Annales Academiae Paedagogicae Cracoviensis
Folia 45 Studia Mathematica VI

Wioletta Syzdek

Submaximal Riemann-Roch expected curves and symplectic packing

Abstract. We study Riemann–Roch expected curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of the Nagata–Biran conjecture. This conjecture predicts that for a sufficiently large number of points multiple points Seshadri constants of an ample line bundle on algebraic surface are maximal. Biran gives an effective lower bound N_0 . We construct examples verifying to the effect that the assertions of the Nagata–Biran conjecture can not hold for small number of points. We discuss cases where our construction fails. We observe also that there exists a strong relation between Riemann–Roch expected curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and the symplectic packing problem. Biran relates the packing problem to the existence of solutions of certain Diophantine equations. We construct such solutions for any ample line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and a relatively small number of points. The solutions geometrically correspond to Riemann–Roch expected curves. Finally we discuss in how far the Biran number N_0 is optimal in the case of $\mathbb{P}^1 \times \mathbb{P}^1$. In fact, we conjecture that it can be replaced by a lower number and we provide an evidence justifying this conjecture.

Introduction

The aim of this paper is to prove that for the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization there exists a constant $R_0 = R_0(a, b)$ such that for $r \geq R_0$ there are no Riemann–Roch expected submaximal curves through r general points (Theorem 3.5).

This fact has consequences for the symplectic packing problem which is strongly connected to the existence of Riemann–Roch expected submaximal curves. More precisely, Biran relates the packing problem to the existence of solutions of certain Diophantine equations ([Bi1] Theorem 6.1.A 2) but the solutions geometrically correspond exactly to Riemann–Roch expected submaximal curves. In particular, Theorem 3.5 implies that for $N \geq R_0$ the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with the polarization (a, b) admits full symplectic packing by N equal

AMS (2000) Subject Classification: 14C20, 14J26.

balls (Theorem 3.18). This improves the result of Biran [[Bi2] Theorem 1.A] as in almost all cases our number R_0 is smaller than Biran's bound N_0. We conjecture that the Biran number N_0 in the Nagata–Biran Conjecture 1.2 can be replaced by R_0.

On the other hand, to complete Theorem 3.5, we are looking for Riemann–Roch expected submaximal curves for $r \leq R_0$ points in general position. We observe that for $r \leq 2 \cdot \lfloor \frac{a}{b} \rfloor + 5$ points we can write down such curves (Proposition 3.7). The cases $r \geq 2 \cdot \lfloor \frac{a}{b} \rfloor + 6$ are more complicated. Only for some polarizations (a, b) we can find Riemann–Roch expected submaximal curves. More precisely, to find them, we construct first a sequence of Riemann–Roch expected curves (Proposition 3.9) and next we compute their submaximality areas, i.e., we estimate polarizations for which curves are submaximal (Lemma 3.10). For $r = 2 \cdot \lfloor \frac{a}{b} \rfloor + 6$ in Proposition 3.11 we give an algorithm producing Riemann-Roch expected submaximal curves. For the number of points r in the range between $2 \cdot \lfloor \frac{a}{b} \rfloor + 6$ and R_0, the situation seems to be hard to control (see Examples 3.13 and 3.14). For such r's for which we found Riemann–Roch expected submaximal curves, we compute Seshadri quotients. If we know that a curve is irreducible, then this quotient is already the Seshadri constant, if not (i.e., a curve can be reducible) then we found only an upper-bound for this constant (Theorem 3.17).

NOTATION: The symbol $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers. For a given real number x we denote by $\lfloor x \rfloor$ its round-down. We work throughout over the field $\mathbb{C}$ of complex numbers. By a polarized variety we mean a pair (X, L) consisting a smooth variety X and an ample line bundle L on it. For a coherent sheaf $\mathcal{F}$ on X we write by $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$ and $h^i(\mathcal{F}) = \dim_{\mathbb{C}} H^i(\mathcal{F})$.

1. Seshadri constants and the Nagata--Biran conjecture

The concept of Seshadri constant was introduced by Demailly in [De]. He associated a real number $\varepsilon(L; x)$ to an ample line bundle L and a point x of an algebraic variety. This number in effect measures how much of positivity of L is concetrated in x. In general, Seshadri constants are very hard to control and their exact value is known only in few examples. In this paper we use a generalized definition of Seshadri constant (see also [Xu]) on a surface X.

DEFINITION 1.1
Let L be a nef line bundle on a smooth projective surface X. The Seshadri constant of L at $x_1, \ldots, x_r \in X$ is the real number

$$\varepsilon(L; x_1, \ldots, x_r) = \inf_{D \cap \{x_1, \ldots, x_r\} \neq \emptyset} \frac{L.D}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} D} ,$$

where the infimum is taken over all reduced and irreducible curves D passing through at least one of the points $x_1, \ldots, x_r$ and $\operatorname{mult} x_i D$ is the multiplicity of the curve D at x_i.

It follows from Kleiman's nefness criterion that $\varepsilon(L; x_1, \ldots, x_r) \leq \sqrt{\frac{L^2}{r}}$. If the value of $\varepsilon(L; x_1, \ldots, x_r)$ is less than the upper bound, then we say that the Seshadri constant of L at $x_1, \ldots, x_r$ is *submaximal*. If $\varepsilon(L; x_1, \ldots, x_r) = \frac{L.D}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} D}$, then we say that the curve D *computes* the Seshadri constant and we call such a curve a *Seshadri curve*. By the *Seshadri quotient* of a curve G at $x_1, \ldots, x_r$ we mean $\frac{L.G}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} G}$.

As a function on X^r the Seshadri constant $\varepsilon(L; \cdot, \ldots, \cdot)$ is semi-continuous and it attains the greatest value at a very general point of X^r (i.e., on the complement of a union of at most countably many Zariski closed subsets). For more details see [Og]. We denote by $\varepsilon(L; r)$ this greatest value. It is conjectured that for r sufficiently large $\varepsilon(L; r)$ has the maximal possible value which is $\varepsilon_{max}(L; r) = \sqrt{\frac{L^2}{r}}$. In fact there are effective predictions of what the least number r should be.

Nagata-Biran Conjecture 1.2
Let (X, L) be a polarized surface. Let k_0 be the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve and let $N_0 := k_0^2 L^2$. With the above assumptions

$$\varepsilon(L; x_1, \ldots, x_r) = \sqrt{\frac{L^2}{r}}$$

for general $x_1, \ldots, x_r \in X$ and $r \geq N_0$.

We should note that in the case when the Seshadri constant $\varepsilon(L; x_1, \ldots, x_r)$ is submaximal, the number of Seshadri curves is bounded.

Proposition 1.3
Let (X, L) be a polarized surface with the Picard number ϱ and let $x_1, \ldots, x_r$ be points in X such that $\varepsilon = \varepsilon(L; x_1, \ldots, x_r)$ is submaximal. There are at most $\varrho + r - 1$ irreducible and reduced Seshadri curves.

Proof. Let $\pi: Y \longrightarrow X$ be the blowing up of X at $x_1, \ldots, x_r$ with exceptional divisors $E_1, \ldots, E_r$ and let $H := \pi^* L$. Suppose that $C_1, \ldots, C_s$ are irreducible and reduced curves computing ε and $\widetilde{C_1}, \ldots, \widetilde{C_s}$ are their proper transforms. The $\mathbb{Q}$-divisor $M := H - \varepsilon \sum_{i=1}^{r} E_i$ is nef and big and for arbitrary $\lambda_i \geq 0$ we have

$$
\begin{aligned}
&M.\Big(\sum_{i=1}^{s}\lambda_i\widetilde{C_i}\Big)\\
&\quad=\sum_{i=1}^{s}\lambda_i\cdot\big(M.\widetilde{C_i}\big)=\sum_{i=1}^{s}\lambda_i\Big(\pi^*L.\widetilde{C_i}-\varepsilon\sum_{j=1}^{r}E_j.\widetilde{C_i}\Big)\\
&\quad=\sum_{i=1}^{s}\lambda_i\Big(\pi^*L.\pi^*C_i-\sum_{k=1}^{r}\operatorname{mult}_{x_k}C_i\cdot(\pi^*L.E_k)-\varepsilon\sum_{j=1}^{r}\operatorname{mult}_{x_j}C_i\Big)\\
&\quad=\sum_{i=1}^{s}\lambda_i\left(L.C_i-0-L.C_i\right)=0.
\end{aligned}
$$

The Hodge Index Theorem implies that the intersection matrix of $\widetilde{C_1},\ldots,\widetilde{C_s}$ is negative definite. Since $\varrho(Y)=\varrho+r$, it must be $s\leq\varrho+r-1$.

We also observe that this upper bound is optimal.

Example 1.4
Let $X=\mathbb{P}^2$ and the number of points be $r=7$. In this case $\varrho=1$ and from the previous proposition we have that the number of irreducible and reduced Seshadri curves is at most 7. On the other hand for any $i\in\{1,\ldots,7\}$ there exists an irreducible cubic D_i with $\operatorname{mult}_{x_i}D_i=2$ and $\operatorname{mult}_{x_j}D_i=1$ for $j\neq i$. Every D_i computes the Seshadri constant $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1);x_1,\ldots,x_7)=\frac{3}{8}<\sqrt{\frac{1}{7}}$. So there are exactly 7 submaximal curves in this case.

Let D be a curve on a surface X passing through $x_1,\ldots,x_r$ with multiplicities $m_1:=\operatorname{mult}_{x_1}D,\ldots,m_r:=\operatorname{mult}_{x_r}D$ respectively. To the curve D we assign its *multiplicity vector* $M_D:=(m_1,\ldots,m_r)\in\mathbb{Z}^r$.

Definition 1.5
A curve D is *almost-homogeneous* if all but at most one of the coordinates of its multiplicity vector M_D are equal. In this case we can also say that the multiplicity vector is almost-homogeneous.

Now using the same arguments as in [Sz] Corollary 4.6, after some elementary calculations we can prove the following

Proposition 1.6
Let (X,L) be a polarized surface with the Picard number ϱ and let $x_1,\ldots,x_r$ be general points on X. If ϱ is equal one or two and the Seshadri constant $\varepsilon(L;x_1,\ldots,x_r)$ is submaximal, then any irreducible and reduced Seshadri curve is almost-homogeneous.

Proof. A submaximal Seshadri constant $\varepsilon(L; x_1, \ldots, x_r)$ implies by the real valued Nakai–Moishezon criterion [CP] that there exists a computing curve.

Let D be an irreducible and reduced Seshadri curve with the multiplicity vector $M_D = (m_1, \ldots, m_r)$. Since the points are general, the monodromy group acts as the full symmetric group S_r , i.e., for $\sigma \in S_r$ there exists a curve D_σ with the multiplicity vector $M_{D_\sigma} = (m_{\sigma(1)}, \ldots, m_{\sigma(r)})$ which is also an irreducible Seshadri curve.

If the curve were not be almost-homogeneous, then it is easy to check that we would get too many Seshadri curves contradicting Proposition 1.3.

2. Seshadri submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$

By the (a, b) *polarization* or by a *curve of type* (a, b) in the product $\mathbb{P}^1 \times \mathbb{P}^1$ we mean a curve of bidegree a, b.

Definition 2.7
Let $D \subset X$ be a curve passing through points $x_1, \ldots, x_r$ with multiplicities $m_1, \ldots, m_r$, respectively. We say that D is Riemann–Roch expected (for short R-R expected) if

$$h^0(\mathcal{O}_X(D)) - \sum_{i=1}^{r} \binom{m_i + 1}{2} > 0.$$

This simply means that a curve D is R-R expected if its existence follows from the naive dimension count (note that it takes at most $\binom{m+1}{2}$ independent linear conditions on a linear system to pass through a given point with multiplicity at least m).

Remark 2.8

(1) On $(\mathbb{P}^2, \mathcal{O}(1))$ we have $N_0 = 9$ and curves computing Seshadri constants for $r \leq N_0$ points are R-R expected.

(2) On $\mathbb{P}^1 \times \mathbb{P}^1$ with the $(1, 1)$ polarization, we have $N_0 = 8$ and again all curves computing Seshadri constant for at most 8 points are R-R expected.

This implies that in these two examples the number N_0 suggested by Biran cannot be lowered. However, there are cases (e.g. $(1, 2)$ polarization, see [S1]) suggesting that the Biran number N_0 might not be optimal even in the simple case of $\mathbb{P}^1 \times \mathbb{P}^1$. We address this question in this article. Before proceeding, we need some more notation. For a vector $M = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geq 0}$ with non-negative entries we define

$$|M| := \sum_{i=1}^{r} m_i,$$
$$\alpha(M) := \max\{|m_i - m_j| : \ i,j = 1,\dots,r\},$$
$$\mathbf{l}(M) := \sum_{i=1}^{r} \binom{m_i+1}{2}.$$

LEMMA 2.9
Let $M_1 = (m,\dots,m,m+\delta) \in \mathbb{Z}^r_{\geq 0}$ with $r \geq 2$ and an integer δ. If $|\delta| = c\cdot r+q$, with $c \in \mathbb{N}$, $0 \leq q < r$ and

$$M_2 = (\underbrace{m+\operatorname{sgn}(\delta)\cdot c,\dots,m+\operatorname{sgn}(\delta)\cdot c}_{r-q},\underbrace{m+\operatorname{sgn}(\delta)\cdot(c+1),\dots,m+\operatorname{sgn}(\delta)\cdot(c+1)}_{q}),$$

then $\mathbf{l}(M_2) \leq \mathbf{l}(M_1)$ and the equality holds if and only if $|\delta| = 0$ or $|\delta| = 1$.

Proof. This is a simple computation.

An obvious consequence of this lemma is

COROLLARY 2.10
Let $\mathcal{M}_p = \{M \in \mathbb{Z}^r_{\geq 0} : \ |M| = p\}$. Let M_0 be an element in $\mathcal{M}_p$ imposing the least theoretical number of conditions i.e. $\mathbf{l}(M_0) = \min\{\mathbf{l}(M) \mid M \in \mathcal{M}_p\}$. Then either $\alpha(M_0) = 0$, or if this is not the case, then $\alpha(M_0) = 1$.

We have also

COROLLARY 2.11
Let (X,L) be a polarized surface with the Picard number $\varrho \leq 2$ and let $x_1,\dots,x_r \in X$ be fixed general points. If $M_D = (m_1,\dots,m_r) \in \mathbb{Z}^r_{\geq 0}$ is a multiplicity vector of a R-R expected submaximal irreducible and reduced curve D at $x_1,\dots,x_r$, and $r \geq 3$, then up to permutation M_D is of the form

$$M_D = (m,\dots,m,m+\delta) \qquad \text{with } \delta \in \{-1,0,1\}.$$

Proof. Since the Picard number $\varrho \leq 2$ and D is a reduced and irreducible submaximal curve, by Proposition 1.6 its multiplicity vector M_D, up to permutation, is of the form

$$M_D = (m,\dots,m,m+\delta).$$

Suppose that $|\delta| \geq 2$. Then as the points are general, we have r different submaximal curves. By Lemma 2.9 there exists also a R-R expected submaximal curve D' with $\alpha(D') \leq 1$. This implies, again by generality of the points $x_1,\dots,x_r$, the existence of at least $\frac{1}{2}(r-1)r$ additional submaximal curves which contradicts Proposition 1.3. Hence $|\delta| \leq 1$.

3. Symplectic packing and the Nagata--Biran conjecture

Let (M,ω) be a closed symplectic 4-manifold and $B(\lambda_q,\omega_{std})$ be the standard closed 4-ball of radius λ_q with $\omega_{std}=dx_1\wedge dy_1+dx_2\wedge dy_2$ the standard symplectic form of $\mathbb{R}^4$. Consider a symplectic packing φ_λ of (M,ω) by N equal balls of radii λ (i.e., $\varphi_\lambda=\coprod_{q=1}^N\varphi_q\colon\coprod_{q=1}^N B(\lambda,\omega_{std})\longrightarrow(M,\omega)$ is an embedding and for all q we have that a restriction to the q-th ball coincides with $\varphi_q\colon B(\lambda,\omega_{std})\longrightarrow(M,\omega)$ and $\varphi_q^*\omega=\omega_{std}$). For a symplectic manifold of finite volume McDuff and Polterovich in [MP] introduced

$$v_N(M,\Omega)=\sup_\lambda\frac{\mathrm{Vol}(\mathrm{Image}\,\varphi_\lambda)}{\mathrm{Vol}(M,\Omega)},$$

where the supremum we take over all $\lambda\in\mathbb{R}_+$ such that φ_λ exists. If $v_N(M,\Omega)=1$ then there exists a *full filling*, in the other case, i.e., $v_N(M,\Omega)<1$, there is a *packing obstruction*.

In [Bi1] Biran proved the following theorem.

THEOREM 3.1 ([Bi1] Theorem 6.1.A 2)
On $\mathbb{P}^1\times\mathbb{P}^1$ with the (a,b) polarization we have

$$v_N=\min\left\{1,\frac{N}{2ab}\cdot\inf_{(\alpha,\beta)\in D_N}\left(\frac{a\alpha+b\beta}{2\alpha+2\beta-1}\right)^2\right\},$$

where D_N is the set of all non-negative solutions $\alpha,\beta,m_1,\ldots,m_N\geq 0$ of the system of Diophantine equations:

$$\begin{cases}2\alpha\beta=\sum_{q=1}^N m_q^2-1,\\ 2\alpha+2\beta=\sum_{q=1}^N m_q+1.\end{cases}$$

In particular on $\mathbb{P}^1\times\mathbb{P}^1$ with the $(1,1)$ polarization we have: $v_1=\frac{1}{2}$, $v_2=1$, $v_3=\frac{2}{3}$, $v_4=\frac{8}{9}$, $v_5=\frac{9}{10}$, $v_6=\frac{48}{49}$, $v_7=\frac{224}{225}$ and $v_N=1$ for any $N\geq 8$ ([Bi1]).

For $\mathbb{P}^2$ there is a similar picture obtained by McDuff and Polterovich in [MP].

Later Biran proved that for a polarized surface (X,L) there exists N_0 such that for all $N\geq N_0$ we have $v_N=1$. More precisely, if we denote by k_0 the smallest integer such that in the linear system $|k_0L|$ there exists a smooth non-rational curve, then $N_0=k_0^2L^2$ (see [Bi2] Theorem 1.A.).

Now we want to study the surface $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of Theorem 3.1. More precisely we want to find a relation between the number v_N and the existence of R-R expected submaximal curves at N points.

First we introduce the following definition.

Definition 3.2
For the (a, b) polarization L on $\mathbb{P}^1 \times \mathbb{P}^1$ we define the following constants:

(1) $N_0 := \begin{cases} 8ab & \text{for } a = 1 \text{ or } b = 1, \\ 2ab & \text{for } a \geq 2 \text{and } b \geq 2, \end{cases}$

(2) $R_0 := \dfrac{3a^2 + 2ab + 3b^2}{2ab} + \dfrac{(a+b)\sqrt{2(a^2+b^2)}}{ab},$

(3) $r_0 := \left\lfloor \dfrac{2(a+b)^2}{ab} \right\rfloor.$

Lemma 3.3
For every positive integers a and b one has

$$r_0 \leq R_0 \tag{3.3.1}$$

and the equality holds only for $a = b$. Moreover we have

$$R_0 \leq N_0, \tag{3.3.2}$$

and the equality holds if and only if $a = 1$ and $b = 1$ or $a = 2$ and $b = 2$.

Proof. Straightforward calculations.

Since the conditions in the last definition are symmetric, we can assume without loss of generality that $a \geq b$. We can write a in the unique way as

$$a = k \cdot b + j, \qquad \text{with } k \geq 1 \text{ and } j \in \{0, \ldots, b-1\}. \tag{3.3.3}$$

We keep this notation for the rest of this article.

Now we compute the value of r_0 .

Lemma 3.4
For any (a, b) polarization we have

$$r_0 = \begin{cases} 2k+4 & \text{for } j \in \left\langle 0, \frac{\sqrt{4k^2+4k-15}-2k+1}{4}\ b\right) \cap \mathbb{N}, \\ 2k+5 & \text{for } j \in \left\langle \frac{\sqrt{4k^2+4k-15}-2k+1}{4}\ b, \frac{1+\sqrt{k^2+2k-3}-k}{2}\ b\right) \cap \mathbb{N}, \\ 2k+6 & \text{for } j \in \left\langle \frac{1+\sqrt{k^2+2k-3}-k}{2}\ b, b-1\right\rangle \cap \mathbb{N}. \end{cases}$$

Proof. Since $a = k \cdot b + j$, from the Definition 3.2 it follows that

$$r_0 = 2k + 4 + \left\lfloor \frac{2kbj + 2b^2 + 2j^2}{b^2k + bj} \right\rfloor .$$

To prove our claim it is enough to show that for all $w \in \langle 0, b-1 \rangle$

$$\frac{2kbw + 2b^2 + 2w^2}{b^2k + bw} < 3, \tag{3.4.1}$$

or equivalently

$$2w^2 + b(2k-3)w - (3k-2)b^2 < 0.$$

This is an elementary calculation.

Now we are in a good position to formulate the following result,

THEOREM 3.5
Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. If L is the (a, b) polarization, then there are no R-R expected submaximal curves on X through $r \geq R_0 = R_0(a, b)$ general points.

Proof. Fix $r \geq R_0$ and suppose to the contrary that $D \subset X$ of type (α, β) is R-R expected and submaximal. We can assume that the multiplicity vector of D is $M_D = (m, \ldots, m, m + \delta)$, where $\delta \in \{-1, 0, 1\}$ and m is a non-negative integer (by Corollary 2.11). Hence the number of independent conditions imposed by M_D is

$$\mathbf{l}(M) = (r-1)\binom{m+1}{2} + \binom{m+\delta+1}{2} = \frac{1}{2}\left(rm^2 + rm + 2m\delta + \delta^2 + \delta\right).$$

Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \beta)) = \alpha\beta + \alpha + \beta + 1$ and D is R-R expected, and by Proposition 1.3 there is no continuous family of submaximal curves, we must have

$$\alpha\beta + \alpha + \beta = \frac{1}{2}\left(rm^2 + rm + 2m\delta + \delta^2 + \delta\right),$$

or equivalently

$$\beta = \frac{rm^2 + rm + 2m\delta + \delta^2 + \delta - 2\alpha}{2(\alpha + 1)}. \tag{3.5.1}$$

The submaximality of D means that

$$\frac{a\beta + \alpha b}{rm + \delta} < \sqrt{\frac{2ab}{r}}. \tag{3.5.2}$$

Substituting $t := \sqrt{r}$, conditions (3.5.1) and (3.5.2) give us the inequality

$$\begin{aligned} &2tb\alpha^2 - (2\sqrt{2ab}t^2m + 2ta - 2tb + 2\sqrt{2ab}\delta)\alpha \\ &\quad + (at^2m + a\delta^2 + at^2m^2 + 2am\delta + a\delta - 2\sqrt{2ab}tm)t - 2\sqrt{2ab}\delta < 0. \end{aligned}$$

We consider it as a quadratic inequality in the variable α. We know that the set of solutions is non-empty, hence

$$\begin{aligned} -2abt^3\left(t-\frac{2(a+b)}{\sqrt{2ab}}\right)m+((a-b)^2 \\ -2ab(1+\delta)\delta)t^2+2\sqrt{2ab}(a+b)\delta t+2ab\delta^2>0. \end{aligned} \tag{3.5.3}$$

If we assume that $t>\frac{2(a+b)}{\sqrt{2ab}}=\sqrt{r_0}$, then (3.5.3) is equivalent to

$$m<\frac{((a-b)^2-2ab(1+\delta)\delta)t^2+2\sqrt{2ab}(a+b)\delta t+2ab\delta^2}{2t^3(abt-\sqrt{2ab}(a+b))}. \tag{3.5.4}$$

In the case $\delta=0$ the inequality (3.5.4) is equivalent to

$$m<\frac{(a-b)^2}{2abt^2-2\sqrt{2ab}(a+b)t}. \tag{3.5.5}$$

If $t\geq\sqrt{R_0}$ then the right side of (3.5.5) is smaller than 1 and it must be $m=0$, but this contradicts the definition of the multiple point Seshadri constant.

In the case $\delta=-1$ the inequality (3.5.4) is equivalent to

$$m<\frac{(a-b)^2t^2-2\sqrt{2ab}(a+b)t+2ab}{2t^3(abt-\sqrt{2ab}(a+b))}. \tag{3.5.6}$$

Since $\sqrt{r_0}\geq 1$, $t\geq\sqrt{r_0}$ implies also $t\geq\dfrac{1}{\sqrt{r_0}}$ and

$$(a-b)^2t^2-2\sqrt{2ab}(a+b)t+2ab\leq(a-b)^2t^2.$$

Applying the last inequality to (3.5.6) we obtain the condition (3.5.5) and we reduce our problem to the previous one.

In the case $\delta=1$, the inequality (3.5.4) is equivalent to

$$m<\frac{((a-b)^2-4ab)t^2+2\sqrt{2ab}(a+b)t+2ab}{2t^3(abt-\sqrt{2ab}(a+b))}. \tag{3.5.7}$$

Since our condition is still symmetric, then without loss of generality we may use notation (3.3.3). We observe that for $t\geq\sqrt{k+4}$ there is the inequality:

$$\frac{((a-b)^2-4ab)t^2+2\sqrt{2ab}(a+b)t+2ab}{2t^3(abt-\sqrt{2ab}(a+b))}\leq\frac{(a-b)^2}{2abt^2-2\sqrt{2ab}(a+b)t}. \tag{3.5.8}$$

If $t\geq\sqrt{R_0}$ then (3.5.8) holds and

$$\frac{(a-b)^2}{2abt^2-2\sqrt{2ab}(a+b)t}<1.$$

In this case it can happen that (3.5.7) has a solution, namely $m = 0$. Since D is R-R expected, (3.5.1) holds and we obtain that only a fiber through one of the points $x_1, \ldots, x_r$ comes into consideration. It is easy to see that the Seshadri quotient given by the fiber is submaximal for at most $2k + 2 - \frac{2}{b}$ points, which by Lemmas 3.3 and 3.4 gives a contradiction with our assumption $t \geq \sqrt{R_0}$.

To complete the picture, we should find R-R expected submaximal curves for $r < R_0$ points. Before we begin, we make an obvious

Observation 3.6
Let (X, L) be a polarized surface. Let $D \subset X$ be a curve which at r points gives the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$. If $\sqrt{\frac{L^2}{r}}$ is non-rational then D is submaximal.

Proof. Let $M_D = (m_1, \ldots, m_r)$ be a multiplicity vector for D. By assumption $\frac{L.D}{\sum_{i=1}^{r} m_i} \leq \sqrt{\frac{L^2}{r}}$. Since the number on the left side is always rational, then the equality can hold only in the case when $\sqrt{\frac{L^2}{r}}$ is rational.

It means only that in practice we will be looking for R-R expected curves which at r points give a Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$.

Analyzing the value of the formula in (3.5.3), for $r \leq 2k + 5$ we find R-R expected curves which give Seshadri quotients at most $\sqrt{\frac{L^2}{r}}$. We observe that these curves depend on k and sometimes on j.

Proposition 3.7
Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization. If $r \leq 2k + 5$, then R-R curves which give Seshadri quotients at most $\sqrt{\frac{L^2}{r}}$ arelike in the following tables:

(a) *Table 1 in the case $k = 1$,*

(b) *Table 2 in the case $k \geq 2$.*

Proof. Since all curves from the tables fulfill the condition (3.5.1), they are R-R expected. One can also check that for appropriate j we have

$$\frac{L.D}{\sum_{i=1}^{r} m_i} \leq \sqrt{\frac{L^2}{r}}.$$

As we observed, R-R expected submaximal curves depend sometimes on j. We see also that only in one case it can happen that for some $r \leq 2k + 5$ and for some polarization we obtain two different types of submaximal curves.

r	Type of curve	$\ldots \le j \le \ldots$		m	δ	The Seshadri quotient	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{2(b+j)b}$
2	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{(b+j)b}$
3	$(1,1)$	0	$b-1$	1	0	$\frac{2b+j}{3}$	$\sqrt{\frac{2(b+j)b}{3}}$
4	$(1,1)$	0	$b-1$	1	-1	$\frac{2b+j}{3}$	$\sqrt{\frac{(b+j)b}{2}}$
5	$(2,1)$	0	$b-1$	1	0	$\frac{3b+j}{5}$	$\sqrt{\frac{2(b+j)b}{5}}$
6	$(2,2)$	0	$\frac{1}{3}b$	1	1	$\frac{4b+2j}{7}$	$\sqrt{\frac{(b+j)b}{3}}$
	$(2,1)$	$\frac{1}{3}b$	$b-1$	1	-1	$\frac{3b+j}{5}$	
7	$(4,4)$	0	$\frac{1}{7}b$	2	1	$\frac{8b+4j}{15}$	$\sqrt{\frac{2(b+j)b}{7}}$
	$(4,3)$	$\frac{1}{7}b$	$\frac{5}{9}b$	2	-1	$\frac{7b+3j}{13}$	
	$(3,1)$	$(3-\sqrt{7})b$	$b-1$	1	0	$\frac{4b+j}{7}$	

Table 1

r	Type of curve	$\ldots \le j \le \ldots$		m	δ	The Seshadri quotient	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{2(kb+j)b}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$2k$	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{\frac{(kb+j)b}{k}}$
$2k+1$	$(k,1)$	0	$b-1$	1	0	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{2(kb+j)b}{2k+1}}$
$2k+2$	$(k,1)$	0	$b-1$	1	-1	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{(kb+j)b}{k+1}}$
$2k+3$	$(k+1,1)$	0	$b-1$	1	0	$\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{2(kb+j)b}{2k+3}}$
$2k+4$	$(k^2+k,k+1)$	0	$\frac{1}{k+2}b$	k	1	$\frac{(k+1)(2kb+j)}{2k^2+4k+1}$	$\sqrt{\frac{(kb+j)b}{k+2}}$
	$(k+1,1)$	$\frac{1}{k+2}b$	$b-1$	1	-1	$\frac{(2k+1)b+j}{2k+3}$	
$2k+5$	$(k+2,1)$	0	$b-1$	1	0	$\frac{2(k+1)b+j}{2k+5}$	$\sqrt{\frac{2(kb+j)b}{2k+5}}$

Table 2

REMARK 3.8
In the case $k = 1$, if we take b such that

$$\left((3-\sqrt{7})b,\ \tfrac{5}{9}b\right) \cap \mathbb{N} \neq \emptyset$$

then for $r = 7$ points and $(3-\sqrt{7})b < j < \frac{5}{9}b$ we have two types of R-R expected submaximal curves coming from type $(3,1)$ and $(4,3)$. The number of submaximal curves is altogether 14. Since we can have at most 8 reduced, irreducible and submaximal, it means that at least one of them is reducible. We see that the curve of type $(3,1)$ is a component of a curve of type $(4,3)$. Moreover, we observe that if $j \leq \frac{3}{8}b$ then $\frac{7b+3j}{13} \leq \frac{4b+j}{7}$.

In all other cases, if different types of R-R expected curves give the same Seshadri quotient, then this quotient is equal to $\sqrt{\frac{L^2}{r}}$, but it means that it is no longer submaximal.

Now we want to show that for $r = 2k+6$ points there exist R-R expected submaximal curves at least for (a,b) such that $r_0 = 2k+6$ (see Lemma 3.4). We observe that R-R expected submaximal curves still depend on k and j and in general case we can not write an explicit form, as we could for $r \leq 2k+5$ points.

We construct now a sequence of curves for which we compute their *submaximality area*, i.e., we estimate polarizations for which our curves are submaximal.

PROPOSITION 3.9
Let $l \in \mathbb{Z}_+$ be a positive integer. Given the following sequences:

$$\begin{aligned} \alpha_1 &:= (l+1)(l+2), & \beta_1 &:= l+2, \\ m_1 &:= l+1, & \delta_1 &:= 1 \end{aligned}$$

and for $n \geq 2$

$$\begin{aligned} \alpha_{n+1} &:= (l+1)\alpha_n - \beta_n + 1, & \beta_{n+1} &:= \alpha_n, \\ m_{n+1} &:= \frac{(2l+4)\alpha_n - 2\beta_n + 1 + \delta_n}{2l+6}, & \delta_{n+1} &:= -\delta_n \end{aligned}$$

we have:

(1) *for every positive integer $n \in \mathbb{Z}_+$ we have $\frac{\alpha_n}{\beta_n} > l$;*

(2) *if $r = 2l+6$ and D_n, with $n \in \mathbb{Z}_+$, is a curve of type (α_n, β_n) with a multiplicity vector $M_{D_n} = (m_n, \ldots, m_n, m_n + \delta_n)$ at r points, then*

$$h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha_n, \beta_n)) = \mathbf{l}(M_{D_n}) + 1. \tag{3.9.1}$$

The condition (3.9.1) *in particular means that the curve D_n is R-R expected.*

Proof. (1) Easy induction on n.

(2) One can check that for all positive integers $n \in \mathbb{Z}_+$ we have

$$2\alpha_n + 2\beta_n - 2(l+3)m_n - \delta_n - 1 = 0. \tag{3.9.2}$$

Due to this equality we observe that for $n \geq 2$ it holds:

$$\alpha_{n+1} = \beta_n + (l+3)\alpha_n - (2l+6)m_n - \delta_n$$

and

$$m_{n+1} = \alpha_n - m_n\,.$$

Using the induction on n we can prove that for all $n \in \mathbb{Z}_+$ the condition (3.9.1) is true.

Now we want to compute submaximality areas for curves $\{D_n\}_{n\in\mathbb{Z}_+}$ from Proposition 3.9. In order to do this, first we prove the following

Lemma 3.10
Let $l, c, z \in \mathbb{Z}_+$ be positive integers with $z \in \langle 0, c-1\rangle$. Let z_n be the smaller solution of the following equation in z

$$\frac{(lc+z)\beta_n + c\alpha_n}{(2l+6)m_n + \delta_n} = \sqrt{\frac{(lc+z)c}{l+3}} \tag{3.10.1}$$

with z as the indeterminate. Then the sequence $\{z_n\}_{n\in\mathbb{N}} \subset \langle 0, c-1\rangle$ is strictly decreasing and

$$\lim_{n\to\infty} z_n = \frac{1+\sqrt{l^2+2l-3}-l}{2}\,c.$$

Proof. Let $\widetilde{z_1}$ and $\widetilde{z_2}$ be solutions of the equation

$$\frac{(lc+z)\beta_{n+1} + c\alpha_{n+1}}{(2l+6)m_{n+1} + \delta_{n+1}} = \sqrt{\frac{(lc+z)c}{l+3}} \tag{3.10.2}$$

with z as the indeterminate. We may assume without loss of generality that $\widetilde{z_1} < \widetilde{z_2}$. By definition we have $z_{n+1} = \widetilde{z_1}$. Using direct calculations we can show that $\widetilde{z_2} = z_n$. Since $\widetilde{z_1} < \widetilde{z_2}$ and n was arbitrary, then the sequence $\{z_n\}_{n\in\mathbb{N}}$ is strongly decreasing.

On the other hand, for every positive integer n we have

$$z_n + z_{n-1} = \frac{(2l+6)[-\alpha_n\beta_n - l\beta_n^2 + (2l+6)m_n^2 + 2m_n\delta_n] + 1}{(l+3)\beta_n^2}\,c.$$

Then from Proposition 3.9 (2) we obtain that

$$\begin{aligned} &z_n + z_{n-1} \\ &\quad = \frac{(2l+6)[-\alpha_n\beta_n - l\beta_n^2 + 2\alpha_n\beta_n + 2\alpha_n + 2\beta_n - (2l+6)m_n - \delta_n - 1] + 1}{(l+3)\beta_n^2}\,c. \end{aligned}$$

By (3.9.2) it follows

$$z_n + z_{n-1} = \frac{(2l+6)\beta_n(\alpha_n - l\beta_n) + 1}{(l+3)\beta_n^2}\, c > 0,$$

where the inequality follows from Proposition 3.9 (1). Since $\{z_n\}_{n\in\mathbb{Z}_+}$ is strongly decreasing, then for every $n \in \mathbb{Z}_+$ we have $z_n > 0$. In particular it means that the sequence $\{z_n\}_{n\in\mathbb{Z}_+}$ is convergent. If this is the case, then

$$\lim_{n\to\infty} z_n = \frac{1}{2}\lim_{n\to\infty}(z_n + z_{n-1}) = \frac{1}{2}c\lim_{n\to\infty}\left(2\frac{\alpha_n}{\beta_n} - 2l + \frac{1}{(l+3)\beta_n^2}\right).$$

From Proposition 3.9 it follows that

$$\lim_{n\to\infty}\beta_n = +\infty \qquad \text{and} \qquad \lim_{n\to\infty}\frac{\alpha_n}{\beta_n} \geq l \geq 1. \tag{3.10.3}$$

Since $\lim_{n\to\infty}\frac{1}{(l+3)\beta_n^2} = 0$ and $\lim_{n\to\infty} z_n$ exists, then also $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$ exists. Let $g := \lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$. We obtain that

$$\begin{aligned} g &= \lim_{n\to\infty}\frac{\alpha_{n+1}}{\beta_{n+1}} = \lim_{n\to\infty}\frac{(l+1)\alpha_n - \beta_n + 1}{\alpha_n} \\ &= \lim_{n\to\infty}\left((l+1) - \frac{\beta_n}{\alpha_n} + \frac{1}{\alpha}\right). \end{aligned} \tag{3.10.4}$$

On the other hand by (3.10.3) we have $g \geq 1$ and hence there exists

$$\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = \frac{1}{g}.$$

Combining this fact with (3.10.4) we obtain our assertion.

As a simple consequence of the previous lemma we obtain the following

PROPOSITION 3.11
Let $r = 2k+6$ be the number of points on $(\mathbb{P}^1 \times \mathbb{P}^1, L)$ with the (a,b) polarization L. Let $\{z_n\}_{n\in\mathbb{N}}$ and (α_n, β_n) with m_n and δ_n be like in Lemma 3.10 and Proposition 3.9 respectively. If for some n_0 there is $z_{n_0} < j < z_{n_0-1}$, then the curve D_{n_0} of type $(\alpha_{n_0}, \beta_{n_0})$ with the multiplicity vector $M_{D_{n_0}} = (m_{n_0}, \ldots, m_{n_0}, m_{n_0} + \delta_{n_0})$ is R-R expected submaximal at r points. If $j = z_{n_0}$ or $j = z_{n_0-1}$ then $\sqrt{\frac{L^2}{r}}$ is rational and D_{n_0} computes this quotient.

Proof. Since $z_{n_0} < j < z_{n_0-1}$, then by Lemma 3.10 we have

$$\frac{(kb+j)\beta_{n_0} + b\alpha_{n_0}}{(2k+6)m_{n_0} + \delta_{n_0}} < \sqrt{\frac{(kb+j)b}{k+3}}.$$

This inequality means that the curve D_{n_0} of type $(\alpha_{n_0}, \beta_{n_0})$ with the multiplicity vector $M_{D_{n_0}} = (m_{n_0}, \ldots, m_{n_0}, m_{n_0} + \delta_{n_0})$ is submaximal. By Proposition 3.9 (2) the curve D_{n_0} is also R-R expected.

If $j = z_{n_0}$ or $j = z_{n_0-1}$ then

$$\frac{(kb+j)\beta_{n_0} + b\alpha_{n_0}}{(2k+6)m_{n_0} + \delta_{n_0}} = \sqrt{\frac{(kb+j)b}{k+3}}$$

and $\sqrt{\frac{(kb+j)b}{k+3}} = \sqrt{\frac{L^2}{r}}$ must be rational. The previous equality also means that the curve of type $(\alpha_{n_0}, \beta_{n_0})$ computes the quotient $\sqrt{\frac{L^2}{r}}$.

In this way we obtain the following:

THEOREM 3.12
Consider the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization. If $r \leq r_0$ (see Definition 3.2 and Lemma 3.4) and $\sqrt{\frac{L^2}{r}}$ is non-rational, then there exist R-R expected submaximal curves at r points.

Proof. If $r_0 \leq 2k + 5$ then expected curves are given in Proposition 3.7.

If $r_0 = 2k + 6$ then by Lemma 3.4 it must be

$$j \in \left\langle \tfrac{1+\sqrt{k^2+2k-3}-k}{2}\, b, b-1 \right\rangle \cap \mathbb{N} \qquad \text{with } k \leq b - 1 + \tfrac{1}{b}.$$

We observe that $\frac{1+\sqrt{k^2+2k-3}-k}{2}\, b$ is an integer only for $k = 1$. In this special case the number $\sqrt{\frac{L^2}{r}}$ is rational.

We should also observe that the sequence from Lemma 3.10 is in fact a partition of the interval

$$\left\langle \tfrac{1+\sqrt{k^2+2k-3}-k}{2}\, b, b-1 \right\rangle.$$

The rest of the proof follows from Proposition 3.11.

Assume that the number of points r is at least $r_0 + 1$ but smaller than R_0. We observe that in this case the situation seems to be out of control. We have conditions (3.5.5), (3.5.6) and (3.5.7) which should eliminate the most of multiplicities m. On the other hand, for r from the neighborhood of r_0 functions on the right side can obtain very high values. We observe that sometimes for $r_0 < r < R_0$ there are no R-R expected submaximal curves.

EXAMPLE 3.13
Let L be $(9, 5)$ polarization. In this case $k = 1$, $b = 5$, $j = 4$ and $R_0 = \frac{68}{15} + \frac{28}{45}\sqrt{53} \approx 9.063$. Analyzing conditions (3.5.5), (3.5.6) and (3.5.7) we obtain

(1) $m < 0$, which is absurd, or (2) $m = 1$, for $\delta = 0$. Since now $r = 9$, in the last case we have only one possibility: $\alpha = 4$ and $\beta = 1$. We see that this curve gives the quotient

$$\frac{L.D}{\sum m_i} = \frac{29}{9} > \sqrt{10} = \sqrt{\frac{L^2}{r}},$$

which is not submaximal.

On the other hand we have:

Example 3.14
Let L be $(3,1)$ polarization. We have $k = 3$, $b = 1$, $j = 0$, $R_0 = 6 + \frac{8}{3}\sqrt{2} \approx 11.962$ and hence $r = 11$. Analyzing the same conditions as in the Example 3.13 we obtain (1) $m < 0$, which is absurd, or (2) $m \leq 3$, for $\delta = 0$. We see that curve D of type $(5,1)$ with $m = 1$ gives the quotient

$$\frac{L.D}{\sum_{i=1}^{11} m_i} = \frac{8}{11} < \sqrt{\frac{6}{11}} = \sqrt{\frac{L^2}{r}},$$

which is submaximal.

These examples show that in general for r in the range between r_0 and R_0 it is difficult to prove for which number of points there are R-R expected submaximal curves. We can only generalized Example 3.14.

Proposition 3.15
Let L be the (a,b) polarization. Let $r = 2k + 2n + 1$ with non-negative integer n. If $(1 + n - \sqrt{2k + 2n + 1})b \leq j \leq b - 1$, then a curve of type $(k + n, 1)$ with $m = 1$ and $\delta = 0$ gives the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$.

Proof. By D we denote a curve of type $(k + n, 1)$. D has the multiplicity vector $M_D = (1, \ldots, 1)$. Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k+n, 1)) = 2k+2n+2 = \mathbf{1}(M_D) - 1$, then D is R-R expected. We compute that

$$\frac{L.D}{\sum_{i=1}^{r} m_i} = \frac{(2k + n)b + j}{2k + 2n + 1},$$

hence $\frac{L.D}{\sum_{i=1}^{r} m_i}$ is at most $\sqrt{\frac{L^2}{r}}$ if and only if

$$(1 + n - \sqrt{2k + 2n + 1})b \leq j \leq b - 1.$$

The Seshadri quotient given by D is submaximal for $j \neq (1+n-\sqrt{2k + 2n + 1})b$.

In this place we should note that $\left\langle (1+n-\sqrt{2k+2n+1})b, b-1\right\rangle \cap \mathbb{N} \neq \emptyset$ only for

$$0 \leq n < \frac{b+\sqrt{2(k+1)b^2-2b}-1}{b}.$$

Now we are in a good position to formulate the following lemma.

LEMMA 3.16
Let D_h be a R-R expected curve of type $(h,1)$ through r points in general position. If $r \geq 2h+1$ and the multiplicity vector $M_{D_h} = (1,\ldots,1)$ then D_h is irreducible.

Proof. We prove this lemma by induction on h.

Step 1. For $h=1$ we have that D_1 is of type $(1,1)$ with the multiplicity vector $M_{D_1} = (1,1,1)$. If D_1 is reducible then D_1 decomposes in the sum of two fibers. Since points are in general position, then the sum of two fibers gives the multiplicity vector $(1,1,0) \neq M_{D_1}$, a contradiction.

Step 2. We assume our thesis for $h < h_0$. We want to show that a curve D_{h_0} of type $(h_0,1)$ through $r \geq 2h_0+1$ points with the multiplicity vector $M_{D_{h_0}} = (1,\ldots,1)$ is irreducible.

We assume to the contrary that D_{h_0} is reducible. Then we take the decomposition on irreducible components. There are two possibilities:

(1) D_{h_0} is the sum of curves of type $(1,0)$ and $(h,1)$ with $h < h_0$, or

(2) D_{h_0} is the sum of curves of type $(1,0)$ and $(0,1)$.

In the first case we have

$$D_{h_0} = (h_0-h)\cdot(1,0)+(h,1).$$

Since points are in general position, then multiplicity vector for a curve $(1,0)$ at r points is $(0,\ldots,0,1)$. Since $(h,1)$ is irreducible, then by the inductive assumption we have that it goes through at least $2h+1$ points with multiplicities 1. Finely we obtain that curves $(h_0-h)\cdot(1,0)$ and $(h,1)$ go through at least $(h_0-h)+(2h+1) = h_0+h+1$ points. Since $h_0+h+1 < 2h_0+1$, the multiplicity vector of the sum of curves $(h_0-h)\cdot(1,0)$ and $(h,1)$ is different from $M_{D_{h_0}}$, a contradiction.

In the second case we have

$$D_{h_0} = h_0\cdot(1,0)+(0,1).$$

Since points are in general position, a multiplicity vector of the sum $h_0\cdot(1,0)+(0,1)$ at r points is $(\underbrace{1,\ldots,1}_{h_0+1},\underbrace{0,\ldots,0}_{r-h_0-1}) \neq M_{h_0}$, a contradiction.

According to this lemma we can say more about Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely we have the following theorem.

THEOREM 3.17
For $(\mathbb{P}^1 \times \mathbb{P}^1, L)$ with L of type (a, b) Seshadri constants are like in the following tables

(1) *Table 3 for $k = 1$,*

(2) *Table 4 for $k \geq 2$.*

r	Type of curve	The submaximality area $\ldots \leq j \leq \ldots$		m	δ	$\varepsilon(L; r)$	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	$= b$	$\sqrt{2(b+j)b}$
2	$(1,0)$	0	$b-1$	0	1	$= b$	$\sqrt{(b+j)b}$
3	$(1,1)$	0	$b-1$	1	0	$= \frac{2b+j}{3}$	$\sqrt{\frac{2(b+j)b}{3}}$
4	$(1,1)$	0	$b-1$	1	-1	$= \frac{2b+j}{3}$	$\sqrt{\frac{(b+j)b}{2}}$
5	$(2,1)$	0	$b-1$	1	0	$= \frac{3b+j}{5}$	$\sqrt{\frac{2(b+j)b}{5}}$
6	$(2,2)$	0	$\frac{1}{3}b$	1	1	$\leq \frac{4b+2j}{7}$	$\sqrt{\frac{(b+j)b}{3}}$
	$(2,1)$	$\frac{1}{3}b$	$b-1$	1	-1	$= \frac{3b+j}{5}$	
7	$(4,4)$	0	$\frac{1}{7}b$	2	1	$\leq \frac{8b+4j}{15}$	$\sqrt{\frac{2(b+j)b}{7}}$
	$(4,3)$	$\frac{1}{7}b$	$\frac{3}{8}b$	2	-1	$\leq \frac{7b+3j}{13}$	
	$(3,1)$	$\frac{3}{8}b$	$b-1$	1	0	$= \frac{4b+j}{7}$	
8	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\sqrt{\frac{(b+j)b}{4}}$
	$(28,21)$	$\frac{15}{49}b$	$\frac{13}{36}b$	12	1	$\leq \frac{49b+21j}{97}$	
	$(21,15)$	$\frac{13}{36}b$	$\frac{11}{25}b$	9	-1	$\leq \frac{36b+15j}{71}$	
	$(15,10)$	$\frac{11}{25}b$	$\frac{9}{16}b$	6	1	$\leq \frac{25b+10j}{49}$	
	$(10,6)$	$\frac{9}{16}b$	$\frac{7}{9}b$	4	-1	$\leq \frac{16b+6j}{31}$	
	$(6,3)$	$\frac{7}{9}b$	$b-1$	2	1	$\leq \frac{9b+3j}{17}$	

Table 3

r	Type of curve	The submaximality area $\dots \le j \le \dots$		m	δ	$\varepsilon(L;r)$	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	$= b$	$\sqrt{2(kb+j)b}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$2k$	$(1,0)$	0	$b-1$	0	1	$= b$	$\sqrt{\frac{(kb+j)b}{k}}$
$2k+1$	$(k,1)$	0	$b-1$	1	0	$= \frac{2kb+j}{2k+1}$	$\sqrt{\frac{2(kb+j)b}{2k+1}}$
$2k+2$	$(k,1)$	0	$b-1$	1	-1	$= \frac{2kb+j}{2k+1}$	$\sqrt{\frac{(kb+j)b}{k+1}}$
$2k+3$	$(k+1,1)$	0	$b-1$	1	0	$= \frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{2(kb+j)b}{2k+3}}$
$2k+4$	$(k^2+k,k+1)$	0	$\frac{1}{k+2}b$	k	1	$\le \frac{(k+1)(2kb+j)}{2k^2+4k+1}$	$\sqrt{\frac{(kb+j)b}{k+2}}$
	$(k+1,1)$	$\frac{1}{k+2}b$	$b-1$	1	-1	$= \frac{(2k+1)b+j}{2k+3}$	
$2k+5$	$(k+2,1)$	0	$b-1$	1	0	$= \frac{2(k+1)b+j}{2k+5}$	$\sqrt{\frac{2(kb+j)b}{2k+5}}$
$2k+6$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\le \frac{(k+2)(2kb+j+b)}{2k^2+8k+7}$	$\sqrt{\frac{(kb+j)b}{k+3}}$
	$(k^2+3k+2,k+2)$	$\frac{k^2+3k+3}{k^2+4k+4}b$	$b-1$	$k+1$	1		
$2k+2n+7$	$(k+n+3,1)$	$(4+n-\sqrt{2k++2n+7})b$	$b-1$	1	0	$= \frac{(2k+3+n)b+j}{2k+2n+7}$	$\sqrt{\frac{2(kb+j)b}{2k+2n+7}}$

Table 4

3.1. Application to the problem of symplectic packing of $\mathbb{P}^1 \times \mathbb{P}^1$

As an application of Theorem 3.5 we prove the following:

THEOREM 3.18
Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization L. For every $N \geq R_0$ the polarized surface (X, L) admits full symplectic packing by N equal balls.

Proof. Fix r a number of points. Let $D \subset X$ of type (α, β) be a R-R expected submaximal curve. Let $M_D = (m, \ldots, m, m+\delta)$, where $\delta \in \{-1, 0, 1\}$ and $m \in \mathbb{Z}$, be its multiplicity vector. Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \beta)) = \alpha\beta + \alpha + \beta + 1$ and D is R-R expected, and by Proposition 1.3 there is no continuous family of submaximal curves, we must have

$$2\alpha\beta + 2\alpha + 2\beta = rm^2 + rm + 2m\delta + \delta^2 + \delta.$$

Rearranging terms on the right side we obtain that

$$rm^2 + rm + 2m\delta + \delta^2 + \delta = \sum_{i=1}^{r} m_i^2 + \sum_{i=1}^{r} m_i \tag{3.18.1}$$

(by m_i we mean the multiplicity D at x_i). By Theorem 3.5 we have that for $r \geq R_0$ points there are no R-R expected submaximal curves. In particular it means that there are no curves such that (3.18.1) becomes true. If this is the case, then the system of Diophantine equations in Theorem 3.1 does not have solutions and by the same theorem for $N \geq R_0$ we have $v_N = 1$.

3.2. Conjecture

As we remarked before, Seshadri constants are known only in few examples and in every such case, the computing curve was R-R expected. We observe also that on $\mathbb{P}^1 \times \mathbb{P}^1$ in that cases when there exists the full filling by N equal balls, there is no R-R expected submaximal curves at N points. This facts give us a reason to formulate the following conjecture.

CONJECTURE 3.19
In the case $\mathbb{P}^1 \times \mathbb{P}^1$ the number N_0 in the Nagata-Biran Conjecture can be replaced by R_0 .

REMARK 3.20
For the (pa, pb) polarization the number N_0, with respect to p, grows like a quadratic function. For the constant R_0 this is not the case. If we look at the Definition 3.2 then we can see, that R_0 is a rational function of a and b of degree 0 so the value of R_0 does not depend on p. In particular, it means that the Biran number N_0 can be optimally applied only for polarizations of type (a, b) with a and b coprime.

Acknowledgments

This article is a part of my PhD thesis. My stay at the University of Essen was made possible by the Leibnitz Preis of Esnault and Viehweg. I am very grateful to H. Esnault, T. Szemberg and E. Viehweg for invitation and helpful discussions. I would also thank H. Tutaj-Gasińska for discussions on symplectic packings.

References

[Bi1] P. Biran, *Symplectic packing in dimension* 4, Geom. Funct. Anal. **7** (1997), no. 3, 420-437.

[Bi2] P. Biran, *A stability property of symplectic packing*, Invent. Math. **136** (1999), no. 1, 123-155.

[CP] F. Campana, Th. Peternell, *Algebraicity of the ample cone of projective varieties*, J. Reine Angew. Math. **407** (1990), 160-166.

[De] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, in: Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math. **1507**, Springer, Berlin, 1992, 87-104.

[MP] D. McDuff, L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994), no. 3, 405-434.

[Og] K. Oguiso, *Seshadri constants in a family of surfaces*, Math. Ann. **323** (2002), no. 4, 625-631.

[S1] W. Syzdek, *Nagata submaximal curves on* $\mathbb{P}^1 \times \mathbb{P}^1$, Ann. Polon. Math. **80** (2003), 223-230.

[Sz] T. Szemberg, *Global and local positivity of line bundles*, Habilitationsschrift, Essen, 2001.

[Xu] G. Xu, *Ample line bundles on smooth surfaces*, J. Reine Angew. Math. **469** (1995), 199-209.

Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
E-mail: syzdek@ap.krakow.pl

Received: 6 January 2007; final version: 21 April 2007;
available online: 23 July 2007.