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Riemann integrability of a nowhere continuous multifunction

Abstract. We present an example of the Riemann integrable multifunction which is discontinuous at each point with respect to the Hausdorff metric. The constructed multifunction is neither lower nor upper semicontinuous.

1. Introduction

The Riemann integral for multifunctions with compact convex values was investigated by A. Dinghas [3] and M. Hukuhara [4]. Some properties of Riemann integral of multifunctions with convex closed bounded values may be found in [5]. The Riemann integrability of multifunctions with compact convex values was presented in [6].

Our main goal is to show that the continuity for almost all $x \in [a, b]$ of a bounded multifunction is not a necessary condition for the Riemann integrability. The same example shows also that the monotonicity does not imply the almost everywhere continuity of multifunctions.

2. Basic definitions

Let X be a real Banach space. Denote by $clb(X)$ the set of all nonempty convex closed bounded subsets of X. For given $A, B \in clb(X)$, we set

$$A + B = \{a + b \mid \ a \in A,\ b \in B\},$$
$$\lambda A = \{\lambda a \mid \ a \in A\} \qquad \text{for } \lambda \geq 0$$

and

$$A \overset{*}{+} B = \operatorname{cl}(A + B),$$

where $\operatorname{cl} A$ means the closure of A in X. It is easy to see that $(clb(X), \overset{*}{+}, \cdot)$ satisfies the following properties

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$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B,$$
$$(\lambda + \mu)A = \lambda A \overset{*}{+} \mu A,$$
$$\lambda(\mu A) = (\lambda\mu)A,$$
$$1 \cdot A = A$$

for each $A, B \in clb(X)$ and $\lambda, \mu \geq 0$. If $A, B, C \in clb(X)$, then the equality $A \overset{*}{+} C = B \overset{*}{+} C$ implies $A = B$, thus $clb(X)$ with addition $\overset{*}{+}$ satisfies the cancellation law (see [1, Theorem II-17] and [8, Corollary 2.3]).

$clb(X)$ is a metric space with the Hausdorff metric h defined by the relation

$$h(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} \|a - b\|$. The metric space $(clb(X), h)$ is complete (see e.g. [1, Theorem II-3]). Moreover, the Hausdorff metric h is translation invariant since

$$h(A \overset{*}{+} C, B \overset{*}{+} C) = h(A + C, B + C) = h(A, B)$$

(cf. [7, Lemma 3], [2, Lemma 2.2]) and positively homogeneous

$$h(\lambda A, \lambda B) = \lambda h(A, B)$$

for all $A, B, C \in clb(X)$ and $\lambda \geq 0$ (cf. [2, Lemma 2.2]).

Lemma 1

Let X be a normed vector space. If $A, B, C \in clb(X)$ and $A \subset B \subset C$, then

$$h(B, C) \leq h(A, C) \qquad \text{and} \qquad h(A, B) \leq h(A, C).$$

Proof. Since $e(B, C) = 0$ and $d(c, B) \leq d(c, A)$, $c \in C$, we have

$$h(B, C) = e(C, B) \leq e(C, A) = h(A, C).$$

The proof of the second inequality is analogous.

Let F be a multifunction defined on the interval $[a, b]$ with nonempty convex closed bounded values in X. A set $\Delta = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \ldots < x_n = b$, is said to be a *partition* of $[a, b]$. For a given partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ we set $\delta(\Delta) = \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}$. Δ' is said to be a *subpartition* of Δ if Δ' is a partition of the same interval and $\Delta \subset \Delta'$. For the partition Δ and for a system $\tau = (\tau_1, \ldots, \tau_n)$ of intermediate points $\tau_i \in [x_{i-1}, x_i]$ we create the *Riemann sum*

$$S(\Delta, \tau) = (x_1 - x_0)F(\tau_1) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(\tau_n).$$

If for every sequence $((\Delta^\nu, \tau^\nu))$, where $\Delta^\nu = \{x_0^\nu, x_1^\nu, \ldots, x_{n_\nu}^\nu\}$ are partitions of $[a,b]$ such that $\lim_{\nu\to\infty} \delta(\Delta^\nu) = 0$ and $\tau^\nu = (\tau_1^\nu, \ldots, \tau_{n_\nu}^\nu)$ are systems of intermediate points ($\tau_i^\nu \in [x_{i-1}^\nu, x_i^\nu]$), the sequence of the Riemann sums $(S(\Delta^\nu, \tau^\nu))$ tends to the same limit I with respect to the Hausdorff metric, then F is said to be *Riemann integrable* over $[a,b]$ and $I =: \int_a^b F(x)\,dx$. Obviously, if the limit I exists, then $I \in clb(X)$.

LEMMA 2
Let X be a real Banach space and $F\colon [a,b] \longrightarrow clb(X)$. Then the following conditions are equivalent:

(i) *F is Riemann integrable on $[a,b]$;*

(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition Δ satisfying $\delta(\Delta) < \delta$, for every subpartition Δ' of Δ and for all corresponding systems τ, τ' of intermediate points, the inequality*

$$h\left(S(\Delta,\tau), S(\Delta',\tau')\right) < \varepsilon$$

is satisfied.

The easy proof is omitted. The completeness of $(clb(X), h)$ is needed only in the proof of sufficiency.

We say that a multifunction $F\colon [a,b] \longrightarrow clb(X)$ is *increasing* if

$$F(s) \subset F(t)$$

holds true for all $a \le s \le t \le b$.

PROPOSITION 1
An increasing multifunction $F\colon [a,b] \longrightarrow clb(X)$ is right-hand side lower semi-continuous at each point of the interval $[a,b)$.

Proof. Let $t_0 \in [a,b)$ and let U be an open subset of X such that $F(t_0) \cap U \neq \emptyset$. Since $F(t_0) \subset F(t)$ when $t > t_0$, $F(t) \cap U \neq \emptyset$ for each $t \in [t_0, b]$ which implies the right-hand side lower semi-continuity of F at t_0.

PROPOSITION 2
An increasing multifunction $F\colon [a,b] \longrightarrow clb(X)$ is left-hand side upper semi-continuous at each point of the interval $(a,b]$.

Proof. Let $t_0 \in (a,b]$ and let U be an open subset of X such that $F(t_0) \subset U$. Since $F(t) \subset F(t_0)$ for $t \in [a, t_0]$, $F(t) \subset U$ for the same t and F is left-hand side upper semi-continuous at t_0.

For an increasing multifunction $F\colon [a,b] \longrightarrow clb(X)$ and for each partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ we may create two sums

$$S(\Delta) := (x_1 - x_0)F(x_1) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(x_n)$$

and

$$s(\Delta) := (x_1 - x_0)F(x_0) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(x_{n-1}).$$

3. Main results

Let Y be a Banach space defined as the set of all bounded functions $x\colon [0,1] \longrightarrow \mathbb{R}$ with the norm $||x|| = \sup_{t\in[0,1]} |x(t)|$.

Let $F\colon [0,1] \longrightarrow 2^Y$ be a multifunction with values defined as follows

$$F(t) := \{x\colon [0,1] \longrightarrow [0,1] \mid x(s) = 0 \text{ for all } s > t\}, \qquad t \in [0,1),$$
$$F(1) := \{x \in Y \mid x\colon [0,1] \longrightarrow [0,1]\}.$$

In particular, $F(0)$ is equal to $\{x\colon [0,1] \longrightarrow [0,1] \mid x(s) = 0 \text{ for each } s \in (0,1]\}$. It is not difficult to see that the set $F(t)$ is an element of $clb(Y)$ for all $t \in [0,1]$.

Now we consider some properties of the multifunction F.

Remark 1
F is increasing on $[0,1]$. Indeed, let $s < t$ and $s,t \in [0,1]$. If $t = 1$, then $F(s) \subset F(1)$ for all $s \in [0,1]$. Assume that $t < 1$ and $x \in F(s)$. We have $x(u) = 0$ for all $u > s$ and, in particular, for all $u > t$. Consequently $F(s) \subset F(t)$.

Remark 2
By Proposition 1 and Remark 1 the multifunction F is right-hand side lower semi-continuous at each point of $[0,1)$. We shall show that it is not left-hand side lower semi-continuous in $(0,1]$. Let $t_0 \in (0,1]$. Define $x(t) = 1$ for $t \in [0,t_0]$ and $x(t) = 0$ for $t \in (t_0, 1]$, if $t_0 \in (0,1)$. Let $S(x, \frac{1}{2})$ be an open ball in Y centered at x with the radius $\frac{1}{2}$. Of course $S(x, \frac{1}{2}) \cap F(t_0) \neq \emptyset$. Now take an arbitrary $s \in [0, t_0)$. If $y \in F(s)$, then

$$1 \geq \|x - y\| = \sup_{u\in[0,1]} |x(u) - y(u)| \geq \sup_{u\in(s,t_0)} |x(u) - y(u)| = 1.$$

Consequently, $\|x - y\| = 1$ and $y \notin S(x, \frac{1}{2})$, i.e., $S(x, \frac{1}{2}) \cap F(s) = \emptyset$ for all $0 \leq s < t_0$.

Remark 3
By Proposition 2 and Remark 1 the multifunction F is left-hand side upper semi-continuous at each point of the interval $(0,1]$. We will show that F is not right-hand side upper semi-continuous in $[0,1)$. Indeed, let $t_0 \in [0,1)$ and let U be an open set defined by $U = \bigcup_{x\in F(t_0)} \{y \in Y \mid \ \|y - x\| < \frac{1}{2}\}$. It is clear that $F(t_0) \subset U$, but $F(t) \not\subset U$ for each $t > t_0$. It is sufficient to choose $z \in F(t)$ such that $z(u) = 1$ for $u \in [t_0, t]$. Thus for each $x \in F(t_0)$

$$\|z - x\| \geq \sup_{u \in [t_0, t]} |z(u) - x(u)| = 1$$

and consequently $z \notin U$.

Remark 4
F is non-continuous at each point of the interval $[0,1]$ with respect to the Hausdorff metric.

By Remark 2 it follows that $h(F(s), F(t)) = 1$ for all $s, t \in [0,1]$ such that $0 \leq s < t \leq 1$. Thus $h(F(s), F(t)) = 1$ if $s \neq t$ and $\lim_{t \to s} h(F(s), F(t)) = 1$ for all $s \in [0,1]$.

Theorem 1
The multifunction F defined by formulas

$$F(t) := \{x \colon [0,1] \longrightarrow [0,1] \mid \ x(s) = 0 \textit{ for all } s > t\}, \qquad t \in [0,1)$$

and

$$F(1) := \{x \in Y \mid \ x \colon [0,1] \longrightarrow [0,1]\}$$

is Riemann integrable on $[0,1]$.

Proof. Let $\varepsilon > 0$ and let $\Delta = \{t_0, t_1, \dots, t_n\}$ be an arbitrary partition of $[0,1]$ such that $\delta(\Delta) < \varepsilon$. It is sufficient (see Lemma 2) to show that for each subpartition Δ' and for each systems of intermediate points τ, τ' corresponding to Δ, Δ', respectively,

$$h\left(S(\Delta, \tau), S(\Delta', \tau')\right) < 2\varepsilon.$$

At first we are going to show that

$$s(\Delta) = \{x \colon [0,1] \longrightarrow [0,1] \mid \ x(t) \in [0, 1 - t_k] \text{ for } t \in (t_{k-1}, t_k], \\ k \in \{1, \dots, n\} \text{ and } x(0) \in [0,1]\}, \tag{1}$$

$$S(\Delta) = \{y \colon [0,1] \longrightarrow [0,1] \mid y(t) \in [0, 1 - t_{k-1}] \text{ for } t \in (t_{k-1}, t_k], \\ k \in \{1, \dots, n\} \text{ and } y(0) \in [0,1]\}. \tag{2}$$

Let us take $a \in s(\Delta)$. We can find n sequences (a_k^ν), such that $a_k^\nu \in (t_k - t_{k-1})F(t_{k-1})$, and $\sum_{k=1}^n a_k^\nu \to a$ if $\nu \to \infty$. Obviously $a_k^\nu(t) \in [0, t_k - t_{k-1}]$ for $t \leq t_{k-1}$ and $a_k^\nu(t) = 0$ if $t > t_{k-1}$. Summing up over $k \in \{1, \dots, n\}$ we have

$$0 \leq \left(\sum_{k=1}^n a_k^\nu \right)(t) = \sum_{k=1}^n a_k^\nu(t) = \sum_{j=k+1}^n a_j^\nu(t) \leq \sum_{j=k+1}^n (t_j - t_{j-1}) = 1 - t_k$$

for each $t \in (t_{k-1}, t_k]$ and

$$0 \le \Big(\sum_{k=1}^{n} a_k^{\nu}\Big)(0) = \sum_{k=1}^{n} a_k^{\nu}(0) \le \sum_{k=1}^{n}(t_k - t_{k-1}) = 1.$$

Thus $\sum_{k=1}^{n} a_k^{\nu}$ belong to the right-hand side of (1) which is a closed set, so a also belongs there.

Conversely, let a belongs to the right-hand side of (1). We define functions $b\colon [0,1] \longrightarrow [0,1]$, $b_k\colon [0,1] \longrightarrow [0,1]$, $k \in \{0,\dots,n-1\}$, by formulas

$$b(t) = \begin{cases} a(t), & t = 0, \\ \dfrac{a(t)}{1-t_k}, & t \in (t_{k-1}, t_k],\ k = 1,\dots,n-1, \\ 0, & t \in (t_{n-1}, 1] \end{cases}$$

and

$$b_k(t) = \begin{cases} b(t), & t \in [0, t_k], \\ 0, & t \in (t_k, 1]. \end{cases}$$

Obviously, $b_k \in F(t_k)$ for each $k \in \{0,\dots,n-1\}$.

For $t \in (t_{j-1}, t_j]$, where $j \in \{1,\dots,n-1\}$, we have

$$\begin{aligned} &[(t_1 - t_0)b_0 + \ldots + (t_n - t_{n-1})b_{n-1}](t) \\ &\quad = [(t_{j+1} - t_j)b_j + \ldots + (t_n - t_{n-1})b_{n-1}](t) \\ &\quad = [(t_{j+1} - t_j)b + \ldots + (t_n - t_{n-1})b](t) \\ &\quad = (1 - t_j)b(t) \\ &\quad = a(t), \end{aligned}$$

and for $u \in (t_{n-1}, t_n]$ the equality

$$[(t_1 - t_0)b_0 + \ldots + (t_n - t_{n-1})b_{n-1}](u) = 0 = a(u)$$

holds. Moreover

$$[(t_1 - t_0)b_0 + \ldots + (t_n - t_{n-1})b_{n-1}](0) = b(0) = a(0).$$

Thus $a \in s(\Delta)$ and the proof of (1) is complete.

The equality (2) can be established similarly.

Since F is increasing (by Remark 1) the following inclusions are valid

$$s(\Delta) \subset S(\Delta, \tau) \subset S(\Delta). \tag{3}$$

We will show that

$$s(\Delta) \subset S(\Delta', \tau') \subset S(\Delta). \tag{4}$$

There is no loss of generality in assuming that $\Delta' = \Delta \cup \{u\}$, where $u \in (t_{n-1}, 1)$ and $\tau' = (\tau_1, \ldots, \tau_{n+1})$, where $\tau_i \in [t_{i-1}, t_i]$, $i \in \{1, \ldots, n-1\}$, $\tau_n \in [t_{n-1}, u]$, $\tau_{n+1} \in [u, 1]$. By definitions of $s(\Delta)$, $S(\Delta)$ and $S(\Delta', \tau')$ we have

$$\begin{aligned} s(\Delta) &= (t_1 - t_0)F(t_0) \overset{*}{+} \ldots \overset{*}{+} (t_{n-1} - t_{n-2})F(t_{n-2}) \\ &\quad \overset{*}{+} (u - t_{n-1})F(t_{n-1}) \overset{*}{+} (t_n - u)F(t_{n-1}) \\ &\subset (t_1 - t_0)F(\tau_1) \overset{*}{+} \ldots \overset{*}{+} (t_{n-1} - t_{n-2})F(\tau_{n-1}) \\ &\quad \overset{*}{+} (u - t_{n-1})F(\tau_n) \overset{*}{+} (t_n - u)F(\tau_{n+1}) \\ &= S(\Delta', \tau') \\ &\subset (t_1 - t_0)F(t_1) \overset{*}{+} \ldots \overset{*}{+} (t_{n-1} - t_{n-2})F(t_{n-1}) \\ &\quad \overset{*}{+} (u - t_{n-1})F(t_n) \overset{*}{+} (t_n - u)F(t_n) \\ &= S(\Delta). \end{aligned}$$

Now, by Lemma 1, (3) and (4) we have

$$\begin{aligned} h\left(S(\Delta, \tau), S(\Delta', \tau')\right) &\le h\left(S(\Delta, \tau), S(\Delta)\right) + h\left(S(\Delta), S(\Delta', \tau')\right) \\ &\le 2h\left(S(\Delta), s(\Delta)\right). \end{aligned}$$

We are going to show that

$$e\left(S(\Delta), s(\Delta)\right) = \delta(\Delta).$$

Let $x_0, y_0 \colon [0,1] \longrightarrow [0,1]$ be defined by

$$x_0(t) = \begin{cases} 1, & t = 0, \\ 1 - t_k\,, & t \in (t_{k-1}, t_k], \end{cases} \qquad y_0(t) = \begin{cases} 1, & t = 0, \\ 1 - t_{k-1}, & t \in (t_{k-1}, t_k]. \end{cases}$$

Obviously $x_0 \in s(\Delta)$ and $y_0 \in S(\Delta)$.

In order to see that

$$\|y_0 - x_0\| = d\left(y_0, s(\Delta)\right) \tag{5}$$

suppose that $x \in s(\Delta)$. Then for $t = 0$ we have

$$y_0(t) - x(t) = 1 - x(t) \ge 0 = y_0(t) - x_0(t)$$

and if $t \in (t_{k-1}, t_k]$, we obtain

$$y_0(t) - x(t) = 1 - t_{k-1} - x(t) \ge 1 - t_{k-1} - (1 - t_k) = y_0(t) - x_0(t).$$

Hence

$$\|y_0 - x\| \ge \|y_0 - x_0\|$$

for each $x \in s(\Delta)$, which completes the proof of (5).

Now for each $y \in S(\Delta)$ we will find $x \in s(\Delta)$ such that

$$\|y_0 - x_0\| \geq \|y - x\| \geq d(y, s(\Delta)). \tag{6}$$

Let x be defined by

$$x(t) = \begin{cases} x_0(t), & y(t) \in [x_0(t), y_0(t)], \\ y(t), & y(t) \in [0, x_0(t)). \end{cases}$$

It is clear that

$$|y_0(t) - x_0(t)| \geq |y(t) - x_0(t)| = |y(t) - x(t)|$$

if $y(t) \in [x_0(t), y_0(t)]$ and

$$|y_0(t) - x_0(t)| \geq 0 = |y(t) - x(t)|$$

for $y(t) \in [0, x_0(t))$. Thus $\|y_0 - x_0\| \geq \|y - x\|$ and (6) holds.

By (5) and (6) we obtain

$$\begin{aligned} e\left(S(\Delta), s(\Delta)\right) = \|y_0 - x_0\| &= \sup_{t \in [0,1]} |y_0(t) - x_0(t)| \\ &= \max_{k \in \{1,\ldots,n\}} t_k - t_{k-1} = \delta(\Delta) \\ &< \varepsilon \end{aligned}$$

and

$$h\left(S(\Delta, \tau), S(\Delta', \tau')\right) < 2\varepsilon,$$

which completes the proof.

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