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Jurij Povstenko

Fractional calculus and diffusive stress

Abstract. The main ideas of fractional calculus are recalled. A quasi-static uncoupled theory of diffusive stresses based on the anomalous diffusion equation with fractional derivatives is formulated.

It is well known that integrating by parts $n-1$ times the calculation of the n-fold primitive of a function $f(t)$ can be reduced to the calculation of a single integral of the convolution type

$$I^n f(t) = \frac{1}{(n-1)!}\int\limits_0^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau, \tag{1}$$

where n is a positive integer.

The Laplace transform rule for an integral (1) can be found in every textbook on this subject

$$\mathcal{L}\{I^n f(t)\} = \frac{1}{s^n}\mathcal{L}\{f(t)\},$$

where s is the transform variable.

The Riemann–Liouville fractional integral is introduced as a natural generalization of the convolution type form (1):

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0,$$

where $\Gamma(\alpha)$ is the gamma function. The Laplace transform rule for the fractional integral reads

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha}\mathcal{L}\{f(t)\}.$$

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^α [15, 6]

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$$D_{RL}^{\alpha} f(t) = D^n I^{n-\alpha} f(t)$$
$$= \begin{cases} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, \mathrm{d}\tau \right], & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n \end{cases}$$

and for its Laplace transform it requires the knowledge of the initial values of the fractional integral $I^{n-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, n-1$:

$$\mathcal{L}\{D_{RL}^{\alpha} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^+) s^{n-1-k}, \qquad n-1 < \alpha < n.$$

An alternative definition of the fractional derivative was proposed by Caputo [2, 3]:

$$D_C^{\alpha} f(t) = I^{n-\alpha} D^n f(t) =$$
$$= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n. \end{cases}$$

For its Laplace transform rule the Caputo fractional derivative requires the knowledge of the initial values of the function $f(t)$ and its integer derivatives of order $k = 1, 2, \ldots, n-1$:

$$\mathcal{L}\{D_C^{\alpha} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} D^k f(0^+) s^{\alpha-1-k}, \qquad n-1 < \alpha < n.$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions [5]. In this paper we shall use the Caputo fractional derivative omitting the index C. The major utility of this type fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the govering equation is of fractional order [11].

The definitions of space-fractional differential operators can be found in [15, 6, 1]. The cumbersome aspects of these operators disappear when one computes their Fourier transforms.

The space-fractional derivative of order β is defined as a pseudo-differential operator with the following rule for the Fourier transform [8]:

$$\mathcal{F}\left\{\frac{\mathrm{d}^\beta f(x)}{\mathrm{d}|x|^\beta}\right\} = -|\xi|^\beta \mathcal{F}\left\{f(x)\right\},$$

where ξ is the transform variable. For the Fourier transform of fractional Laplacian one has [15]

$$\mathcal{F}\big\{(-\Delta)^{\frac{\beta}{2}} f(\mathbf{x})\big\} = |\xi|^\beta \mathcal{F}\left\{f(\mathbf{x})\right\}.$$

If the function $f(\mathbf{x})$ depends only on the radial coordinate $r = |\mathbf{x}|$, then in the two-dimensional case we can obtain the corresponding formula for the Hankel transform

$$\mathcal{H}\big\{(-\Delta)^{\frac{\beta}{2}} f(r)\big\} = |\xi|^\beta \mathcal{H}\left\{f(r)\right\}.$$

A quasi-static uncoupled theory of diffusive stress is governed by the equilibrium equation in terms of displacements [12]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \beta_c K \mathrm{grad}\, c,$$

the stress-strain-concentration relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_c K c)\mathbf{I},$$

and the time-fractional diffusion equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta\, c, \qquad 0 \le \alpha \le 2, \tag{2}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{e}$ the linear strain tensor, c the concentration, a the diffusivity coefficient, λ and μ are Lamé constants, $K = \lambda + \frac{2}{3}\mu$, β_c is the diffusion coefficient of volumetric expansion, $\mathbf{I}$ denotes the unit tensor.

If a bounded solid is considered, the corresponding boundary conditions should be given; for unbounded medium

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x},t) = 0,$$
$$\lim_{|\mathbf{x}|\to\infty} c(\mathbf{x},t) = 0.$$

Equation (2) should also be subject to initial conditions

$$t = 0: \quad c = P(\mathbf{x}), \qquad 0 < \alpha \le 2,$$
$$t = 0: \quad \frac{\partial c}{\partial t} = W(\mathbf{x}), \qquad 1 < \alpha \le 2.$$

The most suitable method to solve the obtained system of equations is the method of integral transforms. For classical domains the exponential, sine and cosine Fourier transforms and the Hankel transform with respect to spatial coordinates can be used. The Laplace transform with respect to time is also extensively employed.

In a two-dimensional medium in an axially symmetric case equation (2) has the form

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right).$$

We choose the initial conditions

$$t = 0: \qquad c = p\,\frac{\delta(r)}{2\pi r}, \qquad 0 < \alpha \le 2,$$

$$t = 0: \qquad \frac{\partial c}{\partial t} = 0, \qquad 1 < \alpha \le 2,$$

where $\delta(r)$ is the Dirac delta function.

The nonzero components of the stress tensor expressed in terms of displacement potential read [9]

$$\sigma_{rr} + \sigma_{\theta\theta} = -2\mu\Delta\Phi,$$

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu\left(\frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right).$$

The displacement potential is determined from the equation

$$\Delta\,\Phi = mc, \qquad m = \frac{1+\nu}{1-\nu}\frac{\beta_c}{3},$$

where ν is the Poisson ratio.

We present results corresponding to $\alpha = \frac{1}{2}$ and obtained using the Laplace transform with respect to time t and the Hankel transform with respect to polar coordinate r:

$$c = \frac{p}{4\pi^{\frac{3}{2}} a\sqrt{t}} \int_0^\infty \exp\left(-u^2 - \frac{\rho^2}{8u}\right)\frac{\mathrm{d}u}{u},$$

$$\sigma_{rr} = -2\mu m\frac{p}{\pi^{\frac{3}{2}} a\sqrt{t}\rho^2}\int_0^\infty \mathrm{e}^{-u^2}\left[1 - \exp\left(-\frac{\rho^2}{8u}\right)\right]\mathrm{d}u,$$

$$\sigma_{\theta\theta} = -2\mu m\frac{p}{\pi^{\frac{3}{2}} a\sqrt{t}}\int_0^\infty \mathrm{e}^{-u^2}\left[\left(\frac{1}{4u} + \frac{1}{\rho^2}\right)\exp\left(-\frac{\rho^2}{8u}\right) - \frac{1}{\rho^2}\right]\mathrm{d}u,$$

where the similarity variable

$$\rho = \frac{r}{\sqrt{a}\,t^{\frac{\alpha}{2}}}$$

has been chosen.

It should be noted that in two-dimensional case the fundamental solution to a Cauchy problem for equation (2) in the case $\alpha = \frac{1}{2}$ has the logarithmical singularity at the origin (in contrast to nonsingular solution of classical diffusion equation in the case $\alpha = 1$).

Additional insights into applications of fractional calculus in continuum mechanics and physics as well as extensive literature on the subject can be found in [10, 13, 4, 7, 14].

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Institute of Mathematics
and Computer Science
Jan Długosz University
al. Armii Krajowej 13/15
PL-42-200 Częstochowa
Poland
E-mail: j.povstenko@ajd.czest.pl

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