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Some remarks on the domination between conjunctions and disjunctions

Abstract. There are diverse domination properties considered in linear programming, game theory, semigroup theory and graph theory. In this paper we present a class of conjunctions which dominate each triangular conorm. Moreover, we give the characterization of such conjunctions.

1. Introduction

The notion of domination was introduced by R.M. Tardiff [11]. in the case of triangle functions. It was generalized by B. Schweizer and A. Sklar [10] to the class of associative binary operations with common domain (and common unit element) in order to construct Cartesian products of probabilistic metric spaces. The domination of t-norms is also used in construction of fuzzy equivalence relations [1] and fuzzy orderings [2]. The domination between aggregation operations is useful in investigation of aggregation procedures preserving $\mathcal{T}$-transitivity of fuzzy relations [8].

Furthermore, the characterization of the relation of domination in a class of operations is a solution of a functional inequality in the class of functions. This inequality is a natural generalization of the equation of bisymmetry. Concerning reflexivity of domination, one can obtain the equation of bisymmetry.

Some particular problems of domination were recently examined (Drewniak et al. [4], Drewniak, Król [5]). The characterization of all t-seminorms dominating each triangular conorm was given by P. Sarkoci (cf. [9]).

In this paper at first, in Section 2, we recall definitions of binary operations which will be used in the sequel. Next, we recall the notion of domination concerning two binary operations (Section 3). In Section 4 we describe the class of conjunctions which dominate each triangular conorm. Section 5 contains the characterization of such conjunctions.

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2. Multivalued conjunctions and disjunctions

In this section we recall definitions of binary operations which will be used in the sequel.

Definition 1 (cf. [3])
A conjunction (disjunction) is any increasing binary operation

$$C\colon [0,1]^2 \longrightarrow [0,1] \qquad (\, D\colon [0,1]^2 \longrightarrow [0,1]\ )$$

fulfilling

$$\begin{aligned} &C(0,0) = C(0,1) = C(1,0) = 0, \qquad C(1,1) = 1 \\ &(\, D(0,1) = D(1,0) = D(1,1) = 1, \qquad D(0,0) = 0\ ). \end{aligned}$$

Example 1
The operation $C\colon [0,1]^2 \longrightarrow [0,1]$ given by formula

$$C(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [\frac{1}{2},1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a conjunction.

Definition 2 (cf. [3])
A t-seminorm (t-semiconorm) is any increasing binary operation T (S): $[0,1]^2 \to [0,1]$ with neutral element 1 (0).

Remark 1
An operation is a t-seminorm (t-semiconorm) iff it is a conjunction (disjunction) with the neutral element 1 (0).

Remark 2
Any t-seminorm T and t-semiconorm S fulfils

$$T(x,y) \le \min(x,y); \quad S(x,y) \ge \max(x,y), \qquad x,y \in [0,1].$$

Example 2
The operation $T\colon [0,1]^2 \longrightarrow [0,1]$ given by formula

$$T(x,y) = \begin{cases} 0 & \text{if } x \in [0,\frac{1}{4}],\ y \in [0,\frac{1}{2}], \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a t-seminorm.

Remark 3
The conjunction from Example 1 is not a t-seminorm.

Definition 3 ([7], Definitions 1.1, 1.13)
An associative, commutative and increasing operation T (S): $[0,1]^2 \longrightarrow [0,1]$ is called a t-norm (t-conorm), if it has the neutral element $e = 1$ ($e = 0$).

Remark 4
An operation is a t-norm (t-conorm) iff it is an associative, commutative t-seminorm (t-semiconorm).

By an order isomorphism we can obtain a new operation from a given one.

Theorem 1 (cf. [7], p. 38)
Let us consider an increasing binary operation $F\colon [0,1]^2 \longrightarrow [0,1]$, *a bijection* $\varphi\colon [0,1] \longrightarrow [0,1]$ *and*

$$F_\varphi(x,y) = \varphi^{-1}(F(\varphi(x),\varphi(y))), \qquad x,y \in [0,1].$$

If φ *is increasing and* F *is a* t*-norm (*t*-conorm),* t*-seminorm (*t*-semiconorm) or conjunction (disjunction) then* F_φ *remains a* t*-norm (*t*-conorm),* t*-seminorm (*t*-semiconorm) or conjunction (disjunction), respectively.*

If φ *is decreasing and* F *is a* t*-norm (*t*-conorm),* t*-seminorm (*t*-semiconorm) or conjunction (disjunction) then* F_φ *changes* φ *into a* t*-conorm (*t*-norm),* t*-semiconorm (*t*-seminorm) or disjunction (conjunction), respectively.*

3. Notion of domination

Now we recall the notion of domination concerning two binary operations.

Definition 4 (cf. [11])
Let $F, G\colon [0,1]^2 \longrightarrow [0,1]$. Operation F dominates operation G ($F \gg G$), iff

$$F(G(a,b),G(c,d)) \geq G(F(a,c),F(b,d))$$

for $a,b,c,d \in [0,1]$.

Lemma 1 ([4])
The operation $F = \min$ *dominates every increasing operation. Every increasing operation dominates* $G = \max$.

New examples of domination can be obtained from given ones by order isomorphisms.

Lemma 2 ([8], Proposition 4.2)
Let us consider increasing binary operations $F, G\colon [0,1]^2 \longrightarrow [0,1]$, *a bijection* $\varphi\colon [0,1] \longrightarrow [0,1]$ *and*

$$F_\varphi(x,y) = \varphi^{-1}(F(\varphi(x),\varphi(y))), \qquad x,y \in [0,1]. \tag{1}$$

If φ is increasing, then $F \gg G \Leftrightarrow F_\varphi \gg G_\varphi$. If φ is decreasing, then $F \gg G \Leftrightarrow F_\varphi \ll G_\varphi$.

Using the decreasing bijection $\varphi(x) = 1-x$, $x \in [0,1]$ in (1) we can consider dominations for dual operations.

COROLLARY 1
We have $F \gg G \Leftrightarrow F' \ll G'$, where

$$F'(x,y) = 1 - F(1-x, 1-y), \qquad x, y \in [0,1].$$

4. Domination in the class of conjunctions and disjunctions

In this section we describe the class of conjunctions which dominate each triangular conorm.

THEOREM 2 ([9])
A t-seminorm C dominates the class of all t-conorms iff

$$C(x,y) \in \{0, x, y\} \qquad \text{for any } x, y \in [0,1]. \tag{2}$$

COROLLARY 2
If a t-seminorm C dominates every t-semiconorm then it fulfils (2).

COROLLARY 3
If a t-seminorm C dominates every disjunction then it fulfils (2).

In Theorem 2 one cannot replace a t-seminorm by an arbitrary conjunction. This is illustrated by the following counterexample.

EXAMPLE 3
The operation given by the formula

$$C(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [\frac{1}{8}, 1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a commutative and associative conjunction with the neutral element $\frac{1}{8}$ such that $C(x,y) \in \{x,y\} \subset \{0,x,y\}$ for any $x, y \in [0,1]$. It does not dominate the t-conorm S_P, where $S_P(x,y) = x + y - xy$, $x, y \in [0,1]$. Indeed, for $x = 0.3$, $y = 0.4$, $u = 0.6$, $v = 0.2$ we have

$$\begin{aligned} C(S_P(x,y), S_P(u,v)) &= C(S_P(0.3, 0.4), S_P(0.6, 0.2)) = C(0.58, 0.68) \\ &= 0.68. \end{aligned}$$

On the other hand we have

$$S_P(C(x,u),C(y,v)) = S_P(C(0.3,0.6),C(0.4,0.2)) = S_P(0.6,0.4) \\ = 0.76.$$

It means, that C does not dominate the t-conorm S_P.

This is why we add an additional assumption concerning a conjunction.

THEOREM 3
If $C \leq \min$ is a conjunction fulfilling (2)*, then it dominates every t-conorm.*

Proof. Let $C \leq \min$ be a conjunction fulfilling (2), S be a t-conorm and $x, y, u, v \in [0,1]$. We denote

$$L = C(S(x,y),S(u,v)), \qquad R = S(C(x,u),C(y,v)).$$

If $C(x,u) = C(y,v) = 0$, then we get $R = S(0,0) = 0 \leq L$. If $C(x,u) = 0$ and $C(y,v) = \min(y,v) > 0$, we have

$$R = S(0,\min(u,v)) = \min(u,v) = C(y,v) \leq L.$$

Similarly in the case $C(y,v) = 0$ and $C(x,y) = \min(x,y) > 0$. Let $C(x,y) = \min(x,y) > 0$ and $C(y,v) = \min(y,v) > 0$. At first we observe that by Remark 2, $L \geq C(\min(x,u),\min(y,v)) \geq C(x,u) \neq 0$. So we have two possibilities $L = S(x,y)$ or $L = S(u,v)$. In both cases we have

$$L \geq S(\min(x,u),\min(y,v)) = R.$$

Thus C dominates every triangular conorm.

Simple computations show the following lemma.

LEMMA 3
Let $C\colon [0,1]^2 \longrightarrow [0,1]$ be an increasing operation. Then $C \leq \min$ and C fulfils (2) *iff*

$$C(x,y) \in \{0,\min(x,y)\} \qquad \textit{for any } x,y \in [0,1]. \tag{3}$$

Directly from Lemma 3 and Theorem 3 we obtain the following result.

THEOREM 4
If C is a conjunction fulfilling the condition (3)*, then it dominates every t-conorm.*

The next example shows that there exist binary operations which fulfil the assumptions of Theorem 4 but are not t-seminorms, so they do not fulfil conditions used in Theorem 2.

Example 4
By Theorem 4 the operation $C\colon [0,1]^2 \longrightarrow [0,1]$ given by formula

$$C(x,y) = \begin{cases} \min(x,y) & \text{if } x \in [\frac{1}{2},1],\ y \in [\frac{3}{4},1], \\ 0 & \text{otherwise} \end{cases}$$

dominates any t-conorm.

By duality (cf. Theorem 1, Corollary 1) we obtain analogous results for disjunctions which are dominated by any t-norm.

Theorem 5
If D is a disjunction fulfilling the condition

$$D(x,y) \in \{\max(x,y),1\} \qquad \textit{for any } x,y \in [0,1],$$

then it is dominated by every t-norm.

5. Characterization of a class of conjunction dominating any t-conorm

Conjunctions from Theorem 4 can be characterized in a way used for uninorms (cf. [6]).

Theorem 6
If C is a conjunction fulfilling (3)*, then there exists a decreasing function $g_C\colon [0,1] \longrightarrow [0,1]$ such that*

$$C(x,y) = \begin{cases} 0 & \textit{if } y < g_C(x), \\ \min(x,y) & \textit{if } y > g_C(x), \\ 0 \textit{ or } \min(x,y) & \textit{if } y = g_C(x). \end{cases} \tag{4}$$

Moreover, for $s \in [0,1]$ let $B_s = \{x:\ g_C(x) = s\}$, $a_s = \inf B_s$, $b_s = \sup B_s$. If $a_s < b_s$, then there exists $c_s \in [a_s, b_s]$ such that

$$C(x,s) = \begin{cases} 0 & \textit{if } x < c_s\,, \\ \min(x,s) & \textit{if } x > c_s\,, \\ 0 \textit{ or } \min(x,s) & \textit{if } x = c_s\,. \end{cases} \tag{5}$$

Proof. Define $g_C(x) = \sup A_x$, where $A_x = \{y \in [0,1]:\ C(x,y) = 0\}$. Of course A_x is non-empty because $C(x,0) \le C(1,0) = 0$, so $0 \in A_x$.

Next we prove, that g_C is decreasing. First we note, that $g_C(0) = 1$, because $C(0,1) = 0$.

Let $x < y$. If $g_C(x) = 1$ then $g_C(x) \ge g_C(y)$. If $g_C(x) < 1$ then $C(x,t) = \min(x,t)$ for all $t > g_C(x)$. So, by the monotonicity of C we have $C(y,t) \ge$

$C(x,t) = \min(x,t) > 0$. Therefore $C(y,t) = \min(y,t)$ for all $t > g_C(x)$. It means, that $g_C(y) \leq g_C(x)$.

Now, let $s \in [0,1]$ be such that $a_s < b_s$. Let $c_s = \sup\{x : C(x,s) = 0\}$. We prove that $c_s \in [a_s, b_s]$. If $c_s < a_s$, then $a_s > 0$ and $s < 1$. Let $t \in (c_s, a_s)$. Then by the monotonicity of the function g_C and by definition of the set B_s we obtain $g_C(c_s) \geq g_C(t) > s$ and by (4), $C(t,s) = 0$, which leads to a contradiction. If $c_s > b_s$ then $b_s < 1$ and $s > 0$. Let $t \in (b_s, c_s)$. Then, by the monotonicity of the function g_C and by definition of the set B_s, we obtain $g_C(c_s) \leq g_C(t) < s$ and by (4), $C(t,s) = \min(t,s) > 0$, which leads to a contradiction. So, $c_s \in [a_s, b_s]$.

Directly by the definition of the point c_s and (3) we obtain (5).

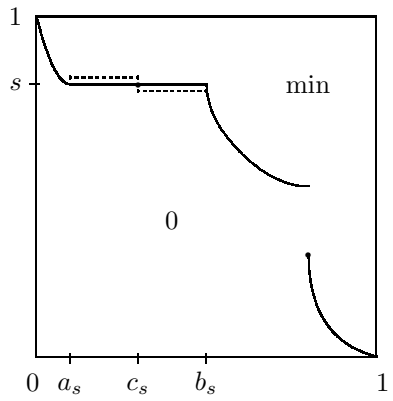

Figure. Structure of operation (4), (5) and graph of g_C

THEOREM 7

Let $g\colon [0,1] \longrightarrow [0,1]$ be a decreasing function. If $C\colon [0,1]^2 \longrightarrow [0,1]$ is the operation given by (4) *with $g_C = g$ and by* (5) *in intervals of constant values of function g and $C(1,1) = 1$, then the operation C is a conjunction fulfilling* (3).

Proof. Directly by (4) we obtain (3). Moreover, $C(1,1) = 1$ and $C(0,0) = C(0,1) = C(1,0) = 0$, because $\min(x,0) = 0$ for all $x \in [0,1]$.

Let $x, y, z \in [0,1]$ and $x < y$.

If $y \leq g(z)$, then $x < g(z)$ and $C(z,x) = 0 \leq C(z,y)$.

If $x \geq g(z)$, then $y > g(z)$ and $C(z,y) = \min(z,y) \geq \min(z,x) \geq C(z,x)$.

Otherwise, we have $C(z,y) = \min(z,y) \geq 0 = C(z,x)$.

Thus operation C is increasing with respect to the second variable. To prove the monotonicity with respect to the first variable we consider a few cases.

If $C(x,z) = \min(x,z)$, then $z \geq g(x)$. By monotonicity of g we have $z \geq g(y)$ and by (4) and (5) we have $C(y,z) = \min(y,z) \geq \min(x,z) = C(x,z)$.

Otherwise, we have $C(x,z) = 0 \leq C(y,z)$. It means that the operation C is increasing with respect to the second variable.

COROLLARY 4
Let $g\colon [0,1] \longrightarrow [0,1]$ *be a decreasing function. The operation* $C\colon [0,1]^2 \longrightarrow [0,1]$ *given by*

$$C(x,y) = \begin{cases} 0 & \text{if } y < g(x), \\ \min(x,y) & \text{if } y \geq g(x) \end{cases}$$

is a conjunction fulfilling (3).

REMARK 5
By duality (cf. Theorem 1, Corollary 1) we may obtain a similar characterization of disjunctions which are dominated by any t-norm.

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