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Boundary value problem and functional equations for overlapping disks

Abstract. We compare applications of the method of functional equations to boundary value problem for circular multiply connected domains and to circular polygons generated by overlapping disks. The second part of the paper is devoted rather to the statement of a problem than its resolution.

1. Introduction

Boundary value problems for circular n-polygons can be solved using Christoffel–Schwarz integral. But it is necessary to find the corresponding accessor parameters. It is possible to do it for polygons with two or three vertexes [7]. In the general case, when $n > 3$, it is necessary to solve a nonlinear system of equations. In the present note, another approach based on functional equations is proposed. Only the question for 2-polygons is discussed. However, it seems that n-polygons can be treated by functional equations as well as discussed 2-polygons.

Boundary value problems for non-overlapping disks (circular multiply connected domains) have been solved in [6]. In the present note, it is proposed to develop the same method to overlapping disks. More questions than answers arise in this case.

Let z be a complex variable on the complex plane $\mathbb{C}$, $a > 0$. Consider two disks of radius r on the complex plane $D_1 = \{z \in \mathbb{C} : \ |z + a| < r\}$ and $D_2 = \{z \in \mathbb{C} : \ |z - a| < r\}$. Let D be the complement of the closures of D_1 and D_2 to the extended complex plane $\widehat{\mathbb{C}}$ and let $D^+ = D \cap \mathbb{C}^+$ be the upper half of D. We have the following two cases displayed in Fig. 1:

i) $r \leq a$,

ii) $r > a$.

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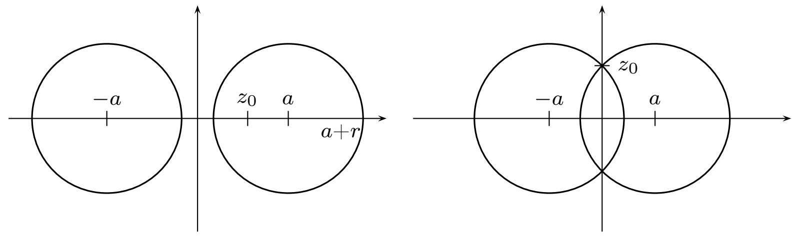

i) Non-overlapping disks ii) Overlapping disks

Figure 1

2. Functional equations [6]

Consider the case i). Following [6] we reduce a boundary value problem to a functional equation. For simplicity, the problem of conformal mapping of D^+ onto $\mathbb{C}^+$ is considered. The desired conformal mapping $\varphi(z) + z$ is determined up to purely imaginary constant by the condition

$$\operatorname{Im}(\varphi(t) + t) = 0, \qquad t \in \partial D^+. \tag{1}$$

Using the symmetry with respect to the real axis one can reduce (1) to the boundary value problem

$$\operatorname{Im}(\varphi(t) + t) = 0, \qquad t \in \partial D \tag{2}$$

with respect to $\varphi(z)$ analytic in doubly connected domain D and continuous in its closure.

Following [6] one can rewrite (2) as the $\mathbb{R}$-linear problem

$$\varphi(t) = \varphi_k(t) + \overline{\varphi_k(t)} - t, \qquad t \in \partial D_k \ (k = 1, 2), \tag{3}$$

where the auxiliary functions $\varphi_k(z)$ are analytic in D_k and continuous in the closures of the disks.

Consider inversions with respect to the circles $|z \pm a| = r$ and their compositions

$$z^*_{(1)} := \frac{r^2}{\overline{z} + a} - a, \quad z^*_{(2)} := \frac{r^2}{\overline{z} - a} + a, \tag{4}$$

$$\alpha(z) = z^*_{(12)} := \left(z^*_{(2)}\right)^*_{(1)}, \ \ \alpha^{-1}(z) = z^*_{(21)} := \left(z^*_{(1)}\right)^*_{(2)}, \ \ z^*_{(121)} := \left(z^*_{(21)}\right)^*_{(1)} \tag{5}$$

and so forth. The functions (4), (5) generate a Schottky type group $\mathcal{S} = \{\gamma_s, \ s \in \mathbb{Z}\}$ each element of which is presented in the form of the composition of inversions (4)

$$\begin{aligned} \gamma_0(z) := z, \quad & \gamma_1(\overline{z}) := z^*_{(1)}, & \gamma_{-1}(\overline{z}) &:= z^*_{(2)} \\ & \gamma_2(z) := \alpha(z), & \gamma_{-2}(z) &:= \alpha^{-1}(z), \\ & \gamma_3(\overline{z}) := \alpha(z^*_{(1)}), & \gamma_{-3}(\overline{z}) &:= \alpha^{-1}(z^*_{(2)}), \ldots \end{aligned} \tag{6}$$

When s is even, γ_s is a Möbius transformation in z. If s is odd, it is a transformation in $\overline{z}$. The number $|s|$ is called the level of the mapping γ_s.

Let us introduce the function

$$\Phi(z) := \begin{cases} \varphi_1(z) - \overline{\varphi_2\left(z^*_{(2)}\right)} - z, & |z+a| \le r, \\ \varphi_2(z) - \overline{\varphi_1\left(z^*_{(1)}\right)} - z, & |z-a| \le r, \\ \varphi(z) - \overline{\varphi_1\left(z^*_{(1)}\right)} - \overline{\varphi_2\left(z^*_{(2)}\right)}, & z \in D. \end{cases}$$

It follows from (3) that the jump of $\Phi(z)$ across the circles $|z \pm a| = r$ is equal to zero. Then the Analytic Continuation Principle implies that $\Phi(z)$ is analytic in the extended complex plane. By the Liouville theorem, $\Phi(z)$ must be constant: $\Phi(z) \equiv c$. The definition of $\Phi(z)$ in $|z \pm a| \le r$ yields the following system of functional equations

$$\varphi_1(z) = \overline{\varphi_2\left(z^*_{(2)}\right)} + z + c, \qquad |z+a| \le r, \tag{7}$$

$$\varphi_2(z) = \overline{\varphi_1\left(z^*_{(1)}\right)} + z + c, \qquad |z-a| \le r. \tag{8}$$

Elimination of φ_2 from (7)-(8) yields the classical iterative functional equation with the shift into domain [3], [5]

$$\varphi_1(z) = \varphi_1[\alpha(z)] + g(z), \qquad |z+a| \le r, \tag{9}$$

where

$$g(z) = z + \overline{z^*_{(2)}} + 2\mathrm{Re}\, c, \qquad |z+a| \le r. \tag{10}$$

Investigate the function

$$\alpha(z) = \frac{(r^2 - 2a^2)z - 2a(r^2 - a^2)}{2az + r^2 - 2a^2}. \tag{11}$$

One can see that $\alpha(z)$ has two fixed points $\pm z_0$, where $z_0 = \sqrt{a^2 - r^2}$ (see Fig. 1. i). Substitution of the attractive fixed point $-z_0 \in D_1$ into (10) yields $\mathrm{Re}\, c = 0$. General solution of (10) has the form

$$\varphi_1(z) = \sum_{k=0}^{\infty} g[\alpha^k(z)] + c_1, \tag{12}$$

where c_1 is an arbitrary constant. Using the definition of Φ in D we obtain up to an additive purely imaginary constant

$$z + \varphi(z) = \sum_{s \in 2\mathbb{Z}} \gamma_s(z) + \sum_{s \in 2\mathbb{Z}+1} \gamma_s(\overline{z}). \tag{13}$$

The series (13) converges absolutely and uniformly in each compact subset of D. One can check directly that (13) satisfies (2).

The series (13) can be directly obtained from (2) by Grave's method of symmetry [6, 1, 4] or from the alternating Schwarz method [6]. The idea of Grave's method consists in the using of all symmetries generating a group and the analytical continuation by symmetry. The derivative of the series (13) is a θ-Poincaré series [6].

The briefly presented method of functional equations is applied to arbitrary number of non-overlapping disks and gives solution to boundary value problems in the form of the general θ-Poincaré series [6].

3. Overlapping disks

The main question of the present note can be stated as follows. Is it possible to extend the method of functional equations to overlapping disks?

We proceed to study two overlapping disks.

3.1. Functional equations

The function $\alpha(z)$ has two neutral fixed points $\pm z_0$, where $z_0 = i\sqrt{r^2 - a^2}$ (see Fig. 1. ii). Consider the conformal mapping

$$\zeta = \frac{z - z_0}{z + z_0}. \tag{14}$$

On the plane ζ the circles $|z \pm a| = r$ becomes the straight lines $\arg\zeta = \pm\theta$, where 2θ is the angle between the circles $|z \pm a| = r$. The function $\alpha(z)$ becomes the composition of two symmetries with respect to the lines $\arg\zeta = \pm\theta$, hence it becomes the transformation $e^{4i\theta}\zeta$. Therefore, the functional equation (9) becomes

$$\psi(\zeta) = \psi[e^{4i\theta}\zeta] + h(\zeta), \tag{15}$$

where $\psi(\zeta) = \varphi_1(z)$. However, the functional equation (15) is not reduced now from the boundary value problem

$$\mathrm{Im}(\varphi(t) + t) = 0, \qquad t \in \partial D, \tag{16}$$

where $D = \{z \in \mathbb{C} : \ |z \pm a| > r\}$. (16) is obtained formally from (9) without construction of $\Phi(z)$.

Formally, (13) satisfies (16). But does it converge? Consider the case when the circles meet at the angle $2\theta = \frac{\pi}{2}$. Then Grave's method yields the group

$$\left\{z, z^*_{(1)}, z^*_{(2)}, z^*_{(21)}\right\} \equiv \{\zeta, e^{\frac{\pi i}{2}}\overline{\zeta}, e^{-\frac{\pi i}{2}}\overline{\zeta}, e^{\pi i}\zeta\}$$

consisting of four elements. Hence (13) have to be replaced by

$$z + \varphi(z) = z + \overline{z^*_{(1)}} + \overline{z^*_{(2)}} + z^*_{(21)}. \tag{17}$$

In this case the functional equation (15) implies the same result (17).

3.2. Schwarz alternating method

Another interesting question related to the Schwarz alternating method arises [2, 6]. It is known that the method converges for overlapping domains. But does it yield the series (13)? In the case $2\theta = \frac{\pi}{2}$ the series (13) diverges. Then, what is the difference between (13) and a series obtained by the Schwarz alternating method?

3.3. Conformal mapping

I think, it is possible to apply the method of conformal mapping to the problem (16) and to compare the result with (13). The domain D is mapped onto the upper half plane $\operatorname{Im} w > 0$ by the function $w = i\zeta^{\frac{\pi}{2\theta}}$, where ζ is given by (14). Then the problem (16) becomes

$$\operatorname{Im}\Phi(w) = G(w), \qquad w \in \mathbb{R}, \tag{18}$$

where

$$G(w) = -\operatorname{Im}\left(z_0 \frac{1 + e^{-i\theta} w^{\frac{2\theta}{\pi}}}{1 - e^{-i\theta} w^{\frac{2\theta}{\pi}}}\right)$$

$\Phi(w)$ is analytic in $\operatorname{Im} w > 0$, continuous in $\operatorname{Im} w \geq 0$ including infinity. Applying the Poisson formula we obtain

$$\Phi(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} G(t) \frac{\eta\, dt}{(t-\xi)^2 + \eta^2}\,, \tag{19}$$

where $w = \xi + i\eta$. When does (19) coincide with (13)? We have to substitute in (13)

$$z = z_0 \frac{1 + e^{-i\theta} w^{\frac{2\theta}{\pi}}}{1 - e^{-i\theta} w^{\frac{2\theta}{\pi}}}\,.$$

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