y)$ is the unique regularly varying at zero solution of the canonical Schröder equation

$$\varphi_f(f(x)\|y) = d \cdot \varphi_f(x\|y), \qquad x > 0, \ \text{where } d = D_0 f$$

equal 1 at y, for arbitrary positive y (for details, see [1]). The following are considerations which were suggested to me by prof. J. Matkowski. Let f and g be commuting Szekeresian homeomorphisms. Then $\varphi_f := \varphi_f(x\|y)$, with fixed y, satisfies

$$d \cdot (\varphi_f \circ g) = \varphi_f \circ f \circ g = (\varphi_f \circ g) \circ f,$$

which means that $\varphi_f \circ g$ is again a regularly varying solution of the canonical Schröder equation for f. By a suitable uniqueness theorem, for some positive constant C

$$\varphi_f \circ g = C \cdot \varphi_f\,.$$

All the facts together show, that φ_f is the Szekeres principal function for g and that $C = D_0 g$. Let us introduce $\rho := \frac{\log D_0 g}{\log D_0 f}$.

COROLLARY
If φ_f is homeomorphic, then there is exactly one regular iteration group containing f and g. Moreover the iterates are given by the formula:

$$f_t = (\varphi_f)^{-1} \circ (d^t \cdot \varphi_f), \qquad t \in (-\infty, \infty),$$

and $f = f_1$ and $g = f_\rho$.

CONJECTURE
If $\rho \notin \mathbb{Q}$, then there is a regular iteration group $(f_t;\ t \in T)$ indexed by the (dense) additive subgroup T generated by 1 and ρ and such that $f = f_1$ and $g = f_\rho$.

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