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# Upper estimates of complexity of algorithms for multi-peg Tower of Hanoi problem

**Abstract**. There are proved upper explicit estimates of complexity of algorithms: for multi-peg Tower of Hanoi problem with the limited number of disks, for Reve's puzzle and for 5-peg Tower of Hanoi problem with the free number of disks.

## 1. Introduction

The Tower of Hanoi is a mathematical game or puzzle, which was invented by the French mathematician Edouard Lucas in 1883 under the pen-name N. Claus [1] described as an “old Indian legend".

According to the legend of the Tower of Hanoi (originally the “Tower of Brahma" in a temple in the Indian city of Benares), the temple priests are to transfer a tower consisting of 64 fragile disks of gold from one part of the temple to another, one disk at a time. The disks are arranged in order, no two of them the same size, with the largest on the bottom and the smallest on top. Because of their fragility, a larger disk may never be placed on a smaller one, and there is only one intermediate location where disks can be temporarily placed. It is said that before the priests complete their task the temple will crumble into dust and the world will vanish in a clap of thunder.

In the basic version, a stack of discs of mutually distinct sizes is arranged on one of three pegs, with the size restriction that no larger disc is atop a smaller disc. The problem is then to move the entire stack of discs to another of the three pegs by moving one disc at a time, and always maintaining the size restriction.

Let us denote by $H_3(n)$ the minimum number of moves needed to solve the puzzle with $n$ discs for three pegs. It is known (E. Lucas)

$$H_3(n) = 2^n - 1. \tag{1}$$

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The Tower of Hanoi is a well known NP problem in recreational mathematics. The problem is isomorphic to finding a Hamiltonian path on an $n$-hypercube.

The literature devoted to the problem is vast and enumerates at least some 200 relevant positions printed in various countries and in various languages (not counting appearances in psychological journals and textbooks in discrete mathematics [6]). Authors of [3] analyse several references from the first edition of the Bibliography of P.K. Stockmeyer, which was posted in 1997, and declare “many papers only rediscover known results".

This problem has also been widely used in the computer science as a paradigmatic teaching example for recursive solution methods. Algorithms of moving discs from one of three pegs to another peg is still used today in many computer science textbooks to demonstrate how to write a recursive algorithm or program. Also these algorithms are often proposed programming on various olympiads and competitions for informaticians. Many computer games use algorithms “The Tower of Hanoi", for example [7].

## 2. The multi-peg Tower of Hanoi problem

One of many possible generalizations of the Tower of Hanoi problem is to increase the number of pegs. The multi-peg Tower of Hanoi problem consists of $k > 3$ pegs $(B_1, B_2, \ldots, B_k)$ mounted on a board together with $n$ discs of different sizes $(1, 2, \ldots, n)$. Initially these discs are placed on one peg $(B_1)$ in order of size, with the largest ($n$-disc) on the bottom. The rules of the problem allow discs to be moved one at a time from one peg to another as long as a largest disc is never placed on top of a smaller disc. The goal of the problem is to transfer all the discs to another peg $(B_2)$ with the minimum number of moves, denoted $H_k(n)$. The function $H_k(n)$ characterize the complexity of the algorithm for the solution of the multi-peg Tower of Hanoi problem.

It is surprising that an optimal solution to the $k$-peg version of the classic Tower of Hanoi problem is unknown for each $k \geq 4$.

Algorithms for transporting $n$ discs from the first peg to $B_2$ for the case of $k > 3$ pegs were investigated by some mathematicians. A well-known result for investigation of our problem is the recurrence formula

$$S(n, k) = 2S(n - i, k) + S(i, k - 1), \tag{2}$$

where $k > 3$ is a number of pegs, $n$ is a number of discs and $S(n, k)$ is the minimal number of moves required for transporting $n$ discs from the first peg to $B_2$.

This formula was independently published in 1941 by two mathematicians Frame and Stewart [5] with the help of algorithms, which modern mathematicians name as the “Frame–Stewart algorithm scheme":

1. Recursively transport a stack of $n-i$ smallest disks from the first peg to a temporal peg, using all $k$ pegs;
2. Transport the remaining stack of $i$ largest discs from the first peg to the final peg, using $(k-1)$ pegs and ignoring the peg occupied by the smaller discs;
3. Recursively transport the smallest $n-i$ discs from the temporal peg to the final peg, using all $k$ pegs.

The Frame–Stewart number denoted $S(n,k)$ or $FS(n,k)$, is the minimum number of moves needed to solve the Tower of Hanoi problem using the above Frame-Stewart algorithm scheme.

It is easily to get an other recurrence formula for the multi-peg Tower of Hanoi problem with the help of a next algorithm:

1. Move $i_k$ smallest discs from the first peg to the peg $B_k$ .
2. Move $i_{k-1}$ next discs from the first peg to the peg $B_{k-1}$ .
3. Move $i_{k-2}$ next discs from the first peg to the peg $B_{k-2}$ .

At last (on step $(k-3)$) move $i_4$ next discs from the first peg to the peg $B_4$ .

We sum our moves for $k-3$ steps and obtain

$$H_k(i_k) + H_{k-1}(i_{k-1}) + \cdots + H_4(i_4).$$

Step $k-2$. Move $i_3$ largest discs, where $i_3 = n - \sum_{j=4}^{k} i_j$ , from the first peg to the peg $B_2$ , using three pegs ($2^{i_3}-1$ moves).

Step $k-1$. Move all discs from pegs $B_4, B_5, \ldots, B_k$ to the peg $B_2$ ($H_k(i_k) + H_{k-1}(i_{k-1}) + \cdots + H_4(i_4)$ moves).

We sum all moves, which needed for transporting of all $n$ discs from the first peg to the peg $B_2$, and obtain

$$H_k(n) = 2\sum_{j=4}^{k} H_j(i_j) + 2^{i_3} - 1, \tag{3}$$

where $i_j \le i_{j+1}$ and $i_3 = n - \sum_{j=4}^{k} i_j$ .

The recurrence formula (3) is known to mathematicians and is published in [3]. It is not comfortable for practical using as it is difficult to find optimal decomposition of the number $n$ into $k-2$ numbers $i_3, i_4, \ldots, i_k$.

Also the following statement is known:

If $k \ge 3$ and $n \le k-1$, then

$$H_k(n) = 2n - 1. \tag{4}$$

If $n = k$, we must place on the peg $B_k$ two smallest (2-disc and 1-disc) discs (three moves). Then we move $n-2$ discs from $B_1$ to $B_2, B_3, \ldots, B_{k-1}$ , where on the each peg is placed one disc and $n$-disc is placed on $B_2$ ($n-2$ moves). Then we move $n-3$ discs from $B_3, \ldots, B_{k-1}$ to $B_2$ ($n-3$ moves). At last we move two discs from $B_k$ to $B_2$ (three moves).

We sum our moves in this case and obtain the estimation

$$H_k(n) = 2n + 1. \tag{5}$$

## 3. The explicit estimate for multi-peg Tower of Hanoi problem with the limited number of disks

With the help of a similar algorithm we get

THEOREM 1
*If $k \geq 3$ and $k \leq n \leq \frac{k(k-1)}{2}$, then*

$$H_k(n) = 4n - 2k + 1. \tag{6}$$

*Proof.* CASE 1. We consider first a case, where $k \leq n \leq 2k-3$.
Let $n = l + k - 1$, with $l \leq k - 2$.
Then we will use the following algorithm of transferring of discs:

1. Move the $k-1$ smallest discs from the first peg to $B_2, B_3, \ldots, B_k$, so that on each peg one disc is placed and 1-disc is placed on $B_2$.
2. Move $l$ smallest discs (1-disc, $2-$ disc, $\ldots$, $l$-disc) from temporal pegs to the peg $B*$, where we placed the $(l+1)$-disc.
3. Move $l$ largest discs from $B_1$ to free pegs $B_2, B_3, \ldots, B_{l+1}$, so that on each peg one disc is placed and $n$-disc is placed on $B_2$.
4. Move $l-1$ largest discs ($(n-1)$-disc, $\ldots$, $(n-l+1)$-disc) from pegs $B_3, \ldots, B_{l+1}$ to the peg $B_2$.
5. Move the remaining stack of $k-2-l$ largest discs to $B_2$.
6. Move the remaining stack of $l+1$ smallest discs from $B*$ to $B_2$.

We sum our moves and obtain

$$H_k(n) = 2k + 4l - 3.$$

From $l = n - k + 1$ we have

$$H_k(n) = 2k + 4(n - k + 1) - 3 = 4n - 2k + 1. \tag{7}$$

CASE 2. In the next case we have $2k - 3 < n \leq \frac{k(k-1)}{2}$.
Let $n = \frac{k(k-1)}{2}$.
Then we will use the following algorithm to transfer the discs:

1. Move $k-1$ smallest discs from the first peg to the peg $B_k$ ($2k-3$ moves).
2. Move $k-2$ next discs from the first peg to the peg $B_{k-1}$ ($2k-5$ moves).

3. Move $k-3$ next discs from the first peg to the peg $B_{k-2}$ $(2k-7$ moves).

At last on stage $k-1$ we move the last $n$-disc from $B_1$ to $B_2$ .
We sum our moves and obtain

$$\frac{((2k-3)+1)(k-1)}{2} = (k-1)^2.$$

Then we must sum moves, which are necessary for transporting $n-1$ discs from pegs $B_3, \ldots, B_k$ to the peg $B_2$ .

It is obvious, that this number of moves is equal to $(k-1)^2 - 1$.

Then we obtain, that the sum of moves needed for transporting $n = \frac{k(k-1)}{2}$ discs from the peg $B_1$ to the peg $B_2$ is equal to

$$H_k(n) = 2(k-1)^2 - 1. \tag{8}$$

If we have to transport $n < \frac{k(k-1)}{2}$ discs from the peg $B_1$ to the peg $B_2$ and

$$\frac{k(k-1)}{2} - n = i,$$

the number of moves in the case

$$2k-3 < n \le \frac{k(k-1)}{2}$$

is equal to

$$H_k(n) = 2(k-1)^2 - 1 - 4i. \tag{9}$$

From $i = \frac{k(k-1)-2n}{2}$ we have

$$\begin{aligned} H_k(n) &= 2(k-1)^2 - 1 - 2(k(k-1) - 2n) \\ &= 2(k-1)^2 - 2k(k-1) + 4n - 1 \\ &= 2(k-1)(k-1-k) + 4n - 1 \\ &= 4n - 2k + 1. \end{aligned} \tag{10}$$

This yields our statement.

## 4. The new estimate for Reve's puzzle

We can deduce a non-recursive (explicit) formula estimating the minimal number of moves required for transporting $n$ discs from the peg $B_1$ to the peg $B_2$ with the help of two subsidiary pegs $B_3$ and $B_4$. The puzzle "The Tower of Hanoi" for $k = 4$ pegs is known as Reve's puzzle. There are several paper published on Reve's puzzle. For example, in [4] there are recursive algorithm computations for $H_4(20)$, $H_4(50)$, $H_4(100)$, $H_4(150)$, $H_4(200)$. However, explicit formula estimating $H_4(n)$ is not stated in that paper.

We will prove the next result.

Theorem 2
*Let $n$ be fixed and $m$ be an integer such that $\frac{m(m-1)}{2} < n \le \frac{(m+1)m}{2}$. Then*

$$H_4(n) = 2^{m-2}(2n - (m-2)^2 - m) + 1. \tag{11}$$

*Proof.* Case 1. $n = \frac{(m+1)m}{2}$ is a triangular number.

We will apply the Frame–Stewart algorithm scheme for transporting of $n$ discs from the $B_1$ to the $B_4$ in the following way:

1. Move $i$ smallest discs from the first peg to the peg $B_4$ , using all four pegs.
2. Move $n - i$ largest discs from the first peg to the peg $B_2$ , using three pegs.
3. Move $i$ smallest discs from the $B_4$ to the peg $B_2$ , using all four pegs.

We sum our moves and obtain

$$H_4(n) = 2H_4(i) + 2^{n-i} - 1. \tag{12}$$

It is obvious, that for transporting of $i$ discs from the $B_1$ to the $B_4$ we can transport $j < i$ smallest discs from the $B_1$ to the $B_3$ , then $i - j$ largest discs from the $B_1$ to the peg $B_4$ and at last transport $j$ smallest discs from the $B_3$ to the $B_4$ . With the help of

$$H_4(i) = 2H_4(j) + 2^{i-j} - 1$$

we obtain

$$H_4(n) = 2(2H_4(j) + 2^{i-j} - 1) + 2^{n-i} - 1.$$

Using the Frame–Stewart algorithm and the formula (12) many times we obtain a next formula for calculation of the minimum number of moves needed to solve the 4-peg Tower of Hanoi problem:

$$H_4(n) = 2^m - 1 + 2(2^{m-1} - 1 + 2(2^{m-2} - 1 + \cdots + 2(2^2 - 1 + 2 \cdot 1)\cdots)), \tag{13}$$

where $m + (m-1) + \cdots + (m - (m-1)) = \frac{(m+1)m}{2} = n$ is a triangular number.

The number of moves necessary for transporting of three smallest discs from the $B_1$ to a temporal peg ($B_3$ or $B_4$) is described in the innest brackets.

Dropping all the brackets in the formula (13) we obtain $m - 1$ summands, which are equal to $2^m$, and one, which is equal to $2^{m-1}$. The others summands of the development of the formula (13) are negative integers, which sum up to

$$1 + 2 + 4 + \cdots + 2^{m-2} = 2^{m-1} - 1.$$

Then we have for triangular numbers the exact formula

$$H_4(n) = (m-1)2^m + 2^{m-1} - 2^{m-1} + 1 = (m-1)2^m + 1. \tag{14}$$

CASE 2. $n$ is nontriangular:

$$\frac{(m-1)m}{2} < n < \frac{(m+1)m}{2}.$$

We estimate the number $H_4(n)$ in the following way:

We observe from (13) that the absence of one (the smallest) disc, that is $n = \frac{(m+1)m}{2} - 1$, on the peg $B_1$, allows to economize $2^{m-1}$ moves needed for transporting discs from the peg $B_1$ to the peg $B_2$ compared to the triangular case $n = \frac{(m+1)m}{2}$.

If we have to transport $\frac{(m+1)m}{2} - n$ discs from the peg $B_1$ to the peg $B_2$ and $n > \frac{(m-1)m}{2}$, then number of "saved" moves is

$$\left(\frac{(m+1)m}{2} - n\right) 2^{m-1} = (m(m+1) - 2n)2^{m-2}.$$

Finally, we get

$$\begin{aligned} H_4(n) &= (m-1)2^m + 1 - (m(m+1) - 2n)2^{m-2} \\ &= (m-1)2^2 2^{m-2} + 1 - (m^2 + m - 2n)2^{m-2} \\ &= 2^{m-2}(4m - 4 - m^2 - m + 2n) + 1 \\ &= 2^{m-2}(2n - (m-2)^2 - m) + 1. \end{aligned}$$

REMARK 1

Poole (1994) and Rangel–Mondragón [8] computed the minimum numer of moves needed to solve the Reve's puzzle by:

$$H_4(n) = 1 + \left[n - \frac{x(x-1)}{2} - 1\right] 2^x \tag{15}$$

with

$$x = \left[\frac{\sqrt{8n-7} - 1}{2}\right]$$

where [ ] is the Gauss bracket.

Of course, formulas (11) and (15) yield identical results for $H_4(n)$ but formula (11) seems to be more comfortable for calculation.

## 5. The explicit estimate for 5-peg Tower of Hanoi

With the help of formulas (2), (6) and (11) we can deduce a nonrecursive formula for an estimation of $H_5(n)$:

$$H_5(n) = H_4(n_4) + 2H_5(n_5), \tag{16}$$

where $n = n_4 + n_5$.

Corollary 1

$$H_5(n) = 2^{m-2}(2n_4 - (m-2)^2 - m) + 8n_5 - 17, \tag{17}$$

*with* $n_4 < n_5 \leq 10$, $n = n_4 + n_5$ *and* $\frac{(m-1)m}{2} < n_4 \leq \frac{(m+1)m}{2}$.

The formula (17) allows to estimate the function $H_5(n)$ for case $11 \leq n \leq 24$.

We can obtain another nonrecursive formula for $H_5(n)$, which applies for $n \geq 11$.

We use (2) and (16) with the following assumptions on the splitting $n = n_4 + n_5$: we take $n_4(m)$ to be the triangular number $\frac{(m+1)m}{4}$ such that $n_4 < n_5$ and the difference $n_5 - n_4$ is minimal among all decompositions of $n$.

Using the Frame–Stewart algorithm and the formula (16) many times we obtain a next formula for calculation of the minimum number of moves needed to solve the 5-peg Tower of Hanoi problem:

$$\begin{aligned} H_5(n) = H_4(n_4(m)) + 2(H_4(n_4(m-1)) + 2(H_4(n_4(m-2)) \\ + \cdots + 2(H_4(n_4(1)) + 2H_5(1)\cdots)), \end{aligned}$$

where $n_4 = n_4(i)$ is a triangular number and $n = \sum_{i=1}^{m} n_4(i) + 1$.

The sum of triangular numbers is called a tetrahedral number.

From (14) we deduce for a tetrahedral number

$$\begin{aligned} H_5(n) &= (m-1)2^m + 1 + 2(m-2)2^{m-1} + 2 + 2^2(m-3)2^{m-2} + 4 \\ &\quad + \cdots + 2^{m-1}(0 \cdot 2 + 1 + 2 \cdot 1)\cdots) \\ &= 2^m((m-1) + (m-2) + \cdots + 1) + (1 + 2 + \cdots + 2^m) \\ &= 2^m\left(\frac{m(m-1)}{2}\right) + 2^{m+1} - 1 = 2^{m-1}m(m-1) + 4 \cdot 2^{m-1} - 1 \\ &= 2^{m-1}(m(m-1) + 4) - 1. \end{aligned} \tag{18}$$

Theorem 3

*If $n$ is a non-tetrahedral number, such that*

$$\sum_{i=1}^{m+1} n_4(i) \geq n > \sum_{i=1}^{m} n_4(i) + 1,$$

*then*

$$H_5(n) = \frac{2^{m-1}}{3}(6n - m^3 - 5m + 6) - 1. \tag{19}$$

*Proof.* From the formula (18) it follows that increasing the number of discs by 1 implies the increase of the number of moves ($H_5(n)$) by $2^m$. Then the number of moves needed for transporting $n$ discs, where $n > \sum_{i=1}^{m} n_4(i) + 1$ is equal to

$$H_5(n) = 2^{m-1}(m(m-1)+4) - 1 + 2^m(n - \sum_{i=1}^{m} n_4(i) - 1).$$

Since

$$\sum_{i=1}^{m} n_4(i) = \frac{(m+2)(m+1)m}{6}$$

we get

$$\begin{aligned} n - \frac{(m+2)(m+1)m}{6} - 1 &= \frac{(6n-(m+2)(m+1)m-6)}{6} \\ &= \frac{6n - m^3 - m^2 - 2m^2 - 2m - 6}{6}. \end{aligned}$$

Hence

$$\begin{aligned} H_5(n) &= 2^{m-1}(m(m-1)+4) - 1 + \frac{2^{m-1}}{3}(6n - m^3 - 3m^2 - 2m - 6) \\ &= \frac{2^{m-1}}{3}(6n - m^3 - 5m + 6) - 1. \end{aligned}$$

Remark 2
Our formula (19) allows to discover errors in results, which published in [2], where $H_5(11) - H_5(10) = 39 - 31 = 8$ and $H_5(n+1) - H_5(n) = 4$ for $n > 11$. It follows from (19), that $H_5(n+1) - H_5(n)$ cannot decrease with increasing the number of discs.

Using the Frame–Stewart algorithm scheme and the formulas (2), (6), and (19) we can obtain formulas for calculation of the minimum number of moves needed to solve the 6-peg Tower of Hanoi problem.

Corollary 2

$$H_6(n) = \frac{2^{m-1}}{3}(6n_5 - m^3 - 5m + 6) + 8n_6 - 23, \tag{20}$$

*where* $n_6 \leq 15$, $n = n_6 + n_5$ ,

$$\sum_{i=1}^{m+1} n_4(i) \geq n_5 > \sum_{i=1}^{m} n_4(i) + 1,$$

$$\frac{(m-1)m}{2} < n_4 \leq \frac{(m+1)m}{2}.$$

The formula (20) allows to estimate the function $H_6(n)$ for case $16 \leq n_6 \leq 33$.

## 6. Conclusion

As a conclusion we can observe with the help of our new formulas values of the function $H_k(n)$ for $n = 64$. Suppose that one move requires one second. Then it's known (E. Lucas) $H_3(n) = 2^{64} - 1$ and the puzzle takes more than 590 000 000 000 years. Then by (11) we have $H_4(64) = 18433$ (only five hours running time). We have by (19) $H_5(64) = 1535$ and $H_6(64) = 673$ and $H_7(64) = 479$ with the help of formulas (2), (6), (19) and (20) for $m = 4$, $n_4 = 10$, $n_5 = 22$, $n_6 = 21$. Next results are: $H_8(64) = 385$, $H_9(64) = 351$, $H_{10}(64) = 313$, $H_{11}(64) = 271$, which we obtain with help of formulas (2) and (6). We can easily calculate the values of $H_k(64)$ for $64 \geq k \geq 12$ with the help of one formula (6). At last $H_{65}(64) = 127$. It is obvious, that the number of moves required for solution of the puzzle in case $n = 64$, $k > 65$ stabilizes.

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