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Another description of continuous solutions of a nonlinear functional inequality

Abstract. The paper gives a general construction of all continuous solutions of inequality (1) fulfilling one of conditions (5) or (26). This paper is a continuation of [3].

1. Introduction

In the paper [3] we considered the problem of existence of the continuous solutions of the functional inequality

$$\psi[f(x)] \le G(x, \psi(x)), \tag{1}$$

where ψ is an unknown function, in the case where continuous solutions of the corresponding functional equation

$$\varphi[f(x)] = G(x, \varphi(x)) \tag{2}$$

depend on an arbitrary function. In particular we proved there Theorems 1 and 5 quoted below.

In the present paper we shall give other descriptions of the general continuous solution of (1) which are more convenient to study, for example, solutions of (1) which are Lipschitzian or possess some asymptotic property (see [2], [1]). We shall also adapt some results from [3] to a more general class of continuous solutions of inequality (1).

We start with reminding some notations and assumptions from [3]. Let $I = (\xi, a)$, where $\xi < a \le \infty$. We assume that

(i) the function $f: I \longrightarrow \mathbb{R}$ is continuous and strictly increasing in I. Moreover, $\xi < f(x) < x$ for all $x \in I$.

Remark 1

Hypothesis (i) implies that $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$. Here f^n denotes the n-th iterate of f.

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As to the function G we assume:

(ii) $G\colon\Omega \longrightarrow \mathbb{R}$ is continuous in an open set $\Omega \subset I \times \mathbb{R}$;

(iii) for every $x \in I$ the set

$$\Omega_x := \{y:\ (x,y) \in \Omega\} \tag{3}$$

is a non-empty open interval and

$$G(x,\Omega_x) \subset \Omega_{f(x)}. \tag{4}$$

Let $J \subset I$ be an open subinterval such that $\xi \in \operatorname{cl} J$. We shall consider solutions ψ of inequality (1) and solutions φ of equation (2) such that their graphs lie in Ω, i.e.,

$$\psi(x), \varphi(x) \in \Omega_x \qquad \text{for } x \in J \subset I. \tag{5}$$

The class of these solutions will be denoted by $\Psi(J)$ and $\Phi(J)$, respectively. Moreover, we denote $I_k := [f^{k+1}(x_0), f^k(x_0)]$ for a fixed $x_0 \in I$ and $k \in \mathbb{N}\cup\{0\}$.

Finally, we consider the sequence $\{g_k\}$ defined by the recursive formula:

$$\begin{cases} g_0(x,y) = y, \\ g_{k+1}(x,y) = G(f^k(x), g_k(x,y)), \quad k \in \mathbb{N}\cup\{0\}. \end{cases} \tag{6}$$

2. Solutions of (1) in the interval $(\xi, x_0]$

Let us assume (i)-(iii). It is known (see [4]) that then continuous solutions of equation (2) depend on an arbitrary function. It means that for any $x_0 \in I$ and an arbitrary continuous function $\varphi_0\colon I_0 \longrightarrow \mathbb{R}$ fulfilling the conditions

$$\varphi_0(x) \in \Omega_x \qquad \text{for } x \in I_0\,, \tag{7}$$

$$\varphi_0[f(x_0)] = G(x_0, \varphi_0(x_0)) \tag{8}$$

there exists exactly one continuous solution $\varphi \in \Phi((\xi, x_0])$ of equation (2) extending φ_0, i.e.,

$$\varphi(x) = \varphi_0(x) \qquad \text{for } x \in I_0\,.$$

A corresponding result for solutions of inequality (1) has been proved in [3]:

THEOREM 1
Let assumptions (i)-(iii) *be fulfilled. Then for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ *fulfilling the conditions*

$$\psi_0[f(x_0)] \le G(x_0, \psi_0(x_0)), \tag{9}$$

$$\psi_0(x) \in \Omega_x\,, \qquad x \in I_0 \tag{10}$$

there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *such that*

$$\psi(x) = \psi_0(x) \qquad \text{for } x \in I_0\,. \tag{11}$$

This solution is given by the formula

$$\psi[f^k(x)] = \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \qquad \text{for } x \in I_0\,,\ k \in \mathbb{N} \cup \{0\}, \tag{12}$$

where $\lambda_k\colon I_k \longrightarrow \mathbb{R}$ *is an arbitrary sequence of continuous functions fulfilling the conditions:*

$$\lambda_0(x) = 0, \qquad x \in I_0\,, \tag{13}$$

$$\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \in \Omega_{f^k(x)}\,, \qquad x \in I_0\,,\ k \in \mathbb{N} \cup \{0\}, \tag{14}$$

$$\begin{aligned}\lambda_k[f^k(x)] + g_k(x, \psi_0(x)) \le G(f^{-k}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))),\\ x \in I_0\,,\ k \in \mathbb{N},\end{aligned} \tag{15}$$

$$\begin{aligned}\lambda_k[f^k(x_0)] + g_k(x_0, \psi_0(x_0)) = \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)]),\\ k \in \mathbb{N}.\end{aligned} \tag{16}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *may be obtained in this manner.*

Now, we shall prove the following corollary from Theorem 1.

Theorem 2

Let assumptions (i)-(iii) *be fulfilled. Then for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ *fulfilling* (9) *and* (10) *there exists a continuous solution* $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *such that* (11) *holds. This solution is given by the formula*

$$\psi(x) = \begin{cases} \psi_0(x), & x \in I_0\,, \\ M_k(\psi_0, \lambda)(x), & x \in I_k\,,\ k \in \mathbb{N}, \end{cases} \tag{17}$$

where the functional sequence of continuous functions $M_k(\psi_0, \lambda)$ *is defined by the recurrence*

$$\begin{cases} M_1(\psi_0, \lambda)(x) = \lambda(x) + G(f^{-1}(x), \psi_0[f^{-1}(x)]), & x \in I_1\,, \\ M_{k+1}(\psi_0, \lambda)(x) = \lambda(x) + G(f^{-1}(x), M_k(\psi_0, \lambda)[f^{-1}(x)]), & x \in I_{k+1} \end{cases} \tag{18}$$

and $\lambda\colon (\xi, f(x_0)] \longrightarrow (-\infty, 0]$ *is an arbitrary continuous function fulfilling the conditions:*

$$M_k(\psi_0, \lambda)(x) \in \Omega_x\,, \qquad x \in I_k\,,\ k \in \mathbb{N}, \tag{19}$$

$$\lambda[f(x_0)] + G(x_0, \psi_0(x_0)) = \psi_0[f(x_0)]. \tag{20}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *may be obtained in this manner.*

Proof. We fix an $x_0 \in I$ and an arbitrary continuous function $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ fulfilling (9) and (10). Moreover, we take a continuous function $\lambda\colon (\xi, f(x_0)] \longrightarrow (-\infty, 0]$ fulfilling (19), (20) and define the function $\psi\colon (\xi, x_0] \longrightarrow \mathbb{R}$ by formula (17). Condition (19) implies that the sequence $M_k(\psi_0, \lambda)$ (and, consequently, the function ψ) is well defined. Now, we define the sequence $\lambda_k\colon I_k \longrightarrow \mathbb{R}$ of continuous functions by formula (13) and

$$\lambda_k(x) := \lambda(x) + G(f^{-1}(x), \psi[f^{-1}(x)]) - g_k(f^{-k}(x), \psi[f^{-k}(x)]), \qquad x \in I_k\,,\ k \in \mathbb{N}. \tag{21}$$

It is obvious that (21) implies that ψ may be represented also by formula (12). Moreover, condition (19) implies (14). We have also the estimate

$$\begin{aligned} \lambda_k[f^k(x)] + g_k(x, \psi_0(x)) &= \lambda_k[f^k(x)] + G(f^{k-1}(x), \psi[f^{k-1}(x)]) \\ &\le G(f^{k-1}(x), \psi[f^{k-1}(x)]) \\ &= G(f^{k-1}(x), \lambda_{k-1}[f^{k-1}(x)] + g_{k-1}(x, \psi_0(x))), \\ &\qquad x \in I_0\,,\ k \in \mathbb{N}, \end{aligned}$$

which implies (15). Finally from (20) we obtain (16) for $k = 1$ and, by virtue of the equalities

$$\begin{aligned} &\lambda_k[f^k(x_0)] + g_k(x_0, \psi_0(x_0)) \\ &\quad = \lambda[f^k(x_0)] + G(f^{k-1}(x_0), \psi[f^{k-1}(x_0)]) \\ &\quad = \lambda[f^{k-1}(f(x_0))] + G(f^{k-2}(f(x_0)), \psi[f^{k-2}(f(x_0))]) \\ &\quad = \lambda_{k-1}[f^{k-1}(f(x_0))] + g_{k-1}(f(x_0), \psi_0[f(x_0)]) \\ &\quad = \lambda_{k-1}[f^k(x_0)] + g_{k-1}(f(x_0), \psi_0[f(x_0)]), \end{aligned}$$

we have (16) for $k \ge 2$. Thus, by virtue of Theorem 1, formula (12) (and, consequently, formula (17)) defines a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality (1).

On the other hand, let us assume that $\psi \in \Psi((\xi, x_0])$ is a continuous solution of (1). It is sufficient to put

$$\psi_0(x) := \psi(x) \qquad \text{for } x \in I_0\,, \tag{22}$$

$$\lambda(x) := \psi(x) - G(f^{-1}(x), \psi[f^{-1}(x)]) \qquad \text{for } x \in (\xi, f(x_0)]. \tag{23}$$

Let us notice that (19) and (20) hold. Moreover, it follows from (1) that the function λ takes nonpositive values only. It is obvious that the solution ψ may be represented by formula (17). We may prove it by simple induction.

Indeed, formulas (22) and (23) imply that for $x \in I_1$:

$$\begin{aligned} \psi(x) &= \lambda(x) + G(f^{-1}(x), \psi[f^{-1}(x)]) = \lambda(x) + G(f^{-1}(x), \psi_0[f^{-1}(x)]) \\ &= M_1(\psi_0, \lambda)(x). \end{aligned}$$

Thus, if we assume that for an arbitrarily chosen integer $k > 1$ we have $\psi(x) =$

$M_k(\psi_0, \lambda)(x)$ for $x \in I_k$, then from (23) we obtain for $x \in I_{k+1}$:

$$\begin{aligned}\psi(x) &= \lambda(x) + G(f^{-1}(x), \psi[f^{-1}(x)]) = \lambda(x) + G(f^{-1}(x), M_k(\psi_0, \lambda)[f^{-1}(x)]) \\ &= M_{k+1}(\psi_0, \lambda)(x).\end{aligned}$$

Consequently, ψ is of the form (17) and this ends the proof of the theorem.

3. Solutions of (1) in the interval I

We assume additionally that:

(iv) for every $x \in I$ the function $G(x, \cdot)$ is invertible,

(v) the function f fulfils the condition $f(I) = I$,

(vi) for every $x \in I$, with Ω_x defined by (3) we have

$$G(x, \Omega_x) = \Omega_{f(x)} . \tag{24}$$

Thanks to these assumptions we may extend the definition (6) to negative indices by putting

$$g_{-k-1}(x, y) := G^{-1}(f^{-k-1}(x), g_{-k}(x, y)), \qquad k \in \mathbb{N} \cup \{0\}, \tag{25}$$

where $G^{-1}(x, \cdot)$ denotes the inverse of the function $G(x, \cdot)$. It is obvious (by virtue of (4) and (24)) that the sequences (6) and (25) are well defined. We may also consider intervals I_k for $k \in \mathbb{Z}$.

If we assume (i)-(vi), then for an arbitrary $x_0 \in I$ every continuous function $\varphi_0 \colon I_0 \longrightarrow \mathbb{R}$ fulfilling (7), (8) may be extended to a continuous solution $\varphi \in \Phi(I)$ of equation (2). For inequality (1) the following theorem has been also formulated in [3].

Theorem 3
Let assumptions (i)-(vi) *be fulfilled. Then, for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0 \colon I_0 \longrightarrow \mathbb{R}$ *fulfilling* (9) *and* (10) *there exists a continuous solution* $\psi \in \Psi(I)$ *of inequality* (1) *such that* (11) *holds. This solution is given by formulas* (12) *and*

$$\psi[f^{-k}(x)] = l_k[f^{-k}(x)] + g_{-k}(x, \psi_0(x)) \qquad \textit{for } x \in I_0 , \ k \in \mathbb{N},$$

where $\lambda_k \colon I_k \longrightarrow \mathbb{R}$, $l_k \colon I_{-k} \longrightarrow \mathbb{R}$ *are arbitrary sequences of continuous functions fulfilling conditions* (13)-(16) *and, additionally, the following conditions*

$$l_0(x) = 0, \qquad x \in I_0 ,$$

$$l_k[f^{-k}(x)] + g_{-k}(x, \psi_0(x)) \in \Omega_{f^{-k}(x)} , \qquad x \in I_0 , \ k \in \mathbb{N},$$

$$l_{k+1}[f^{-k+1}(x)] + g_{-k+1}(x, \psi_0(x)) \le G(f^{-k}(x), l_k[f^{-k}(x)] + g_{-k}(x, \psi_0(x))),$$
$$x \in I_0\,,\ k \in \mathbb{N},$$

$$l_{k+1}[f^{-k+1}(x_0)] + g_{-k+1}(x_0, \psi_0(x_0)) = l_k[f^{-k+1}(x_0)] + g_{-k}(f(x_0), \psi_0[f(x_0)]),$$
$$k \in \mathbb{N}.$$

Moreover, we may obtain in this way all continuous solutions $\psi \in \Psi(I)$ *of inequality* (1)*.*

Theorems 2 and 3 also imply the following theorem.

THEOREM 4
Under assumptions (i)-(vi) *for any* $x_0 \in I$ *and for an arbitrary continuous function* $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ *fulfilling the conditions* (9), (10)*, there exists a continuous solution* $\psi \in \Psi(I)$ *of inequality* (1) *such that* (11) *holds. This solution is given by formulas* (17) *and*

$$\psi(x) := P_k(\psi_0, \lambda)(x), \qquad x \in I_{-k}\,,\ k \in \mathbb{N},$$

where the functional sequences of continuous functions $M_k(\psi_0, \lambda)$*,* $P_k(\psi_0, \lambda)$*, are defined by formula* (18) *and by*

$$\begin{cases} P_1(\psi_0, \lambda)(x) = G^{-1}(x, \psi_0[f(x)] - \lambda[f(x)]), & x \in I_{-1}\,, \\ P_{k+1}(\psi_0, \lambda)(x) = G^{-1}(x, P_k(\psi_0, \lambda)[f(x)]), & x \in I_{-k-1}\,,\ k \in \mathbb{N} \end{cases}$$

and $\lambda\colon I \longrightarrow (-\infty, 0]$ *is an arbitrarily chosen continuous function fulfilling conditions* (19), (20) *together with*

$$\psi_0(x) - \lambda(x) \in \Omega_x\,, \qquad x \in I_0\,,$$
$$P_k(\psi_0, \lambda)(x) \in \Omega_x\,, \qquad x \in I_{-k}\,,\ k \in \mathbb{N}.$$

Moreover, all continuous solutions $\psi \in \Psi(I)$ *of inequality* (1) *may be obtained in this manner.*

The proof of the above theorem runs analogously to that of Theorem 2 and is therefore omitted.

4. Main result

Here we shall characterize continuous solutions ψ of inequality (1) which fulfil, for arbitrarily chosen $x_0 \in I$, the additional condition

$$\psi[f(x)] \in G(x, \Omega_x), \qquad x \in (\xi, x_0]. \tag{26}$$

We replace (iv) by a stronger assumption

(vii) For every $x \in I$ the function $G(x, \cdot)$ is strictly increasing.

In the paper [3] we considered continuous solutions ψ of (1) which fulfil the following condition:

$$L_k^\psi[f(x)] \in G(x, \Omega_x), \qquad x \in (\xi, x_0],\ x_0 \in I,\ k \in \mathbb{N} \cup \{0\}, \tag{27}$$

where the sequence $\{L_k^\psi\}$ was defined by the recurrence

$$\begin{cases} L_0^\psi(x) = \psi(x), \\ L_{k+1}^\psi(x) = G^{-1}(x, L_k^\psi[f(x)]), \qquad k \in \mathbb{N} \cup \{0\}. \end{cases}$$

It is obvious, by virtue of (27), that the above sequence is well defined.

The following theorem has been proved in [3].

Theorem 5

Let assumptions (i)-(iii) *and* (vii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 \colon I_0 \longrightarrow \mathbb{R}$ fulfilling* (9), (10) *and, moreover, the condition*

$$\psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}) \tag{28}$$

there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality (1) *fulfilling* (11) *and* (27). *This solution is given by the formula*

$$\psi[f^k(x)] = g_k(x, \gamma_k(x) + \psi_0(x)) \qquad \text{for } x \in I_0\,,\ k \in \mathbb{N} \cup \{0\}, \tag{29}$$

where $\{\gamma_k\}$ is an arbitrary sequence of continuous functions defined in I_0 and fulfilling the conditions:

$$\gamma_0(x) = 0, \qquad x \in I_0\,,$$

$$\{\gamma_k\} \text{ is decreasing in } I_0\,,$$

$$\gamma_k(x) + \psi_0(x) \in \Omega_x\,, \qquad \text{for } x \in (f(x_0), x_0],\ k \in \mathbb{N} \cup \{0\},$$

$$\gamma_k[f(x_0)] + \psi_0[f(x_0)] \in G(x_0, \Omega_{x_0}), \qquad k \in \mathbb{N},$$

$$g_k(x_0, \gamma_k(x_0) + \psi_0(x_0)) = g_{k-1}(f(x_0), \gamma_{k-1}[f(x_0)] + \psi_0[f(x_0)]), \qquad k \in \mathbb{N}.$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ of inequality (1), *fulfilling* (27) *may be obtained in this manner.*

If we replace in the above theorem condition (27) by (26) then we obtain:

Theorem 6

Let assumptions (i)-(iii) *and* (vii) *be fulfilled. Then for any $x_0 \in I$ and for an arbitrary continuous function $\psi_0 \colon I_0 \longrightarrow \mathbb{R}$ fulfilling* (9), (10), (28), *there exists a continuous solution $\psi \in \Psi((\xi, x_0])$ of inequality* (1) *fulfilling* (11) *and* (26). *This solution is given by the formula*

$$\psi(x) = \begin{cases} \psi_0(x), & x \in I_0\,, \\ R_k(\psi_0, \gamma)(x), & x \in I_k\,,\ k \in \mathbb{N}, \end{cases} \tag{30}$$

where the sequence $\{R_k(\psi_0, \gamma)\}$ *of continuous functions is defined recursively by*

$$\begin{cases} R_1(\psi_0, \gamma)(x) = G(f^{-1}(x), \gamma[f^{-1}(x)] + \psi_0[f^{-1}(x)]), \\ \hfill x \in I_1\,, \\ R_{k+1}(\psi_0, \gamma)(x) = G(f^{-1}(x), \gamma[f^{-1}(x)] + R_k(\psi_0, \gamma)[f^{-1}(x)]), \\ \hfill x \in I_{k+1}\,,\ k \in \mathbb{N}, \end{cases} \tag{31}$$

and $\gamma\colon (\xi, x_0] \longrightarrow (-\infty, 0]$ *is an arbitrary continuous function fulfilling the conditions:*

$$\gamma(x) + R_k(\psi_0, \gamma)(x) \in \Omega_x\,, \qquad x \in I_k\,,\ k \in \mathbb{N} \cup \{0\}, \tag{32}$$

$$\psi_0[f(x_0)] = G(x_0, \gamma(x_0) + \psi_0(x_0)). \tag{33}$$

Moreover, all continuous solutions $\psi \in \Psi((\xi, x_0])$ *of inequality* (1) *fulfilling* (26) *may be obtained in this manner.*

Proof. Similarly as in the proof of Theorem 2 we fix an $x_0 \in I$ and an arbitrary continuous function $\psi_0\colon I_0 \longrightarrow \mathbb{R}$ fulfilling (9), (10) and (28). Moreover, let $\gamma\colon (\xi, x_0] \longrightarrow (-\infty, 0]$ be a continuous function fulfilling (32), (33) and define a function $\psi\colon (\xi, x_0] \longrightarrow \mathbb{R}$ by formula (30). Condition (32) implies that the sequence $\{R_k(\psi_0, \gamma)\}$ is well defined. It is also clear that the following equalities

$$R_{k+1}(\psi_0, \gamma)[f^{k+1}(x_0)] = R_k(\psi_0, \gamma)[f^k(x_0)], \qquad k \in \mathbb{N}, \tag{34}$$

hold. Thus (34) together with (33) imply that the function ψ is well defined. Since $R_k(\psi_0, \gamma)$ are continuous functions (by the continuity of the given functions f, γ, ψ_0, G), so is ψ.

It is obvious that ψ may be represented by the following form, equivalent to (30),

$$\psi[f(x)] = G(x, \gamma(x) + \psi(x)), \qquad x \in (\xi, x_0]. \tag{35}$$

Equality (35) implies condition (26) and, moreover, we obtain that ψ fulfils inequality (1) by virtue of (vii) and the fact that γ takes nonpositive values only.

On the other hand let us assume that $\psi \in \Psi((\xi, x_0])$ is a continuous solution of (1) that fulfils (26). It is sufficient to define ψ_0 by (22) and to put

$$\gamma(x) := G^{-1}(x, \psi[f(x)]) - \psi(x), \qquad x \in (\xi, x_0]. \tag{36}$$

Let us notice that (9), (10), (28), (32) and (33) hold. It is obvious that the solution ψ may be represented by (35) and, consequently, by (30). This ends the proof of the theorem.

Remark 2
If we are confined to solutions ψ of inequality (1) fulfilling (27), then formulas (29) and (30) are equivalent. Indeed, if we define solution ψ fulfilling (27) by formula (30) then we may define the sequence $\{\gamma_k\}$ from Theorem 5 by the formula

$$\gamma_k(x) = L_k^{\psi}(x) - \psi_0(x), \qquad x \in I_0\,,\ k \in \mathbb{N}.$$

Conversely, if we define a solution ψ by formula (29), then we may define the function γ by (36) and the functional sequence $\{R_k(\psi_0, \gamma)\}$ of continuous functions by the recurrent formula (31).

Remark 3
It is known (see [3]) that contrary to the situation with continuous solutions of equation (2) in I, a continuous function ψ_0 fulfilling (9), (10) and (28) cannot be extended uniquely to a continuous solution ψ of inequality (1) fulfilling (26).

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