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Iterative roots of homeomorphisms possessing periodic points

Abstract. In this paper we give necessary and sufficient conditions for the existence of orientation-preserving iterative roots of a homeomorphism with a nonempty set of periodic points. We also give a construction method for these roots.

1. Introduction

The problem of the existence of *iterative roots* of a given function F, i.e., the solution of the following equation $G^m = F$, where $m \geq 2$ is an integer, has been considered for nearly two hundred years (see for example [1], [10], [12], [14], [15], [25]). There are also some results for some homeomorphisms of the unit circle S^1, e.g., homeomorphisms with an irrational rotation number (see [18], [24]), for the identity function (see [11]) and for some other homeomorphisms with a rational rotation number (see [16], [19], [20]). In particular, [16] relates the existence of an iterative root of F to the existence of an iterative root of $F_{|\operatorname{Per} F}$, where $\operatorname{Per} F := \{z \in S^1 \mid \exists k \in \mathbb{N} \ \ F^k(z) = z\}$. More precisely, an orientation-preserving homeomorphism $F\colon S^1 \longrightarrow S^1$ such that $F^n(z) = z$ for $z \in \operatorname{Per} F$, has an iterative root of order m if and only if there exists an iterative root $\psi\colon \operatorname{Per} F \longrightarrow \operatorname{Per} F$ of order m of $F_{|\operatorname{Per} F}$ such that

(i) ψ preserves orientation;

(ii) for any connected component $\overrightarrow{(u, v)}$ of $S^1 \setminus \operatorname{Per} F$, $\overrightarrow{(\psi(u), \psi(v))}$ and $\overrightarrow{(u, v)}$ are both increasing (or both decreasing) arcs of F^n.

Recall that an arc $\overrightarrow{(u, v)}$, where $u, v \in \operatorname{Per} F$ and $\overrightarrow{(u, v)} \cap \operatorname{Per} F = \emptyset$, is called increasing (resp. decreasing) arc of F^n if there is an $x \in \overrightarrow{(u, v)}$ such that $F^n(x) \in \overrightarrow{(x, v)}$ (resp. $F^n(x) \in \overrightarrow{(u, x)}$).

This paper answers the question when iterative roots of the function $F_{|\operatorname{Per} F}$ exist and generalizes results from [20]. For this purpose we apply the method

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which is used for the construction of the iterative roots of a homeomorphism with an irrational rotation number (i.e., the method that uses a solution of some Schröder equation, see [18]).

2. Preliminaries

We begin with recalling some definitions and notations. For any $u, w, z \in S^1$ there exist unique $t_1, t_2 \in \langle 0, 1)$ such that $we^{2\pi i t_1} = z$, $we^{2\pi i t_2} = u$. Define

$$u \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

(see [2]). Some properties of this relation can be found in [3], [4] and [5].

We say that a function $F\colon A \longrightarrow S^1$, where $A \subset S^1$, *preserves orientation* if for any $u, w, z \in A$ such that $u \prec w \prec z$ we have $F(u) \prec F(w) \prec F(z)$.

For every orientation-preserving homeomorphism $F\colon S^1 \longrightarrow S^1$ there exists a unique (up to translation by an integer) homeomorphism $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, called the *lift* of F, such that $F\left(e^{2\pi i x}\right) = e^{2\pi i f(x)}$ and $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$. Moreover, the limit

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} \,(\mathrm{mod}\ 1), \qquad x \in \mathbb{R}$$

always exists and does not depend on x and the choice of f. This number is called the *rotation number* of F (see [9]). It appears that a homeomorphism $F\colon S^1 \longrightarrow S^1$ preserves orientation if and only if f is a strictly increasing function (see for example [4]). Moreover, $\alpha(F)$ is a rational number if and only if $\operatorname{Per} F \neq \emptyset$ (see for example [9]).

Let us introduce a classification of orientation-preserving homeomorphisms. Namely, for $n \in \mathbb{N}$ and $q \in \{0, 1 \ldots, n-1\}$ such that $\gcd(q, n) = 1$ denote by $\mathcal{F}_{q,n}$ the set of all orientation-preserving homeomorphisms F of the circle with $\alpha(F) = \frac{q}{n}$. From now on writing $F \in \mathcal{F}_{q,n}$ without any additional assumptions on q and n, we mean that the numbers q and n are such that $n \in \mathbb{N}$, $q \in \{0, \ldots, n-1\}$ and $\gcd(q, n) = 1$.

Finally, for any distinct $u, z \in S^1$ put $\overrightarrow{(u, z)} := \{w \in S^1 \mid u \prec w \prec z\}$ (such a set is said to be an *open arc*) and $\overrightarrow{\langle u, z)} := \overrightarrow{(u, z)} \cup \{u\}$.

Remark 1
If $F \in \mathcal{F}_{q,n}$, then $\operatorname{Per} F = \{z \in S^1 \mid F^n(z) = z\}$ and n is the minimal number such that $F^n(z) = z$ for $z \in \operatorname{Per} F$. In fact, notice that $\alpha(F^n) = n\alpha(F)\,(\mathrm{mod}\ 1) = 0$. Therefore F^n has a fixed point (see [9], Ch. 3, § 3). The assertion follows from the fact that every two periodic points of an orientation-preserving homeomorphism have the same period (see for example [17], p. 16). Now suppose that $F^m(z) = z$ for an $m \in \{1, \ldots, n-1\}$ and a $z \in \operatorname{Per} F$. Then $m\frac{q}{n}\,(\mathrm{mod}\ 1) = 0$. Thus n divides m, a contradiction.

For any $F \in \mathcal{F}_{q,n}$ define the following set

$$\mathcal{M}_F := \{u \in \operatorname{Per} F \mid \exists w \in \operatorname{Per} F,\ w \neq u : \ \overrightarrow{(u,w)} \cap \operatorname{Per} F = \emptyset\}.$$

Such a set is F-*invariant* (i.e., $F(\mathcal{M}_F) = \mathcal{M}_F$). It may happen that $\mathcal{M}_F = \emptyset$ (if $\operatorname{Per} F = S^1$), $\mathcal{M}_F = \operatorname{Per} F$ (for example, if $\operatorname{Per} F$ is finite) or $\emptyset \subsetneq \mathcal{M}_F \subsetneq \operatorname{Per} F$ (for example, if $\operatorname{int}(\operatorname{Per} F) \neq \emptyset$). Moreover, if $\mathcal{M}_F \neq \emptyset$, then $S^1 \setminus \operatorname{Per} F \neq \emptyset$. Since $\operatorname{Per} F$ is closed, we have that $S^1 \setminus \operatorname{Per} F$ is a sum of pairwise disjoint open arcs. Denote the family of these arcs by $\mathcal{A}_F$. For every $\overrightarrow{(u,w)} \in \mathcal{A}_F$, where $u, w \in \operatorname{Per} F$, put $l\left(\overrightarrow{(u,w)}\right) := u$ and observe that l maps bijectively $\mathcal{A}_F$ onto $\mathcal{M}_F$. Setting $I_u := l^{-1}(u)$ for $u \in \mathcal{M}_F$ we have

$$S^1 \setminus \operatorname{Per} F = \bigcup_{u \in \mathcal{M}_F} I_u .$$

For the convenience of the reader we recall the relevant, slightly modified material from [21].

Proposition 1

Let $F \in \mathcal{F}_{q,n}$ be such that $\operatorname{Per} F \neq S^1$ *and let $I \in \mathcal{A}_F$. Then $\overrightarrow{(z, F^n(z))} \subset I$ for every $z \in I$ or $\overrightarrow{(F^n(z), z)} \subset I$ for every $z \in I$.*

Moreover, if $\overrightarrow{(z, F^n(z))} \subset I$ (resp. $\overrightarrow{(F^n(z), z)} \subset I$) for a $z \in I$, then $\overrightarrow{(z_1, F^n(z_1))} \subset F(I)$ (resp. $\overrightarrow{(F^n(z_1), z_1)} \subset F(I)$) for all $z_1 \in F(I)$.

We also recall a sketch of the proof. Assume $z \in I \in \mathcal{A}_F$. Then $F^n(z) \in I$ and $z \neq F^n(z)$. Therefore $\overrightarrow{(z, F^n(z))} \subset I$ or $\overrightarrow{(F^n(z), z)} \subset I$. Suppose that $\overrightarrow{(z, F^n(z))} \subset I$. Since F preserves orientation we have

$$\overrightarrow{(F^{ln}(z), F^{n(l+1)}(z))} \subset I \qquad \text{for all } l \in \mathbb{Z}.$$

Moreover, $\bigcup_{l\in\mathbb{Z}} \overrightarrow{\langle F^{ln}(z), F^{(l+1)n}(z))} = I$. Now fix $u \in I$. We may assume $u \neq F^{ln}(z)$ for $l \in \mathbb{Z}$. Then $u \in \overrightarrow{(F^{nj}(z), F^{n(j+1)}(z))}$ for some $j \in \mathbb{Z}$. Hence $F^n(u) \in \overrightarrow{(F^{n(j+1)}(z), F^{n(j+2)}(z))}$, as F preserves orientation. This gives $\overrightarrow{(u, F^n(u))} \subset I$.

For the second assertion suppose that $\overrightarrow{(z, F^n(z))} \subset I$ for an $z \in I$. Let $z_1 \in F(I)$ be fixed. Then there exists a $z_0 \in I$ such that $F(z_0) = z_1$ and $\overrightarrow{(z_0, F^n(z_0))} \subset I$. Hence $\overrightarrow{(z_1, F^n(z_1))} = F\left(\overrightarrow{(z_0, F^n(z_0))}\right) \subset F(I)$. This ends the sketch of the proof.

Now we present some results concerning the Schröder equation

$$\psi \circ F = s\psi, \tag{1}$$

where $s \in S^1$ and $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism with a rational rotation number. It is a known fact (see for example [9], [17] or [22]) that if F is a homeomorphism with an irrational rotation number and $s = e^{2\pi i \alpha(F)}$, then (1) has a continuous solution $\psi\colon S^1 \longrightarrow S^1$. If F is a homeomorphism with a rational rotation number and such that $\operatorname{card}(\operatorname{Per} F) \leq \aleph_0$, then the only continuous solutions of (1) are constant functions. Of course, in this case $s = 1$ (see Theorem 4.1 in [7]). On the other hand, it follows from Theorem 4.2 in [7] that, if F is an orientation-preserving homeomorphism such that $\operatorname{Per} F = S^1$ and $F \neq \mathrm{id}_{S^1}$, then there exists a constant $s \neq 1$ for which (1) has a homeomorphic and orientation-preserving solution $\psi\colon S^1 \longrightarrow S^1$. The following theorem generalizes the results from Theorem 4.2 in [7].

Theorem 1
Let $n > 1$ and $F \in \mathcal{F}_{q,n}$. There exists an orientation-preserving continuous mapping $\psi\colon \operatorname{Per} F \longrightarrow S^1$ such that

$$\psi(F(z)) = e^{2\pi i \alpha(F)}\psi(z), \qquad z \in \operatorname{Per} F. \tag{2}$$

The solution of (2) *depends on an arbitrary function.*

The proof of the above theorem is based on Theorem 4.2 from [7] and the following observation.

Lemma 1
For any $F \in \mathcal{F}_{q,n}$, where $n > 1$, with $\operatorname{Per} F \neq S^1$ there exist infinitely many homeomorphisms $\hat{F} \in \mathcal{F}_{q,n}$ such that $\operatorname{Per} \hat{F} = S^1$ and $\hat{F}(z) = F(z)$ for $z \in \operatorname{Per} F$.

Proof. Fix $F \in \mathcal{F}_{q,n}$ such that $\operatorname{Per} F \neq S^1$. Define the equivalence relation on $\mathcal{M}_F$:

$$p \sim q \iff \exists k \in \mathbb{Z} \;\; p = F^k(q).$$

By $E_\sim$ denote the set of class representatives. In other words, we decompose $\mathcal{M}_F$ onto cycles of F. Let $\phi_{p,k}\colon I_{F^k(p)} \longrightarrow I_{F^{k+1}(p)}$ for all $p \in E_\sim$ and $k \in \{0, \ldots, n-2\}$ be arbitrary orientation-preserving homeomorphisms. Put

$$\phi_{p,n-1}(z) := \phi_{p,0}^{-1} \circ \phi_{p,1}^{-1} \circ \ldots \circ \phi_{p,n-2}^{-1}(z), \qquad z \in I_{F^{n-1}(p)}. \tag{3}$$

It is easy to see that $\phi_{p,n-1}\colon I_{F^{n-1}(p)} \longrightarrow I_p$ for $p \in E_\sim$ are orientation-preserving homeomorphisms. Let $z \in S^1 \setminus \operatorname{Per} F$. There exist a unique $p \in E_\sim$ and $k \in \{0, \ldots, n-1\}$ such that $z \in I_{F^k(p)}$. Set

$$\phi(z) := \phi_{p,k}(z).$$

and observe that ϕ maps $S^1 \setminus \operatorname{Per} F$ onto $S^1 \setminus \operatorname{Per} F$ and

$$\phi^n(z) = \begin{cases} \phi_{p,n-1} \circ \ldots \circ \phi_{p,0}(z), & k = 0, \\ \phi_{p,k-1} \circ \ldots \circ \phi_{p,0} \circ \phi_{p,n-1} \circ \ldots \circ \phi_{p,k}(z), & k \neq 0. \end{cases}$$

This and (3) give $\phi^n(z) = z$ for $z \in S^1 \setminus \operatorname{Per} F$.

Now we show that ϕ preserves orientation. To do this, observe that for every $z \in I_p$, where $p \in \mathcal{M}_F$, we have $\phi(z) \in I_{F(p)}$. Fix $u, w, z \in S^1 \setminus \operatorname{Per} F$ such that $u \prec w \prec z$. Notice that if $\{u, w, z\} \subset I_p$ for a $p \in \mathcal{M}_F$, then the definition of ϕ gives $\phi(u) \prec \phi(w) \prec \phi(z)$. Now assume that there exist distinct $p, q \in \mathcal{M}_F$ such that exactly one element from the set $\{u, w, z\}$ belongs to I_p and the rest of them belong to I_q. In view of Lemma 2 in [4], it is sufficient to consider only the case: $\overrightarrow{(z, u)} \subset I_p$ and $w \in I_q$. Hence $\overrightarrow{(\phi(z), \phi(u))} \subset I_{F(p)}$ and $\phi(w) \in I_{F(q)}$. Since $I_{F(q)} \cap I_{F(p)} = \emptyset$, we have $\phi(u) \prec \phi(w) \prec \phi(z)$. Finally, let $\operatorname{card}(\mathcal{M}_F) \geq 3$ and let $u \in I_p, w \in I_q$ and $z \in I_t$, where $p, q, t \in \mathcal{M}_F$ are such that $p \neq q \neq t \neq p$. The arcs I_p, I_q and I_t are pairwise disjoint, so we have $p \prec q \prec t$. Hence $F(p) \prec F(q) \prec F(t)$. On the other hand, $\phi(u) \in I_{F(p)}$, $\phi(w) \in I_{F(q)}$ and $\phi(z) \in I_{F(t)}$. Thus $\phi(u) \prec \phi(w) \prec \phi(z)$, as $I_{F(p)}$, $I_{F(q)}$ and $I_{F(t)}$ are pairwise disjoint arcs.

Define the function $\hat{F} \colon S^1 \longrightarrow S^1$ as follows:

$$\hat{F}(z) := \begin{cases} F(z), & z \in \operatorname{Per} F, \\ \phi(z), & z \in S^1 \setminus \operatorname{Per} F. \end{cases}$$

Clearly, $\hat{F}$ is a surjection. To show that $\hat{F}$ is an orientation-preserving homeomorphism it is sufficient to prove that it preserves orientation. Similarly as above fix $u, w, z \in S^1$ such that $u \prec w \prec z$. By virtue of Lemma 2 in [4] it is enough to consider three cases:

(i) $\operatorname{card}(\operatorname{Per} F) \geq 3$ and $u, w, z \in \operatorname{Per} F$ or $u, w, z \in S^1 \setminus \operatorname{Per} F$ (this one is clear).

(ii) $u, z \in \operatorname{Per} F$ and $w \in S^1 \setminus \operatorname{Per} F$. There exists a $p \in \mathcal{M}_F \cap \overrightarrow{\langle u, z\rangle}$ such that $w \in I_p$ and $\hat{F}(w) = \phi(w) \in I_{F(p)}$. Thus $F(p) \in \overrightarrow{\langle F(u), F(z)\rangle}$. Consequently, $I_{F(p)} \subset \overrightarrow{(F(u), F(z))}$. Finally, $\hat{F}(u) \prec \hat{F}(w) \prec \hat{F}(z)$, as $\hat{F}_{|\operatorname{Per} F} = F$.

(iii) $u, z \in S^1 \setminus \operatorname{Per} F$ and $w \in \operatorname{Per} F$. In this case it may happen that $u, z \in I_p$ for a $p \in M_F$ or $u \in I_p$ and $z \in I_q$ for some $p, q \in \mathcal{M}_F$, $p \neq q$. Suppose that $u, z \in I_p$ for a $p \in M_F$. Then $\overrightarrow{(z, u)} \subset I_p$ and $w \notin I_p$. Hence $\overrightarrow{(\hat{F}(z), \hat{F}(u))} = \overrightarrow{(\phi(z), \phi(u))} \subset I_{F(p)}$ and $\hat{F}(w) = F(w) \notin I_{F(p)}$. Thus $\hat{F}(u) \prec \hat{F}(w) \prec \hat{F}(z)$. Now suppose that $u \in I_p$ and $z \in I_q$ for some $p, q \in \mathcal{M}_F, p \neq q$. Then $p \prec u \prec w$ and $w \prec z \prec p$. A similar reasoning to this in (ii) yields $\hat{F}(p) \prec \hat{F}(u) \prec \hat{F}(w)$ and $\hat{F}(w) \prec \hat{F}(z) \prec \hat{F}(p)$. Hence, by Lemma 1 in [3], we obtain $\hat{F}(u) \prec \hat{F}(w) \prec \hat{F}(z)$.

Finally, notice that $\hat{F}_{|O(z)} = F_{|O(z)}$, where $O(z) := \{z, F(z), \ldots, F^{n-1}(z)\}$ for $z \in \operatorname{Per} F$. Thus $\alpha(F) = \alpha(\hat{F})$. Consequently, $\hat{F} \in \mathcal{F}_{q,n}$, and the proof is completed.

Now we give the *proof of Theorem 1.* To do this fix $F \in \mathcal{F}_{q,n}$, where $n > 1$. Notice that if $\operatorname{Per} F = S^1$, then, in view of Theorem 4.2 in [7], there exist an orientation-preserving homeomorphism (depending on an arbitrary function) $\psi\colon S^1 \longrightarrow S^1$ and a $q' \in \{1, \ldots, n-1\}$ with $\gcd(q', n) = 1$ such that

$$\psi(F(z)) = e^{2\pi \mathrm{i} \frac{q'}{n}} \psi(z), \qquad z \in S^1.$$

The equality $\alpha(F) = \frac{q'}{n}$ follows from the fact that the homeomorphism ψ conjugates F and the rotation $R(z) = e^{2\pi \mathrm{i} \frac{q'}{n}} z$ and ψ is an orientation-preserving homeomorphism (see Theorem 1 in [8]). Henceforth assume that $\operatorname{Per} F \neq S^1$. Let $\hat{F}$ be an orientation-preserving homeomorphism, which exists by Lemma 1, and let $\hat{\psi}\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphic solution of

$$\hat{\psi}(\hat{F}(z)) = e^{2\pi \mathrm{i} \alpha(F)} \hat{\psi}(z), \qquad z \in S^1.$$

Put $\psi := \hat{\psi}_{|\operatorname{Per} F}$. Observe that $\psi\colon \operatorname{Per} F \longrightarrow S^1$ is the desired solution of (2).

Definition 1
Given $F \in \mathcal{F}_{q,n}$ put

$$\mathcal{M}_F^+ := \left\{ p \in \mathcal{M}_F \mid \overrightarrow{(z, F^n(z))} \subset I_p \text{ for } z \in I_p \right\}$$

and

$$\mathcal{M}_F^- := \left\{ p \in \mathcal{M}_F \mid \overrightarrow{(F^n(z), z)} \subset I_p \text{ for } z \in I_p \right\}.$$

Notice that $\mathcal{M}_F^+ \cap \mathcal{M}_F^- = \emptyset$. Indeed, if $p \in \mathcal{M}_F^+ \cap \mathcal{M}_F^-$, then for any $z \in I_p$ we would have $\overrightarrow{(F^n(z), z)} \subset I_p$ and $\overrightarrow{(z, F^n(z))} \subset I_p$. Hence $S^1 = I_p$, a contradiction.

Remark 2
From Proposition 1 we get $\mathcal{M}_F^+ \cup \mathcal{M}_F^- = \mathcal{M}_F$ and $F(\mathcal{M}_F^+) \subset \mathcal{M}_F^+$. This inclusion and the fact that $\mathcal{M}_F^+ \subset \operatorname{Per} F$ yield

$$\mathcal{M}_F^+ = F^{n-1}(F(\mathcal{M}_F^+)) \subset F(\mathcal{M}_F^+).$$

Thus for any $F \in \mathcal{F}_{q,n}$, we have $\mathcal{M}_F^+ \cup \mathcal{M}_F^- = \mathcal{M}_F$ and $F(\mathcal{M}_F^+) = \mathcal{M}_F^+$.

Since for all $F \in \mathcal{F}_{q,n}$ the sets $\operatorname{Per} F$, $\mathcal{M}_F$, $\mathcal{M}_F^+$ and $\mathcal{M}_F^-$ are invariant sets of F we have the following result.

Remark 3
Let $F \in \mathcal{F}_{q,n}$, $n > 1$, $\psi\colon \operatorname{Per} F \longrightarrow S^1$ be an orientation-preserving continuous solution of (2) and let $X \in \{\operatorname{Per} F, \mathcal{M}_F, \mathcal{M}_F^+, \mathcal{M}_F^-\}$. Then

$$\psi(X) = e^{2\pi \mathrm{i}\alpha(F)}\psi(X).$$

3. Main results

Here we give necessary and sufficient conditions for the existence of orientation-preserving continuous iterative roots of order $m > 2$ of a mapping $F \in \mathcal{F}_{q,n}$. Throughout this section we will assume that $n > 1$. We begin with the following observation.

Lemma 2
Let $m \geq 2$ be an integer and let $F \in \mathcal{F}_{q,n}$. Suppose that the equation

$$G^m(z) = F(z), \qquad z \in S^1 \tag{4}$$

has an orientation-preserving continuous solution. Then there are an orientation-preserving continuous solution of (2) *and a $j \in \{0, \dots, m-1\}$ such that*

$$e^{2\pi \mathrm{i}\frac{\alpha(F)+j}{m}}\psi(X) = \psi(X), \tag{5}$$

where $X \in \{\operatorname{Per} F, \mathcal{M}_F, \mathcal{M}_F^+, \mathcal{M}_F^-\}$.

Proof. Since G satisfies (4), we have $\alpha(F) = m\alpha(G) \pmod 1$. This yields $\frac{\alpha(F)+j}{m} = \alpha(G)$ for a $j \in \{0, \dots, m-1\}$. Theorem 1 implies the existence of an orientation-preserving continuous solution of the following equation

$$\psi(G(z)) = e^{2\pi \mathrm{i}\frac{\alpha(F)+j}{m}}\psi(z), \qquad z \in \operatorname{Per} G. \tag{6}$$

Thus

$$\psi(G^m(z)) = \psi(F(z)) = e^{2\pi \mathrm{i}\alpha(F)}\psi(z), \qquad z \in \operatorname{Per} G.$$

Hence and from the fact that $\operatorname{Per} F = \operatorname{Per} G$ implies $\mathcal{M}_F = \mathcal{M}_G$, we get that ψ is a solution of (2) satisfying (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F\}$. Moreover, $\alpha(G) = \frac{\alpha(F)+j}{m} = \frac{q'}{nl}$, where $q' := \frac{q+jn}{\gcd(q+jn,m)}$, $l := \frac{m}{\gcd(q+jn,m)}$ and $\gcd(q', nl) = 1$, so $G \in \mathcal{F}_{q'nl}$. Hence, if $\operatorname{Per} F \neq S^1$, then $p \in \mathcal{M}_G^+$ gives $\overrightarrow{(z, G^{nl}(z))} \subset I_p$ for every $z \in I_p$. Since

$$G^{kln}(z) \in I_p \quad \text{and} \quad \overrightarrow{(G^{kln}(z), G^{(k+1)nl}(z))} \subset I_p \qquad \text{for } k \in \mathbb{Z},$$

we have $\overrightarrow{(z, G^{nm}(z))} \subset I_p$. Consequently, $p \in \mathcal{M}_F^+$. Whence $\mathcal{M}_G^+ \subset \mathcal{M}_F^+$. Similarly, $\mathcal{M}_G^- \subset \mathcal{M}_F^-$, so $\mathcal{M}_F \setminus \mathcal{M}_F^- = \mathcal{M}_F^+ \subset \mathcal{M}_G^+ = \mathcal{M}_G \setminus \mathcal{M}_G^-$. Finally, $\mathcal{M}_F^+ = \mathcal{M}_G^+$ and $\mathcal{M}_F^- = \mathcal{M}_G^-$. In view of the above facts and Remark 3 equality (5) holds for $X \in \{\operatorname{Per} F, \mathcal{M}_F, \mathcal{M}_F^+, \mathcal{M}_F^-\}$.

COROLLARY 1
Let $F \in \mathcal{F}_{q,n}$. If $G: S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism satisfying (4) *for an integer $m \geq 2$, then $\mathcal{M}_F = \mathcal{M}_G$, $\mathcal{M}_F^+ = \mathcal{M}_G^+$ and $\mathcal{M}_F^- = \mathcal{M}_G^-$.*

Now suppose that $F \in \mathcal{F}_{q,n}$ is such that $\operatorname{Per} F \neq S^1$, $m > 1$ is an integer and $\psi: \operatorname{Per} F \longrightarrow S^1$ is an orientation-preserving continuous solution of (2) satisfying (5) for $X = \operatorname{Per} F$ and a $j \in \{0, \dots, m-1\}$. This fact yields equality (5) for $X = \mathcal{M}_F$. Indeed, put

$$h_\psi(z) := \psi^{-1}\left(e^{2\pi \mathrm{i} \frac{\alpha(F)+j}{m}} \psi(z)\right), \qquad z \in \operatorname{Per} F. \tag{7}$$

It is easy to see that $h_\psi: \operatorname{Per} F \longrightarrow \operatorname{Per} F$ is an orientation-preserving homeomorphism. Notice that $z \in \operatorname{Per} F \setminus \mathcal{M}_F \neq \emptyset$ if and only if there exist a $w \in \operatorname{Per} F \setminus \{z\}$ and $z_n \in \overrightarrow{(z,w)} \cap \operatorname{Per} F$ for $n \in \mathbb{N}$ such that $z_n \to z$ as $n \to \infty$. This is equivalent to $h_\psi^{-1}(z_n) \to h_\psi^{-1}(z)$ as $n \to \infty$ and $h_\psi^{-1}(z_n) \in \overrightarrow{(h_\psi^{-1}(z), h_\psi^{-1}(w))} \cap \operatorname{Per} F$, which gives $h_\psi^{-1}(z) \in \operatorname{Per} F \setminus \mathcal{M}_F$ or equivalently $z \in h_\psi(\operatorname{Per} F \setminus \mathcal{M}_F)$. Hence $h_\psi(\mathcal{M}_F) = \mathcal{M}_F$.

However, (5) with $X = \operatorname{Per} F$ does not imply (5) for $X \in \{\mathcal{M}_F^+, \mathcal{M}_F^-\}$. An example of a function $F \in \mathcal{F}_{1,2}$ such that $\operatorname{Per} F = \mathcal{M}_F = \{1, \mathrm{i}, -1, -\mathrm{i}\}$, $\mathcal{M}_F^+ = \{1, -1\}$ may be given. Put $\psi(z) = z$ for $z \in \operatorname{Per} F$. Then ψ is a solution of (2) satisfying (5) for $m = 2$, $j = 0$ and $X \in \{\operatorname{Per} F, \mathcal{M}_F\}$, but $e^{2\pi \mathrm{i} \frac{1}{4}} \mathcal{M}_F^+ \neq \mathcal{M}_F^+$. Therefore assume subsidiarily that (5) holds for $X = \mathcal{M}_F^+$ and introduce the equivalence relation ρ on $\mathcal{M}_F$:

$$(p,q) \in \rho \iff \exists k \in \mathbb{Z} \;\; q = H_\psi^k(p), \qquad p, q \in \mathcal{M}_F\,, \tag{8}$$

where $H_\psi := h_{\psi|\mathcal{M}_F}$ and h_ψ is given by (7). Let W_ρ be the set of class representatives of ρ.

Notice that (5) with $X = \mathcal{M}_F^+$ yields $[p]_\rho \subset \mathcal{M}_F^+$ or $[p]_\rho \subset \mathcal{M}_F^-$ for all $p \in W_\rho$.

DEFINITION 2
Let $F \in \mathcal{F}_{q,n}$ be such that $\operatorname{Per} F \neq S^1$, $m > 1$ be an integer, $\psi: \operatorname{Per} F \longrightarrow S^1$ be an orientation-preserving continuous solution of (2) satisfying (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and a $j \in \{0, \dots, m-1\}$ and let W_ρ be the set of class representatives of the relation ρ given by (8). Put

$$m' := \gcd(q + jn, m), \quad l := \frac{m}{m'} \quad \text{and} \quad n' := nl. \tag{9}$$

Let $(z_{p,k})_{k\in\mathbb{Z}}$ for $p \in W_\rho$ be sequences such that the points $z_{p,dn'+r} \in I_{H_\psi^r(p)}$ for $r \in \{0, \dots, l-1\}$ and $d \in \{0, \dots, m'-1\}$ are arbitrary fixed and such that

$$\begin{aligned} &H_\psi^r(p) \prec z_{p,r} \prec z_{p,n'+r} \prec \ldots \prec z_{p,(m'-1)n'+r} \prec F^n(z_{p,r}), \quad \text{if } p \in \mathcal{M}_F^+ \\ &\text{or} \\ &H_\psi^r(p) \prec F^n(z_{p,r}) \prec z_{p,(m'-1)n'+r} \prec \ldots \prec z_{p,n'+r} \prec z_{p,r}, \quad \text{if } p \in \mathcal{M}_F^- \end{aligned} \tag{10}$$

and the remaining points are given by

$$z_{p,k+m} := F(z_{p,k}), \qquad k \in \mathbb{Z},\ p \in W_\rho\,. \tag{11}$$

Now we show that the above sequences are well defined and we prove some of their properties.

Lemma 3
Under assumptions of Definition 2, for all $i \in \mathbb{Z}$ and $p \in W_\rho$ there exist unique $s \in \{0,\dots,m'-1\}$, $r' \in \{0,\dots,l-1\}$ and $k \in \mathbb{Z}$ such that $z_{p,i} = F^k(z_{p,sn'+r'})$. Moreover,

$$\{z_{p,dn'+r}\}_{d\in\mathbb{Z}} \subset I_{H^r_\psi(p)}, \qquad p \in W_\rho\,,\ r \in \{0,\dots,n'-1\}, \tag{12}$$

and for any $p \in W_\rho$, $[p]_\rho \subset \mathcal{M}_F^+$ (resp. $[p]_\rho \subset \mathcal{M}_F^-$) if and only if

$$z_{p,an'+r} \prec z_{p,bn'+r} \prec z_{p,cn'+r} \quad (\textit{resp. } z_{p,cn'+r} \prec z_{p,bn'+r} \prec z_{p,an'+r}) \tag{13}$$

for any $r \in \{0,\dots,n'-1\}$ and all $a,b,c \in \mathbb{Z}$ such that $a<b<c$.

Proof. Fix $p \in W_\rho$ and $i \in \mathbb{Z}$. Write $i = dn' + r$, where $d \in \mathbb{Z}$ and $r \in \{0,\dots,n'-1\}$. If $d \in \{0,\dots,m'-1\}$ and $r \in \{0,\dots,l-1\}$, then by Definition 2, $s=d$, $r'=r$, $k=0$ and obviously $z_{p,dn'+r} \in I_{H^r_\psi(p)}$.

Suppose that $d \in \mathbb{Z}\setminus\{0,\dots,m'-1\}$ and $r \in \{0,\dots,l-1\}$. Put $t = \left[\frac{d}{m'}\right]$ ($[x]$ denotes the integer part of x), $k = tn$, $s = d - tm'$ and $r' = r$. Notice that $d = tm' + s$, $s \in \{0,\dots,m'-1\}$ and by (11),

$$F^{tn}(z_{p,sn'+r}) = z_{p,sn'+r+mtn} = z_{p,(tm'+s)n'+r} = z_{p,dn'+r}\,. \tag{14}$$

Since $F^{tn}(I_u) = I_u$ for $u \in \mathcal{M}_F$ and $z_{p,sn'+r} \in I_{H^r_\psi(p)}$, by (14) we have $z_{p,dn'+r} \in I_{H^r_\psi(p)}$.

Finally assume that $d \in \mathbb{Z}$ and $r \in \{l,\dots,n'-1\}$. As $\gcd(q,n)=1$ and $m' = \gcd(q+jn,m)$ we have $\gcd(m',n)=1$. Hence there exists a unique $b \in \{1,\dots,n-1\}$ such that $m'b = 1 \,(\mathrm{mod}\ n)$. Set $a_r := \left[\frac{r}{l}\right]$, $r' = r - a_r l$ and $k_r := a_r b \,(\mathrm{mod}\ n)$. Thus $m'k_r = a_r \,(\mathrm{mod}\ n)$ which, in view of the fact that $r = a_r l + r'$, gives $mk_r + r' = r (\mathrm{mod}\ n')$ and, in consequence,

$$mk_r + r' = xn' + r \qquad \text{for some } x \in \mathbb{Z}. \tag{15}$$

This time put $t_r := \left[\frac{d-x}{m'}\right]$, $k = k_r + t_r n$ and $s = d - x - t_r m'$. Then

$$F^{k_r+t_rn}(z_{p,sn'+r'}) = z_{p,\left(d-\frac{k_rm+r'-r}{n'}\right)n'+r'+k_rm} = z_{p,dn'+r}\,.$$

Since $r' \in \{0,\dots,l-1\}$ and $d-x \in \mathbb{Z}$, we obtain $z_{p,(d-x)n'+r'} \in I_{H^{r'}_\psi(p)}$. To prove $z_{p,dn'+r} \in I_{H^r_\psi(p)}$ it is enough to show that $F^{k_r}(H^{r'}_\psi(p)) = H^r_\psi(p)$. Notice that from (7),

$$H^m_\psi(z) = \psi^{-1}(e^{2\pi i \frac{q}{n}}\psi(z)) = F(z), \qquad z \in \mathcal{M}_F\,.$$

This, (15) and the fact that $H^{xn'}_\psi(p) = p$ yield

$$F^{k_r}\big(H^{r'}_\psi(p)\big) = H^{mk_r+r'}_\psi(p) = H^{xn'+r}_\psi(p) = H^r_\psi(p).$$

The proof of the remaining part of the lemma runs in the same way as the proof of the second assertion of Lemma 7 in [20] (it is enough to take $H^r_\psi(p)$, r_1, and k_r instead of $a_{R_{N_F}(i+rk'q')}$, $R_l(r)$ and p_r, respectively).

Let $(z_{p,k})_{k\in\mathbb{Z}}$, where $p \in W_\rho$, be the family of sequences given by (10) and (11). Define the following families of arcs:

$$L_{p,k} := \begin{cases} \overrightarrow{\langle z_{p,k}, z_{p,k+n'}\rangle}, & p \in \mathcal{M}^+_F, \\ \overrightarrow{\langle z_{p,k+n'}, z_{p,k}\rangle}, & p \in \mathcal{M}^-_F \end{cases} \qquad \text{for } k \in \mathbb{Z},\ p \in W_\rho\,. \tag{16}$$

From Lemma 3 it follows that

$$F\left(L_{p,k}\right) = L_{p,k+m}, \qquad k \in \mathbb{Z},\ p \in W_\rho\,.$$

Lemma 4
Under assumptions of Definition 2 if for any $p \in W_\rho$ the sequences $(z_{p,k})_{k\in\mathbb{Z}}$ are given by (10) *and* (11) *and $\{L_{p,k}\}_{k\in\mathbb{Z}}$ are the families of arcs defined by* (16)*, then*

$$\bigcup_{d\in\mathbb{Z}} L_{p,dn'+r} = I_{H^r_\psi(p)}\,, \qquad r \in \{0,\ldots,n'-1\}. \tag{17}$$

Proof. Fix $r \in \{0,\ldots,n'-1\}$ and suppose that $p \in W_\rho \cap \mathcal{M}^+_F$. From (13) we have $z_{p,dn'+r} \in \overrightarrow{\langle z_{p,(d-1)n'+r}, z_{p,(d+1)n'+r}\rangle}$ for $d \in \mathbb{Z}$. Hence by (12) and (16),

$$L_{p,dn'+r} \subset \overrightarrow{\langle z_{p,(d-1)n'+r}, z_{p,(d+1)n'+r}\rangle} \subset I_{H^r_\psi(p)}\,, \qquad d \in \mathbb{Z}.$$

Thus

$$\bigcup_{d\in\mathbb{Z}} L_{p,dn'+r} \subset I_{H^r_\psi(p)}\,.$$

To prove the converse inclusion fix $z \in I_{H^r_\psi(p)}$. By Lemma 4 in [21] (see also Remark 3 in [20]) we have

$$I_{H^r_\psi(p)} = \bigcup_{k\in\mathbb{Z}} \overrightarrow{\langle F^{kn}(z_{p,r}), F^{(k+1)n}(z_{p,r})\rangle}\,.$$

Hence $z \in \overrightarrow{\langle F^{k_0 n}(z_{p,r}), F^{(k_0+1)n}(z_{p,r})\rangle}$ for a $k_0 \in \mathbb{Z}$. On the other hand, by (11) and (13),

$$\begin{aligned}\overrightarrow{\langle F^{k_0 n}(z_{p,r}), F^{(k_0+1)n}(z_{p,r})\rangle} &= \overrightarrow{\langle z_{p,k_0 nm+r}, z_{p,(k_0+1)nm+r}\rangle} \\ &= \bigcup_{s=0}^{m'} L_{p,k_0 nm+sn'+r} \\ &\subset \bigcup_{k\in\mathbb{Z}} L_{p,kn'+r}\,.\end{aligned}$$

This ends the proof.

Theorem 2

Let $F \in \mathcal{F}_{q,n}$ be such that $\operatorname{Per} F \neq S^1$, *$m \geq 2$ be an integer and let $\psi\colon \operatorname{Per} F \longrightarrow S^1$ be an orientation-preserving continuous solution of* (2) *satisfying* (5) *for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and a $j \in \{0,\dots,m-1\}$. Suppose that W_ρ is the selector of ρ given by* (8), *$(z_{p,k})_{k\in\mathbb{Z}}$ for $p \in W_\rho$ are the families of sequences given by* (10) *and* (11) *and $\{L_{p,k}\}_{k\in\mathbb{Z}}$ for $p \in W_\rho$ are the families of arcs defined by* (16). *If $G_{p,k}\colon L_{p,k} \longrightarrow L_{p,k+1}$ for $k \in \{0,1,\dots,m-2\}$ and $p \in W_\rho$ are orientation-preserving surjections, then there exists a unique orientation-preserving homeomorphism $G : S^1 \longrightarrow S^1$ satisfying* (4) *and such that*

$$G_{|L_{p,k}} = G_{p,k} \qquad \textit{for } p \in W_\rho \textit{ and } k \in \{0,1,\dots,m-2\}.$$

Moreover, $\alpha(G) = \frac{q+jn}{nm}$.

Proof. Some parts of the proof of this theorem are similar to the proof of Theorem 5 from [20]. Here we give only the sketch of these parts. For the details we refer the reader to [20]. Fix $p \in W_\rho$ and orientation-preserving surjections $G_{p,k}\colon L_{p,k} \longrightarrow L_{p,k+1}$ for $k \in \{0,1,\dots,m-2\}$. Put

$$G_{p,m-1} := F \circ G_{p,0}^{-1} \circ G_{p,1}^{-1} \circ \ldots \circ G_{p,m-2}^{-1}\,. \tag{18}$$

For the remaining integers k there exist unique $d \in \mathbb{Z} \setminus \{0\}$ and an $r \in \{0,1,\dots,m-1\}$ such that $k = md + r$. For such k's define

$$G_{p,k} = G_{p,md+r} := F^d \circ G_{p,r} \circ F^{-d}_{|L_{p,k}}. \tag{19}$$

It might be shown that $G_{p,k}(L_{p,k}) = L_{p,k+1}$ for $k \in \mathbb{Z}$ and $G_{p,k}\colon L_{p,k} \longrightarrow L_{p,k+1}$ for $k \in \mathbb{Z}$ are orientation-preserving surjections.

Now fix $z \in S^1 \setminus \operatorname{Per} F$. There exist a $p \in W_\rho$ and an $r \in \{0,\dots,n'-1\}$, where n' is determined by (9), such that $z \in I_{H_\psi^r(p)}$. By (17), $z \in L_{p,dn'+r}$ for some $d \in \mathbb{Z}$. Notice that such a d is unique. Indeed, the assumption

$L_{p,cn'+r} \cap L_{p,dn'+r} \neq \emptyset$ for some $c, d \in \mathbb{Z}$, $c \neq d$, contradicts (13). Define a function $\widetilde{G}: S^1 \setminus \operatorname{Per} F \longrightarrow S^1 \setminus \operatorname{Per} F$ as follows:

$$\widetilde{G}(z) := G_{p,dn'+r}(z), \quad z \in L_{p,dn'+r}\,,\ p \in W_\rho\,,\ d \in \mathbb{Z},\ r \in \{0, \ldots, n'-1\}. \tag{20}$$

Notice that for every $u \in \mathcal{M}_F$ there exist unique $p \in W_\rho$ and $r \in \{0, \ldots, n'-1\}$ such that $u = H_\psi^r(p)$. Therefore by (20), (17) and the properties of $G_{p,k}$ we have

$$\begin{aligned}\widetilde{G}(I_u) &= \widetilde{G}\big(I_{H_\psi^r(p)}\big) = \widetilde{G}\left(\bigcup_{d\in\mathbb{Z}} L_{p,dn'+r}\right) = \bigcup_{d\in\mathbb{Z}} L_{p,dn'+r+1} = I_{H_\psi^{r+1}(p)}\\ &= I_{H_\psi(u)}\end{aligned}$$

(if $r+1 = n'$ we use the equality $H_\psi^{n'}(p) = p$).

It is easy to see that $\widetilde{G}: S^1 \setminus \operatorname{Per} F \longrightarrow S^1 \setminus \operatorname{Per} F$ is a surjection. By induction it can be proved that $\widetilde{G}$ satisfies

$$\widetilde{G}^m(z) = F(z), \qquad z \in S^1 \setminus \operatorname{Per} F. \tag{21}$$

Moreover, using the same method as in the proof of Theorem 5 in [20] (the proof of 1^o) it can be shown that $\widetilde{G}$ preserves orientation on every I_p for $p \in \mathcal{M}_F$.

We are now in a position to define the solution of (4). Namely, put

$$G(z) = \begin{cases} \widetilde{G}(z), & z \in S^1 \setminus \operatorname{Per} F,\\ h_\psi(z), & z \in \operatorname{Per} F,\end{cases} \tag{22}$$

where h_ψ is defined by (7). It is easy to see that G maps S^1 onto itself. Furthermore, setting $F = h_\psi$ and $\phi = \widetilde{G}$ and repeating the same argument as in the proof of Lemma 1 (i.e., the proof of the fact that $\hat{F}$ preserves orientation) one can obtain that G preserves orientation. Since S^1 is a closed set, it follows that G is an orientation-preserving homeomorphism. Moreover, (7) and (21) imply that G satisfies (4).

It remains to show that $\alpha(G) = \frac{q+jn}{nm}$. From Lemma 1 there exists an orientation-preserving homeomorphism $\hat{G}$ such that $\alpha(\hat{G}) = \alpha(G)$, $\hat{G}(z) = G(z)$ for $z \in \operatorname{Per} F = \operatorname{Per} G$ and $\operatorname{Per} \hat{G} = S^1$. From Theorem 4.2 in [7] it follows that $\hat{G}$ is conjugated to a rotation. On the other hand, by (22), $\hat{G}(z) = h_\psi(z)$ for $z \in \operatorname{Per} F$. By (7) we get that $\hat{G}$ is conjugated to $R(z) = e^{2\pi \mathrm{i} \frac{q+jn}{mn}} z$, $z \in S^1$. Hence $\alpha(\hat{G}) = \frac{q+jn}{mn}$ (see Theorem 1 in [8]), and the assertion follows.

Remark 4

Suppose that $F \in \mathcal{F}_{q,n}$ is such that $\operatorname{Per} F \neq S^1$. Then every continuous and orientation-preserving solution G of (4) with $\alpha(G) = \frac{\alpha(F)+jn}{mn}$, where $j \in$

$\{0, \dots, m-1\}$, may be obtained by the method described in the proof of Theorem 2. Indeed, suppose that $G\colon S^1 \longrightarrow S^1$ is a solution of (4) for an integer $m \geq 2$. Then $\alpha(G) = \frac{\alpha(F)+jn}{mn}$ for a $j \in \{0, \dots, m-1\}$. Furthermore, by (4), $\operatorname{Per} F = \operatorname{Per} G$, $\mathcal{A}_F = \mathcal{A}_G$ and, by Corollary 1, $\mathcal{M}_F = \mathcal{M}_G$, $\mathcal{M}_F^+ = \mathcal{M}_G^+$ and $\mathcal{M}_F^- = \mathcal{M}_G^-$. Lemma 2 implies that there exists an orientation-preserving continuous mapping $\psi\colon \operatorname{Per} F \longrightarrow S^1$ satisfying (6). Put $h_\psi := G_{|\operatorname{Per} G}$ and $H_\psi := G_{|\mathcal{M}_G}$. By (6), h_ψ satisfies (7) and $H_\psi = h_{\psi|\mathcal{M}_G}$. Notice that

$$G(I_p) = I_{G(p)} = I_{H_\psi(p)}\,, \qquad p \in \mathcal{M}_G\,. \tag{23}$$

Let ρ be the relation on $\mathcal{M}_G = \mathcal{M}_F$ given by (8) and let W_ρ be its selector. Fix $p \in W_\rho$, $z_{p,0} \in I_p$ and put

$$z_{p,k} := G^k(z_{p,0}), \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{24}$$

Obviously, $(z_{p,k})_{k\in\mathbb{Z}}$ satisfies (11). Moreover, (23) and the fact that $H^{n'} = \mathrm{id}_{\mathcal{M}_F}$, where n' is given in (9), yield

$$\begin{aligned} z_{p,dn'+r} = G^{dn'+r}(z_{p,0}) \in I_{H_\psi^{dn'+r}(p)} &= I_{H_\psi^r(p)}, \\ d \in \mathbb{Z},\ r &\in \{0, \dots, n'-1\}. \end{aligned} \tag{25}$$

By Definition 1, since n' is the minimal number such that $G^{n'}(z) = z$ for $z \in \operatorname{Per} G$ and $\mathcal{M}_F^+ = \mathcal{M}_G^+$, we have $\overrightarrow{\langle z_{p,0}, z_{p,n'}\rangle} \subset I_p$, if $p \in \mathcal{M}_G^+$ and $\overrightarrow{\langle z_{p,n'}, z_{p,0}\rangle} \subset I_p$, if $p \in \mathcal{M}_G^-$. Hence in view of (24), (25) and the fact that G preserves orientation we get

$$\overrightarrow{\langle z_{p,(d+1)n'+r}, z_{p,dn'+r}\rangle} \subset I_{H_\psi^r(p)}\,, \qquad (\text{resp. } \overrightarrow{\langle z_{p,dn'+r}, z_{p,(d+1)n'+r}\rangle} \subset I_{H_\psi^r(p)})$$

for $d \in \mathbb{Z}$, $r \in \{0, \dots, n'-1\}$ and $p \in \mathcal{M}_G^+$ (resp. $p \in \mathcal{M}_G^-$). Consequently,

$$H_\psi^r(p) \prec z_{p,r} \prec z_{p,n'+r} \prec \ldots \prec z_{p,(m'-1)n'+r} \prec G^{m'n'}(z_{p,r}) = F^n(z_{p,r})$$

$$(\text{resp. } H_\psi^r(p) \prec F^n(z_{p,r}) = G^{m'n'}(z_{p,r}) \prec z_{p,(m'-1)n'+r} \prec \ldots \prec z_{p,n'+r} \prec z_{p,r}).$$

Let $\{L_{p,k}\}_{k\in\mathbb{Z}}$ be defined by (16). Notice that

$$G(L_{p,k}) = L_{p,k+1}\,, \qquad k \in \mathbb{Z}. \tag{26}$$

Now put

$$G_{p,k} := G_{|L_{p,k}}\,, \qquad p \in W_\rho\,,\ k \in \mathbb{Z}. \tag{27}$$

From (4), (26) and (27) we have

$$F_{|L_{p,0}} = G_{p,m-1} \circ G_{p,m-2} \circ \ldots \circ G_{p,1} \circ G_{p,0}\,, \qquad p \in W_\rho\,,$$

thus (18) holds. Furthermore, (4) implies $G \circ F = F \circ G$. Thus $G \circ F^k = F^k \circ G$ for any $k \in \mathbb{Z}$. From this, (26) and (27) we get (19).

Theorem 2 and Remark 4 solve the problem of the existence of iterative roots of homeomorphisms having the set of periodic points different from the whole circle. Notice that if $F \in \mathcal{F}_{q,n}$ is such that $\operatorname{Per} F = S^1$, then taking $G := h_\psi$, where h_ψ is defined by (7), we get $G^m = F$. To sum up, we have obtained the following result.

Theorem 3
Let $m \geq 2$ be an integer and let $F \in \mathcal{F}_{q,n}$. Equation (4) has orientation-preserving and continuous solution if and only if an orientation-preserving continuous solution $\psi\colon \operatorname{Per} F \longrightarrow S^1$ of (2) satisfies (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and for a $j \in \{0, \dots, m-1\}$. Moreover, if $\operatorname{Per} F \neq S^1$, then for all ψ and j satisfying (5) for $X \in \{\operatorname{Per} F, M_F^+\}$ there exist infinitely many solutions of (4).

The following remark results from the above theorem. It answers the question of the existence of the iterative roots of the mapping $F_{|\operatorname{Per} F}$, where $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism having periodic points.

Remark 5
Let $m \geq 2$ be an integer and let $F \in \mathcal{F}_{q,n}$. The mapping $F_{|\operatorname{Per} F}\colon \operatorname{Per} F \longrightarrow \operatorname{Per} F$ has continuous and orientation-preserving iterative roots of order m if and only if some orientation-preserving continuous solution $\psi\colon \operatorname{Per} F \longrightarrow S^1$ of (2) satisfies

$$e^{2\pi \mathrm{i} \frac{\alpha(F)+j}{m}} \psi(\operatorname{Per} F) = \psi(\operatorname{Per} F)$$

for some $j \in \{0, \dots, m-1\}$.

We conclude with an observation concerning homeomorphisms with a finite and non-empty set of periodic points.

Theorem 4
Suppose that $F \in \mathcal{F}_{q,n}$ is such that $1 < \operatorname{card}(\operatorname{Per} F) =: N_F < \infty$ and $m \geq 2$ is an integer. Let moreover ψ_1 and ψ_2 be orientation-preserving continuous solutions of (2) satisfying (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and a $j \in \{0, \dots, m-1\}$ and let $h_{\psi_1}, h_{\psi_2}\colon \operatorname{Per} F \longrightarrow \operatorname{Per} F$ be defined by (7). Then $h_{\psi_1}(z) = h_{\psi_2}(z)$ for $z \in \operatorname{Per} F$.

In the proof of Theorem 4 we will use the following proposition, which is a slightly modified Theorem 3 from [21] (see also Theorem 2 in [20]).

Proposition 2
Suppose that $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism such that $1 < \operatorname{card}(\operatorname{Per} F) =: N_F < \infty$. Let $z_0 \in \operatorname{Per} F$ be an arbitrary element and let $z_1, \dots, z_{N_F-1} \in \operatorname{Per} F$ satisfy the following condition:

$$\operatorname{Arg}\frac{z_p}{z_0} < \operatorname{Arg}\frac{z_{p+1}}{z_0}, \qquad p \in \{0, \dots, N_F - 2\}.$$

Then $\alpha(F) = \frac{q}{n}$, *where* $0 \leq q < n$ *and* $\gcd(q,n) = 1$, *if and only if*

$$F(z_p) = z_{(p+k_F q) \pmod{N_F}}, \qquad p \in \{0, \dots, N_F - 1\},$$

where $k_F := \frac{N_F}{n}$.

Proof of Theorem 4. In view of Theorem 2 there exist orientation-preserving homeomorphisms G_1 and G_2 such that $\operatorname{Per} G_i = \operatorname{Per} F$, $G_i^m = F$ and $\alpha(G_i) = \frac{q+jn}{mn} = \frac{q'}{n'}$ for $i \in \{1,2\}$, where $q' := \frac{q+jn}{m'}$ and m', n' are given in (9). Moreover, $G_i(z) = h_{\psi_i}(z)$ for $z \in \operatorname{Per} F$ and $i \in \{1,2\}$. Let $z_0, \dots, z_{N_F-1} \in \operatorname{Per} F$ be defined as in Proposition 2 and let $K := \frac{N_F}{n'} = k_{G_1} = k_{G_2}$. By Proposition 2 we have

$$\begin{aligned} h_{\psi_1}(z_p) = G_1(z_p) = z_{(p+Kq') \pmod{N_F}} = G_2(z_p) \\ = h_{\psi_2}(z_p) \end{aligned}$$

for every $p \in \{0, \dots, N_F - 1\}$. Thus the assertion follows.

The property described in Theorem 4 does not have to occur for homeomorphisms with infinitely many periodic points. For example, let $F(z) = e^{\pi \mathrm{i}} z$ for $z \in S^1$ and let $m = 2$. Then $F \in \mathcal{F}_{1,2}$, $\mathcal{M}_F^+ = \emptyset$ and $\operatorname{Per} F = S^1$. Put $\psi_1(z) = z$ for $z \in S^1$ and $\psi_2(e^{2\pi \mathrm{i} x}) = e^{2\pi \mathrm{i} d(x)}$ for $x \in \langle 0, 1)$, where

$$d(x) = \begin{cases} -2x^2 + 2x, & x \in \langle 0, \frac{1}{2}), \\ -2\left(x - \frac{1}{2}\right)^2 + 2\left(x - \frac{1}{2}\right) + \frac{1}{2}, & x \in \langle \frac{1}{2}, 1). \end{cases}$$

Notice that ψ_1 and ψ_2 satisfy (2) and (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and $j = 0$, but $h_{\psi_1} \neq h_{\psi_2}$.

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