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On type sequences and Arf rings

Abstract. In this article in Section 2 we give an explicit description to compute the type sequence $\mathrm{t}_1, \dots, \mathrm{t}_n$ of a semigroup Γ generated by an arithmetic sequence (see 2.7); we show that the i-th term t_i is equal to 1 or to the type τ_Γ, depending on its position. In Section 3, for analytically irreducible ring R with the branch sequence $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$, starting from a result proved in [4] we give a characterization (see 3.6) of the "Arf" property using the type sequence of R and of the rings R_j, $1 \le j \le m-1$. Further, we prove (see 3.9, 3.10) some relations among the integers $\ell^*(R)$ and $\ell^*(R_j)$, $1 \le j \le m-1$. These relations and a result of [6] allow us to obtain a new characterization (see 3.12) of semigroup rings of minimal multiplicity with $\ell^*(R) \le \tau(R)$ in terms of the Arf property, type sequences and relations between $\ell^*(R)$ and $\ell^*(R_j)$, $1 \le j \le m-1$.

0. Introduction

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the $\mathfrak{m}$-adic completion $\hat{R}$ of R is a domain or, equivalently, the integral closure $\overline{R}$ of R in its quotient field $\mathrm{Q}(R)$ is a discrete valuation ring and a finite R-module. We further assume that R is residually rational, i.e., R and $\overline{R}$ have the same residue field. A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

Let $v\colon \mathrm{Q}(R) \longrightarrow \mathbb{Z} \cup \{\infty\}$ be the discrete valuation of $\overline{R}$ and let $\mathfrak{C} := \mathrm{ann}_R(\overline{R}/R) = \{x \in R \mid x\overline{R} \subseteq R\}$ be the conductor ideal of R in $\overline{R}$. Then the value semigroup $v(R) = \{v(x) \mid x \in R,\ x \neq 0\}$ is a numerical semigroup, that is, $\mathbb{N} \setminus v(R)$ is finite and therefore $v(R) = \{0 = \mathrm{v}_0, \mathrm{v}_1, \dots, \mathrm{v}_{n-1}\} \cup \{z \in \mathbb{N} \mid z \ge c\}$, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \dots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ are elements of $v(R)$, $n := n(R) = \ell(R/\mathfrak{C})$ and the integer $c = c(R) := \ell_{\overline{R}}(\overline{R}/\mathfrak{C})$ is also determined by $\mathfrak{C} = \{x \in \mathrm{Q}(R) \mid v(x) \ge\ c\}$ or, equivalently $\mathfrak{C} = (\mathfrak{m}_{\overline{R}})^c$.

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In [11] Matsuoka studied the degree of singularity $\delta = \delta(R) := \ell(\overline{R}/R) = \operatorname{card}(\mathbb{N} \setminus v(R))$ of R by introducing the saturated chain of fractionary ideals

$$\mathfrak{C} = \mathfrak{A}_n \subsetneq \ldots \subsetneq \mathfrak{A}_1 = \mathfrak{m} \subsetneq \mathfrak{A}_0 = R \subsetneq \mathfrak{A}_1^{-1} \subsetneq \ldots \subsetneq \mathfrak{A}_n^{-1} = \overline{R},$$

where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq \mathrm{v}_i\}$ and $\mathfrak{A}_i^{-1} = (R : \mathfrak{A}_i)$, $i = 0, 1, \ldots, n$. Moreover, each $\mathfrak{A}_i^{-1}$, $i = 0, \ldots, n$ is an overring of R which satisfies the assumptions that we assume for R. The sequence $\mathrm{t}_i = \mathrm{t}_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \ldots, n$, is called the *type sequence* of R.

Various algebraic and geometric properties of the ring R are described by some numerical invariants, for example, the degree of singularity and the type sequence. Several authors have studied these numerical invariants (see for example [1], [2], [4], [5], [16]). The first term t_1 is the Cohen–Macaulay type of R and the sum $\sum_{i=1}^{n} \mathrm{t}_i$ is the degree of singularity of R. Further, the "Gorensteinness" and "almost Gorensteinness" are characterized by type sequences (see 1.2). It is worth noting here that if R is a semigroup ring, then the above properties correspond to the properties "symmetric" and "pseudo-symmetric" of numerical semigroups, respectively. These properties are of a special interest (see [7], [17]), since each numerical semigroup can be expressed as an intersection of numerical semigroups that are either symmetric or pseudo-symmetric. Furthermore, if R is analytically irreducible, then the property "Arf" can be described by its type sequence and each term t_i is related to the i-th term in the "branch sequence" of R (see § 4).

In this article we prove the following results:

(1) If Γ is a numerical semigroup generated by an arithmetic sequence, then we explicitly compute the type sequence (see 2.7) and give (see 2.9) a characterization of almost-Gorensteinness of the semigroup ring $R = K[\![\Gamma]\!]$. This is achieved by studying (see 2.6) the "holes" in Γ by using the explicit description (see 2.5) of the standard basis and the type of the numerical semigroup generated by arithmetic sequence given in [14] and [13], respectively.

(2) If R is analytically irreducible, then we relate the degree of singularity of R to the branch sequence $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$, starting from a result proved in [4] we give a characterization (see 3.6) of the "Arf" property using the type sequence (see 1.3) of R and of the rings R_j , $1 \leq j \leq m-1$. Further, we prove (see 3.9, 3.10) some relations among the integers $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$. These relations and a result of [6] allow us to obtain a new charaterization (see 3.12) of semigroup rings of minimal multiplicity with $\ell^*(R) \leq \tau(R)$ in terms of the Arf property, type sequences and relations between $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$.

In Section 4, we also give some illustrative examples to describe our methods.

1. Preliminaries – Assumptions and Notation

Throughout this article we make the following assumptions and notation.

1.1. Notation
Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of all natural numbers and all integers, respectively. Note that we assume $0 \in \mathbb{N}$. Further, for $a, b \in \mathbb{N}$, we denote $[a, b] := \{r \in \mathbb{N} \mid a \leq r \leq b\}$ and $\mathbb{N}_a := \{n \in \mathbb{N} \mid n \geq a\}$.

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the integral closure $\overline{R}$ of R in its quotient field $\mathrm{Q}(R)$ is a discrete valuation ring and is a finite R-module. We further assume that R is residually rational, i.e., the residue field $k_{\overline{R}}$ of $\overline{R}$ is equal to the residue field k_R of R. A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

1.2. Minimal reductions and reduction number
If k_R is infinite, then for every non-zero ideal $\mathfrak{a}$ of R there exists $x \in \mathfrak{a}$ such that xR is a minimal reduction if $\mathfrak{a}$, i.e., $x\mathfrak{a}^m = \mathfrak{a}^{m+1}$ for some $m \in \mathbb{N}$. The natural number $r(\mathfrak{a}) := \min\{m \in \mathbb{N} \mid x\mathfrak{a}^m = \mathfrak{a}^{m+1}\}$ is called the *reduction number* of $\mathfrak{a}$ (see [12]). In particular, if $\mathfrak{a} = \mathfrak{m}$, then $r(\mathfrak{m})$ is called *reduction number* of R. By replacing R by the local ring $R[X]_{\mathfrak{m}[X]}$ of $R[X]$ at the prime ideal $\mathfrak{m}[X]$, we may assume that k_R is infinite and hence assume that a minimal reduction xR of $\mathfrak{m}$ exists.

We shall now recall the notions of *type sequences* and *almost Gorenstein rings.*

1.3. Type sequences — almost Gorenstein rings
Let R be as in 1.1 and let $v(R)$ be its numerical semigroup, $c = c(v(R))$ be the conductor of $v(R)$, $n = n(R) = \ell(R/\mathfrak{C}) = \mathrm{card}(v(R) \setminus \mathbb{N}_c)$ and $\delta = \delta(R) = \ell(\overline{R}/R) = \mathrm{card}(\mathbb{N} \setminus v(R))$ be the degree of singularity of R (see [11]). Let $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ be elements of $v(R)$ such that $v(R) \setminus \mathbb{N}_c = \{\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$. Note that (see [11]) $\delta(R)$ is the sum of n positive integers $\mathrm{t}_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \ldots, n$, where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq \mathrm{v}_i\}$ and $\mathfrak{A}_i^{-1} := (R : \mathfrak{A}_i) := \{x \in \mathrm{Q}(R) \mid x\mathfrak{A}_i \subseteq R\}$. The first positive integer $\mathrm{t}_1(R) = \ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type τ_R of R. The sequence $\mathrm{t}_1(R), \mathrm{t}_2(R), \ldots, \mathrm{t}_n(R)$ is called the *type sequence* of R. Several authors have studied the properties of type sequences (see [1], [5]). The term "type sequence" is chosen since the first term $\mathrm{t}_1(R) = \ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type of R. Further, we have $1 \leq \mathrm{t}_i(R) \leq \tau_R$ for every $i = 1, \ldots, n$ (see [11, §3, Proposition 2 and Proposition 3]) and hence (see also [5, Proposition 2.1]) $\ell^*(R) \leq (\tau_R - 1)\,(\ell(R/\mathfrak{C}) - 1)$, where $\ell^*(R) := \tau_R \cdot \ell(R/\mathfrak{C}) - \ell(\overline{R}/R)$. Moreover, the equality holds if and only if $\ell(\overline{R}/R) = \tau_R + \ell(R/\mathfrak{C}) - 1$, or equivalently $\mathrm{t}_i(R) = 1$ for $i = 2, \ldots, n$.

Type sequence of a numerical semigroup Γ can also be defined analogously: Let $c = c(\Gamma) \in \mathbb{N}$ be the conductor of Γ and let $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ are elements of Γ. Further, for $i = 0, \ldots, n$, let $\Gamma_i := \{h \in \Gamma \mid h \geq \mathrm{v}_i\}$, $\Gamma(i) := \{x \in \mathbb{Z} \mid x + \Gamma_i \subseteq \Gamma\}$ and let $\mathrm{t}_i = \mathrm{card}\,(\Gamma(i) \setminus \Gamma(i-1))$. Then $\Gamma = \Gamma(0) \subseteq \Gamma(1) \subseteq \ldots \subseteq \Gamma(n-1) \subseteq \Gamma(n) = \mathbb{N}$ and the sequence t_i, $i = 1, \ldots, n$ is called the *type sequence* of Γ. In particular, the cardinality t_1 of the set $\mathrm{T}(\Gamma) := \Gamma(1) \setminus \Gamma$ is called the *Cohen–Macaulay type* of the semigroup Γ.

The type sequence of a ring R need not be the same as the type sequence of the numerical semigroup $v(R)$ of R (see for example [5]). However, if $R = K[\![\Gamma]\!]$ is the semigroup ring of a numerical semigroup Γ over a field K, then the type sequence of R is equal to the type sequence of its semigroup $v(R) = \Gamma$.

A ring R in 1.1 is called *almost Gorenstein* if the type sequence of R is $\{\tau_R, 1, 1, \ldots, 1\}$, or equivalently, $\ell^*(R)$ attains its upper bound, i.e., $\ell(\overline{R}/R) = \tau_R - 1 + \ell(R/\mathfrak{C})$. It is clear that Gorenstein rings are almost Gorenstein but not conversely (see [16, (1.2)-(1)]).

2. The type sequence of a semigroup generated by an arithmetic sequence

Let R be as in 1.1. In addition to the notations of Section 1, we also fix the following:

2.1. NOTATION
Put $\Gamma := v(R)$ and let $\Gamma_i := v(\mathfrak{A}_i)$, $\Gamma(i)$ and t_i, $i = 1, \ldots, n$ be as in 1.3.

In order to compute type sequences explicitly, we need to study the "*holes*" of Γ, i.e. elements of $\mathbb{N} \setminus \Gamma$. The positions of the holes will therefore determine the type sequence of Γ. To make these things more precise first let us make the following:

2.2. DEFINITION
An element $z \in \mathbb{Z} \setminus \Gamma$ is called a *hole of first type* (respectively, *hole of second type*) of Γ if $c - 1 - z \in \Gamma$ (respectively, if $c - 1 - z \notin \Gamma$). Then $\Gamma' := \{z \in \mathbb{Z} \setminus \Gamma \mid c - 1 - z \in \Gamma\} = \{c - 1 - h \mid h \in \Gamma\}$ is the set of holes of first type of Γ and $\Gamma'' := \{z \in \mathbb{Z} \setminus \Gamma \mid c - 1 - z \notin \Gamma\}$ is the set of holes of second type of Γ. Therefore $\mathbb{Z} = \Gamma \uplus \Gamma' \uplus \Gamma''$. Further, it is easy to see that:

$$\text{(2.2.a)} \qquad \begin{cases} \Gamma' \cap \mathbb{N} = \{c - 1 - \mathrm{v}_i \mid i \in [0, n-1]\}; \ |\Gamma' \cap \mathbb{N}| = n = c - \delta, \\ \Gamma'' \subseteq \mathbb{N} \setminus \Gamma, \ c - 1 \notin \Gamma'' \quad \text{and} \quad \mathrm{T}(\Gamma) \subseteq \{c - 1\} \cup \Gamma''. \end{cases}$$

In particular, Γ is symmetric if and only if $\Gamma'' = \emptyset$. For this reason the cardinality of Γ'' is called the *symmetry-defect* of Γ.

The following lemma describes the holes of first type of Γ.

2.3. Lemma
$(\Gamma(i) \setminus \Gamma(i-1)) \cap \Gamma' = \{c - 1 - \mathrm{v}_{i-1}\}$ *for each* $i = 1, \ldots, n$.

Proof. Easy to verify (this essentially follows from [11, Proposition 2]).

In order to describe the holes of second type, we assume that Γ is generated by an arithmetic sequence (the description of the holes of second type in the general case is given in § 2 and § 3 of [15]). For this in addition to the notation in 2.1 and 2.2, we further fix the following notation:

2.4. Notation
Let $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 1$ be such that $\gcd(m, d) = 1$ and let p be an integer $p \geq 1$, $m_i := m + id$ for $i = 0, 1, \ldots, p+1$. Let $\Gamma := \sum_{i=0}^{p+1} \mathbb{N}m_i$ be the semigroup generated by the arithmetic sequence $m_0, m_1, \ldots, m_{p+1}$.

For any positive natural number $k \in \mathbb{N}^+$, let $q_k \in \mathbb{N}$ and $r_k \in [1, p+1]$ be the unique integers defined by the equation $k = q_k(p+1) + r_k$. We put $q := q_{m-1}$ and $r := r_{m-1} - 1$. Therefore $q \in \mathbb{N}$, $r \in [0, p]$ and $m - 2 = q(p+1) + r$.

Put $s_0 = 0$ and $s_k := m_{r_k} + q_k m_{p+1} = (1 + q_k)m + (r_k + q_k(p+1))\, d$ for $k \in [1, m-1]$. Further, we put $\mathrm{S}_1 := \{m_i + j m_{p+1} \mid i \in [1, p+1]$ and $j \in [0, q-1]\}$ and $\mathrm{S}_2 := \{m_i + q m_{p+1} \mid i \in [1, r+1]\}$. Note that $\mathrm{S}_1 = \emptyset$, if $q = 0$.

Let $0 = \mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ be elements of Γ such that $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$. For $i \in [0, n]$, the element $\mathrm{v}_i \in \Gamma$ is called the *i-th element* of Γ.

2.5. Proposition
With the notations as in 2.4 *we have:*

(1) *The standard basis* $\mathrm{S} := S_m(\Gamma)$ *with respect to the multiplicity* $m = m_0$ *of* Γ *is*
$$\mathrm{S} = \{s_k \mid k \in [0, m-1]\} = \{0\} \cup \mathrm{S}_1 \cup \mathrm{S}_2 \,.$$

(2) *The conductor* $c := c(\Gamma)$ *and the degree of singularity* $\delta := \delta(\Gamma)$ *of* Γ *are*
$$c = (m-1)(d+q)+q+1 \quad \textit{and} \quad \delta = \left((m-1)(d+q) + (r+1)(q+1)\right)/2.$$

(3) *The set* $\mathrm{T} := T(\Gamma) = \Gamma(1) \setminus \Gamma = \{m_i + q m_{p+1} - m_0 \mid i \in [1, r+1]\} = \{c - 1 - (r - i + 1)d \mid i \in [1, r+1]\}$. *In particular, the Cohen–Macaulay type of* Γ *is* $\tau = \tau_\Gamma = r + 1$.

Proof. (1) and (3) are special cases of the general results proved in [14, (3.5)] and [13, § 5]. (2) is proved in [18, § 3, Supplement 6].

Now we give an explicit description of the positions of the holes of second type of Γ.

2.6. Lemma
With the notations as in 2.1, 2.2 *and* 2.4*, we have:*

(1) $\mathrm{card}(\Gamma'') = (q+1)r$.

(2) $\Gamma'' = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma,\ x \neq c-1 \text{ and } j \in [0,q]\}$.

(3) *For each* $j \in [0,q]$*, there exists a unique integer* $\mathrm{i}(j) \in [0, n-1]$ *such that* $jm_{p+1} = \mathrm{v}_{\mathrm{i}(j)}$ *is the* $\mathrm{i}(j)$*-th element of* Γ*. Moreover,*

$$\Gamma\left(\mathrm{i}(j)+1\right) \setminus \Gamma\left(\mathrm{i}(j)\right) = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma\}.$$

In particular, $\mathrm{card}\left(\Gamma\left(\mathrm{i}(j)+1\right) \setminus \Gamma\left(\mathrm{i}(j)\right)\right) = \tau_\Gamma = r+1$.

Proof. (1) Immediate from 2.5-(2). (2) Easy to verify using 2.5-(3). For the proof of (3) see [15, § 2 and § 3].

Now we give an explicit description of the type sequence of a semigroup generated by an arithmetic sequence.

2.7. Theorem
Let $m, d \in \mathbb{N}$*,* $m \geq 3$*,* $d \geq 1$ *be such that* $\gcd(m,d) = 1$ *and let* p *be an integer with* $1 \leq p \leq m-2$*. Let* $\Gamma := \sum_{k=0}^{p+1} \mathbb{N} m_k$ *be the semigroup generated by the arithmetic sequence* $m_k := m + kd$*,* $k = 0, 1, \ldots, p+1$*. Let* $q \in \mathbb{N}$ *and* $r \in [0,p]$ *be the unique integers defined by the equation* $m - 2 = q(p+1) + r$*. Further, let* $c \in \Gamma$ *be the conductor of* Γ*,* $\mathbb{N}_c = \{z \in \mathbb{N} \mid z \geq c\}$ *and let* $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$ *with* $\mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$*. Then the* i*-th term* $\mathrm{t}_i = \mathrm{t}_i(\Gamma)$ *of the type sequence* $(\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n)$ *of* Γ *is*

$$t_i = \begin{cases} 1, & \text{if } \mathrm{v}_{i-1} \neq jm_{p+1} \text{ for every } j \in [0,q], \\ r+1, & \text{if } \mathrm{v}_{i-1} = jm_{p+1} \text{ for some } j \in [0,q]. \end{cases}$$

Proof. If $\mathrm{v}_{i-1} \neq jm_{p+1}$ for every $j \in [0,q]$, then $\Gamma(i) \setminus \Gamma(i-1) = \{c - 1 - v_{i-1}\}$ by 2.6-(1), (2), (3) and hence $\mathrm{card}\left(\Gamma(i) \setminus \Gamma(i-1)\right) = 1$. If $\mathrm{v}_{i-1} = jm_{p+1}$ for some $j \in [0,q]$, then $\mathrm{card}\left(\Gamma(i) \setminus \Gamma(i-1)\right) = r+1$ by 2.6-(3).

2.8. Corollary
In addition to the notations and assumptions as in 2.7*, further assume that* $d = 1$*. Then the* i*-th term* t_i *of the type sequence* $(\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n)$ *of* Γ *is*

$$t_i = \begin{cases} r+1, & \text{if } i = \binom{j+1}{2}(p+1) + j + 1 \text{ for some } j \in [0,q], \\ 1, & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that for every $j \in [0, q]$, we have

$$\mathrm{i}(j) = \mathrm{card}\Big(\biguplus_{t=0}^{j} \Gamma^{(t)}\Big) = \sum_{t=0}^{j} (t(p+1)+1) = \binom{j+1}{2}(p+1) + j + 1$$

and jm_{p+1} is the $(\mathrm{i}(j)-1)$-th element $\mathrm{v}_{\mathrm{i}(j)-1}$ in Γ. Now the assertion is clear from 2.7.

2.9. Corollary
Let m, d, p, q, r and Γ be as in 2.7 and let $R := K[\![\Gamma]\!]$ be the semigroup ring of Γ over a field K. Then

(1) *R is Gorenstein if and only if $r = 0$.*

(2) *Assume that R is not Gorenstein. Then R is almost Gorenstein if and only if $m = p + 2$. Moreover, in this case we have $\tau_R = m - 1$.*

Proof. (1) Note that $\tau_R = r + 1$ by 2.5-(3). Therefore R is Gorenstein if and only if $r + 1 = \tau_R = 1$, i.e., $r = 0$.

(2) R is almost Gorenstein if and only if the type sequence of R is $\tau_R = r+1, 1, \ldots, 1$ or equivalently (by 2.7) $q = 0$, i.e. $m - 2 = r$. Now, since $m \geq p + 2$ and $r \leq p$, we have $m - 2 = r$ if and only if $m - 2 = p$.

3. Numerical invariants of analytically irreducible Arf rings

In this section we first recall some definitions and results proved in [9] on blowing-up and Arf rings. These results hold more generally, for semi-local 1-dimensional Cohen–Macaulay rings.

Let R be a semi-local Cohen–Macaulay ring of dimension 1 and let $\mathfrak{m}$ be the (Jacobson) radical of R. Let $\overline{R}$ be the integral closure of R in its total quotient ring $\mathrm{Q}(R)$. An ideal $\mathfrak{a}$ in R is called *open* if it is open in the $\mathfrak{m}$-adic topology on R, or, equivalently, $\mathfrak{m}^n \subseteq \mathfrak{a}$ for some $n \geq 1$, or, equivalently, the length $\ell(R/\mathfrak{a})$ is finite. For any two R-submodules M, N of $\overline{R}$, we put $(M : N) := \{y \in \overline{R} \mid yN \subseteq M\}$.

For an open ideal $\mathfrak{a}$ in R, let $\mathrm{B}(\mathfrak{a}) := \cup_{n\in\mathbb{N}}(\mathfrak{a}^n : \mathfrak{a}^n)$. The ring $\mathrm{B}(\mathfrak{a})$ is called the *blowing-up* of R along $\mathfrak{a}$ or the *first neighbourhood ring* of $\mathfrak{a}$.

3.1. Proposition ([9, Proposition 1.1])
For an open ideal $\mathfrak{a}$ in R, the ring $\mathrm{B}(\mathfrak{a})$ is a finitely generated R-module and $R \subseteq \mathrm{B}(\mathfrak{a}) \subseteq \overline{R}$. Moreover, if R is local and if $\mathfrak{a}$ is a $\mathfrak{m}$-primary ideal which is not principal, then $R \subsetneq \mathrm{B}(\mathfrak{a})$. In particular, if R is local and if R is not a discrete valuation ring, then $R \subsetneq \mathrm{B}(\mathfrak{m})$. Furthermore, there exists a non-zero divisor $x \in \mathfrak{a}$ such that $\mathrm{B}(\mathfrak{a}) = R[\frac{z_1}{x}, \ldots, \frac{z_r}{x}]$, where $z_1, \ldots, z_r$ is a generating set for the ideal $\mathfrak{a}$. In particular, $\mathfrak{a}\,\mathrm{B}(\mathfrak{a}) = x\,\mathrm{B}(\mathfrak{a})$.

An open ideal $\mathfrak{a}$ in R is called *stable* in R if $\mathrm{B}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$, or, equivalently, $\mathfrak{a}\,\mathrm{B}(\mathfrak{a}) = \mathfrak{a}$. It is clear that if $\mathfrak{a}$ is an open ideal in R, then $\mathfrak{a}^n$ is stable for some $n > 0$ and if $\mathfrak{a}^n$ is stable, then $\mathfrak{a}^m$ is stable for every $m \geq n$.

Recall that an ideal $\mathfrak{a}$ of R is said to be *integrally closed* in R if $\mathfrak{a} = \overline{\mathfrak{a}} := \{z \in R \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0 \text{ with } a_j \in \mathfrak{a}^j \text{ for every } j = 1, \ldots, n\}$.

Now we recall the definition of an *Arf ring* studied by Lipman in [9].

3.2. Branch sequence and Arf rings

Let R be a ring as above. Since $\overline{R}$ is a finite R-module, there exists a finite sequence

$$R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R} \tag{3.2.1}$$

of one dimensional semi-local noetherian rings such that for each $1 \leq i \leq m$, the ring R_i is obtained from R_{i-1} by blowing up the radical of R_{i-1}. For each maximal ideal $\mathfrak{n}$ of $\overline{R}$, every local ring $R'_i := (R_i)_{\mathfrak{n} \cap R_i}$ is called *infinitely near to* R. For each $i = 0, \ldots, m$, the multiplicity and the residue field of the local ring R'_i are denoted by $e(R'_i)$ and k_i, respectively. The sequence $R'_0, R'_1, \ldots, R'_m$ is called the *branch sequence of* R *along* $\mathfrak{n}$ and the sequence of pairs $((e(R'_i), [k_i : k_0]),\ i = 0, \ldots, m$ is called the *multiplicity sequence* of R, where $[k_i : k_0]$ denotes the degree of the field extension $k_i | k_0$ (see [9, pp. 661, 669]. In particular, if R is analytically irreducible, residually rational and $R \neq \overline{R}$, then each R_i in (3.2.1) is also analytically irreducible, residually rational; if $\mathfrak{m}_i$ is the maximal ideal of R_i, then the ring R_i is obtained from R_{i-1} by blowing up $\mathfrak{m}_{i-1}$. Further, $R_i = R'_i$ for each $i = 0, \ldots, m$, since $\overline{R}$ is local and $\mathfrak{n}$ is the only maximal ideal in $\overline{R}$.

A semi-local Cohen–Macaulay ring of dimension 1 is called an *Arf ring* if every integrally closed open ideal in R is stable, or, equivalently (see [9, Theorem 2.2]), if A is any local ring infinitely near to R, then A has maximal embedding dimension, i.e., $\mathrm{embdim}(A) = e(A)$. In particular, if a local ring R is Arf, then R has maximal embedding dimension.

In the Proposition 3.3 below, we recall some conditions for a 1-dimensional Cohen–Macaulay local ring R which are equivalent to the equality $\mathrm{embdim}(R) = e(R)$.

3.3. Proposition

Let $(R, \mathfrak{m})$ be a one dimensional local Cohen–Macaulay ring and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then the following statements are equivalent:

(i) $\mathrm{B}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$, *i.e., $\mathfrak{a}$ is stable.*

(ii) *There exists $z \in \mathfrak{a}$ such that $z\mathfrak{a} = \mathfrak{a}^2$.*

In particular, the maximal ideal $\mathfrak{m}$ is stable $\iff$ $\mathrm{embdim}(R) = e(R)$ $\iff$ $\tau_R = e(R) - 1$.

Proof. For the equivalence of (i) and (ii) see [9, 1.8] and [12, 5.1]. If $\mathfrak{a} = \mathfrak{m}$, then the equivalence: $\mathfrak{m}$ is stable $\iff$ $\operatorname{embdim}(R) = e(R)$ is proved in [9, 1.8 and 1.10]. Therefore to complete the proof is it enough to show that: $\tau_R = e_0(R) - 1 \iff x\mathfrak{m} = \mathfrak{m}^2$ for some $x \in \mathfrak{m}$. Let $x \in \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. Then, since R is Cohen–Macaulay, $\ell(R/xR) = e(R)$ and from $xR \subseteq \ldots \subseteq (xR : \mathfrak{m}) \subseteq \ldots \subseteq \mathfrak{m} \subsetneq R$ we have $\tau_R = \ell\left((R : \mathfrak{m})/R\right) = \ell\left((xR : \mathfrak{m})/xR\right) \leq \ell(R/xR) - 1 = e(R) - 1$. Moreover, the equality $\tau_R = e(R) - 1 \iff \ell\left((xR : \mathfrak{m})/xR\right) = \ell(R/xR) - 1 \iff \ell\left(R/(xR : \mathfrak{m})\right) = 1 = \ell(R/\mathfrak{m}) \iff (xR : \mathfrak{m}) = \mathfrak{m} \iff x\mathfrak{m} = \mathfrak{m}^2$.

The following result proved in [4] (see also [5]) shows how the property Arf is described by the type sequence of its value semigroup.

3.4. Proposition ([4, Theorem 1.7-(5)])
Let $(R, \mathfrak{m})$ be a one dimensional noetherian local analytically irreducible, residually rational domain. Let v be the discrete valuation of $\overline{R}$ and let $v(R) = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\} \cup \mathbb{N}_c$ be the value semigroup of R, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n = c$, $\mathfrak{C}$ is the conductor of $\overline{R}$ over R, $n := n(R) = \ell(R/\mathfrak{C})$ and $c = c(R) := \ell(\overline{R}/\mathfrak{C})$. If R is an Arf ring, then $\mathrm{t}_i = \mathrm{v}_i - \mathrm{v}_{i-1} - 1$ is the i-th term in the type sequence of R.

Now we recall the following characterization of Arf rings given in [9].

3.5. Proposition ([9, Theorem 2.2 and Corollary 3.8])
Let $(R, \mathfrak{m})$ be a one dimensional noetherian local analytically irreducible ring and let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. Then R is an Arf ring if and only if $\operatorname{embdim}(R_j) = e(R_j)$ for each $j = 0, \ldots, m$. Moreover, if R is complete with algebraically residue field k, then R is an Arf ring if and only if the value semigroup $v(R)$ of R is $\{0, e(R_0), e(R_0) + e(R_1), \ldots, e(R_0) + \cdots + e(R_{m-2})\} \cup \mathbb{N}_c$, where $c = e(R_0) + \cdots + e(R_{m-2}) + e(R_{m-1})$.

Under the assumptions of 3.5 we can characterize Arf rings using the type sequences of R and of each term in the branch sequence of R.

3.6. Theorem
Let $(R, \mathfrak{m})$ be a complete local analytically irreducible domain with algebraically closed residue field k. Let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. For each $j = 0, \ldots, m-1$, let $\mathfrak{C}_j$ be the conductor of $\overline{R}$ over R_j, and let $n_j = n(R_j)$, $c_j = \ell(\overline{R}/\mathfrak{C}_j)$ and $\mathrm{t}_i(R_j)$ be the i-th term in the type sequence of R_j. Then: R is an Arf ring if and only if for each $j = 0, \ldots, m-1$ and $i = 1, \ldots, n_j$, we have $n_j = m - j$ and $\mathrm{t}_i(R_j) = e(R_{j+i-1}) - 1 = \mathrm{t}_{i+1}(R_{j-1})$.

Proof. ($\Rightarrow$): By the assumptions on R and 3.5, for each $j = 0, \ldots, m-1$ we have R_j is an Arf complete domain with integral closure $\overline{R}$, the same residue field k, $R_j \subsetneq R_{j+1} \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ is the branch sequence of R_j and the value semigroup $v(R_j)$ is $\{0, \mathrm{v}_{1,j}, \mathrm{v}_{2,j}, \ldots, \mathrm{v}_{m-j-1,j}\} \cup \mathbb{N}_{c_j}$, where $\mathrm{v}_{i,j} = e(R_j) + \cdots + e(R_{j+i-1})$, $i = 1, \ldots, m-j-1$ and $c_j = e(R_j) + \cdots + e(R_{m-1})$. Therefore we have $n_j = n(R_j) = (m-j-1)+1 = m-j$. Further, for each $j = 0, \ldots, m-1$, if $\{\mathrm{t}_i(R_j) \mid 1 \le i \le m-j\}$ is the type sequence of R_j, then by 3.4 we have $\mathrm{t}_i(R_j) = \mathrm{v}_{i,j} - \mathrm{v}_{i-1,j} - 1 = e(R_{j+i-1}) - 1 = \mathrm{v}_{i+1,j-1} - \mathrm{v}_{i,j-1} - 1 = \mathrm{t}_{i+1}(R_{j-1})$ for every $1 \le i \le m-j$.

($\Leftarrow$): For each $j = 0, \ldots, m-1$, by assumption, in particular, we have $\tau_{R_j} = \mathrm{t}_1(R_j) = e(R_j) - 1$. Therefore $\mathrm{emdim}(R_j) = e(R_j)$ by 3.3 and hence R is an Arf ring by 3.5.

In particular, for the ready reference we note the following formulas for the i-th term t_i in the type sequence of R, in terms of the types, the multiplicities and the lengths arising from the terms of the branch sequence of R.

3.7. Corollary

Let $(R, \mathfrak{m})$ be an Arf complete local domain with algebraically closed residue field k and let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. Then: $m = n = n(R)$ and for each $i = 1, \ldots, n$, the i-th term t_i in the type sequence of R is given by: $\mathrm{t}_i = \tau(R_{i-1}) = e(R_{i-1}) - 1 = \ell(R_i/R_{i-1})$.

3.8. Corollary

Let $(R, \mathfrak{m})$ be an Arf complete local domain with algebraically closed residue field k and let $\mathrm{B} = \mathrm{B}(\mathfrak{m})$ be the blowing up of R along $\mathfrak{m}$. If $\mathrm{t}_1, \ldots, \mathrm{t}_n$ is the type sequence of R, then $\mathrm{t}_2, \ldots, \mathrm{t}_n$ is the type sequence of B.

Recall that several authors (see for example [6], [16] and references in them) have tried to characterize rings for which the inequality $\ell(\overline{R}/R) \le \tau_R \cdot \ell(R/\mathfrak{C})$ is an equality or to give a classification of the rings according to the value of the integer $\ell^*(R) := \tau_R \cdot \ell(R/\mathfrak{C}) - \ell(\overline{R}/R)$. Now, using the special properties of Arf rings and 3.6 we give some relations between $\ell^*(R)$, the terms in the type sequence of R, $\ell^*(R_j)$ and $e(R_j)$, where $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ is the branch sequence of R. More precisely:

3.9. Theorem

Let $(R, \mathfrak{m})$ be a complete local analytically irreducible domain with algebraically closed residue field k. Let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R and let $e_j = e(R_j)$ be the multiplicity of R_j, $j = 0, \ldots, m$. Let $\mathrm{t}_1, \ldots, \mathrm{t}_n$ be the type sequence of R. Then:

(1) *$\ell^*(R_{m-1}) = 0$ and $\ell^*(R_j) = \sum_{i=j+1}^{m-1} (m-i) \cdot (\mathrm{t}_i - \mathrm{t}_{i+1})$ for $1 \le j \le m-2$.*

(2) *For* $j = 0, \dots, m-2$, *we have* $\ell^*(R) = \ell^*(R_j) + \sum_{i=1}^{j}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}) = \ell^*(R_j) + \sum_{i=1}^{j}(m-i)\cdot(e_{i-1} - e_i)$.

Proof. We shall use the notation as in 3.6. Note that for every $0 \le j \le m$, $n_j = m - j$; in particular, $n = n(R) = n(R_0) = m$. Further, $\mathrm{t}_{j+1}, \dots, \mathrm{t}_m$ is the type sequence of R_j ; in particular, t_m is the type sequence of R_{m-1} and hence $n_{m-1} = n(R_{m-1}) = 1$ and $\ell^*(R_{m-1}) = 0$. Now, for $0 \le j \le m-2$, we have

$$\begin{aligned}\ell^*(R_j) &= \tau(R_j)\cdot\ell(R_j/\mathfrak{C}_j) - \ell(\overline{R}/R_j) = \mathrm{t}_{j+1}\cdot n_j - \sum_{i=j+1}^{m}\ell(R_i/R_{i-1})\\ &= \mathrm{t}_{j+1}(m-j) - \sum_{i=j+1}^{m}\mathrm{t}_i = \sum_{i=j+2}^{m}(\mathrm{t}_{j+1} - \mathrm{t}_i) = \sum_{i=j+1}^{m-1}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}).\end{aligned}$$

This proves (1). Now, since $\mathrm{t}_i = e(R_{i-1}) - 1 = e_{i-1} - 1$ by 3.7, we have $\mathrm{t}_i - \mathrm{t}_{i+1} = e_{i-1} - e_i$ for every $1 \le i \le m-1$ and hence by (1), we have

$$\begin{aligned}\ell^*(R) = \ell^*(R_0) &= \sum_{i=1}^{m-1}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}) = \sum_{i=1}^{j}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}) + \ell^*(R_j)\\ &= \sum_{i=1}^{j}(m-i)\cdot(e_{i-1} - e_i) + \ell^*(R_j).\end{aligned}$$

This proves (2).

3.10. Corollary
With the same assumptions and notation as in 3.9, *we have:*

(1) $e_j \le e_{j-1}$ *and* $\ell^*(R_j) \le \ell^*(R)$ *for every* $j = 1, \dots, m-1$.

(2) *For* $1 \le j \le m-2$, $\ell^*(R_j) = \ell^*(R)$ *if and only if* $e_0 = \ldots = e_{j-1} = e_j$.

Proof. Note that the inequality $e_j \le e_{j-1}$ holds for every analytically irreducible domain. Therefore by 3.9-(2) $\ell^*(R_j) \le \ell^*(R)$ for every $j = 1, \dots, m-2$ and by 3.9-(1) $\ell^*(R_{m-1}) = 0 \le \ell^*(R)$.

(2) Since $m - i > 0$ for every $1 \le i \le j \le m-2$, by 3.9-(2) $\ell^*(R_j) = \ell^*(R)$ if and only if $e_{j-1} = e_j$ for every $j = 1, \dots, m-2$.

Now for complete semigroup rings R such that $\ell^*(R) \le \tau_R$ and $\tau_R = e(R)-1$ using [6, Corollary 2.14], we give another characterization involving the type sequence of R and the type sequences of the rings R_j in the branch sequence of R, Arf rings, $\ell^*(R)$, $\ell^*(R_j)$, $1 \le j \le m-1$ (see 3.12 below). First we shall prove the following lemma concerning two special types of semigroup rings considered in [6, Corollary 2.14].

3.11. Lemma

Let Γ be a numerical semigroup and let $R = K[\![\Gamma]\!]$ be the semigroup ring of Γ over a field K. Let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R and let $e_j = e(R_j)$, $j = 0, \ldots, m-1$.

(1) *Suppose that Γ is generated by $e, pe+1, pe+2, \ldots, pe+(e-1)$, where e, p are positive integers with $e \geq 3$. Then $m = p$, R is an Arf ring and $e_j = e(R) = e$ for every $j = 0, \ldots, p-1$.*

(2) *Suppose that Γ is generated by $e, pe-a, pe-a+1, \ldots, pe-a+(a-1)$, where e, p, a are positive integers with $e \geq 3, p \geq 2$ and $1 \leq a \leq e-1$. Then $m = p$, R is an Arf ring, $e_j = e(R) = e$ for every $j = 0, \ldots, p-2$ and $e_{p-1} = e-a$.*

Proof. (1) It is easy to check that $\operatorname{emdim}(R) = e(R) = e$; in fact the e elements $e, pe+1, pe+2, \ldots, pe+(e-1)$ form a minimal set of generators for the semigroup Γ and $e < pe+1$. For $j = 0, \ldots, p-1$, let Γ_j be the semigroup generated by $e, (p-j)e+1, (p-j)e+2, \ldots, (p-j)e+(e-1)$ and let $\Gamma_p = \mathbb{N}$. Then it is easy to verify that the sequence $R = K[\![\Gamma_0]\!] \subsetneq K[\![\Gamma_1]\!] \subsetneq \ldots \subsetneq K[\![\Gamma_{p-1}]\!] \subsetneq K[\![\Gamma_p]\!] = \overline{R}$ is the branch sequence of R. Therefore $m = p$ and $\operatorname{emdim}(R_j) = e = e_j$ for each $j = 0, \ldots, p-1$ and hence R is Arf by 3.5.

(2) For $j = 0, \ldots, p-2$, let Γ_j be the semigroup generated by $e, (p-j)e-a, (p-j)e-a+1, \ldots, (p-j)e-a+(e-1)$ (note that this is a minimal set of generators for Γ_j), Γ_{p-1} generated by $e-a, e-a+1, \ldots, e, e+1, \ldots, 2e-a-1$ (note that $e-a < e$ and that $e-a, e-a+1, 2e-2a-1$ is a minimal set of generators for Γ_{p-1}) and let $\Gamma_p = \mathbb{N}$. Then it is easy to verify that the sequence $R = K[\![\Gamma_0]\!] \subsetneq K[\![\Gamma_1]\!] \subsetneq \ldots \subsetneq K[\![\Gamma_{p-2}]\!] \subsetneq K[\![\Gamma_{p-1}]\!] \subsetneq K[\![\Gamma_p]\!] = \overline{R}$ is the branch sequence of R and $\operatorname{emdim}(R_j) = e = e_j$ for each $j = 0, \ldots, p-2$, $\operatorname{emdim}(R_{p-1}) = e-a = e_{p-1}$ and hence R is Arf by 3.6.

3.12. Theorem

Let Γ be a numerical semigroup of multiplicity e and type τ_Γ. Let $R = K[\![\Gamma]\!]$ be the semigroup ring of Γ over a field K and let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. Let $\mathrm{t}_1 = \tau_\Gamma, \mathrm{t}_2, \ldots, \mathrm{t}_n$ be the type sequence of R. For a natural number $a \leq \mathrm{t}_1$, the following statements are equivalent:

(i) $\ell^*(R) = a$ *and* $\operatorname{emdim}(R) = e(R)$.

(ii) *R is an Arf ring and*

$$\mathrm{t}_i = \begin{cases} e-1, & \text{if } 1 \leq i \leq m \text{ and } a = 0, \\ e-1, & \text{if } 1 \leq i \leq m-1 \text{ and } a > 0, \\ e-a-1, & \text{if } i = m \text{ and } a > 0. \end{cases}$$

(iii) *R is an Arf ring and*

$$\ell^*(R) = \ell^*(R_j) = \begin{cases} 0, & \textit{if } 1 \le j \le m-1 \textit{ and } a = 0, \\ a, & \textit{if } 1 \le j \le m-2 \textit{ and } a > 0, \end{cases}$$

and if $a > 0$, then $\ell^*(R_{m-1}) = 0$.

Proof. (i) ⇒ (ii): Note that by 3.3 $\operatorname{emdim}(R) = e(R) \iff \tau_R = e(R) - 1$. Therefore by [6, Corollary 2.14] the value semigroup of R is:

$$v(R) = \Gamma = \begin{cases} \mathbb{N}e + \sum_{i=1}^{e-1} \mathbb{N}(pe+i), & \text{if } a = 0 \text{ (see 3.11-(1))}, \\ \mathbb{N}e + \sum_{i=0}^{a-1} \mathbb{N}(pe-a+i), & \text{if } a > 0 \text{ (see 3.11-(2))}. \end{cases}$$

In particular, $n = n(R) = m = p$ and R is an Arf ring (see 3.11). Further, by 3.7 and 3.11, i-th term t_i in the type sequence of R is given by

$$\mathrm{t}_i = \begin{cases} e-1, & \text{if } 1 \le i \le m \text{ and } a = 0, \\ e-1, & \text{if } 1 \le i \le m-1 \text{ and } a > 0, \\ e-a-1, & \text{if } i = m \text{ and } a > 0. \end{cases}$$

(ii) ⇒ (iii): If $a = 0$, then $\ell^*(R) = 0$ and by 3.9-(2) $\ell^*(R_j) = 0$ for every $j = 1, \ldots, m-1$. If $a > 0$, then by 3.9, we have $\ell^*(R_{m-1}) = 0$ and $\ell^*(R) = \mathrm{t}_{m-1} - \mathrm{t}_m = a = \ell^*(R_j)$ for every $j = 1, \ldots, m-2$.

(iii) ⇒ (i): Clearly $\ell^*(R) = a$ by (iii) and since R is an Arf ring, we have $\operatorname{emdim}(R) = e(R)$.

4. Examples

In this section we give some examples of Arf rings and some of not Arf rings. In the following examples R denote the semigroup ring $K[\![\Gamma]\!]$ of the semigroup Γ over a field K. Note that in this case each term R_j in the branch sequence of R is also semigroup ring; in fact, if Γ is generated by $n_1, n_2, \ldots, n_p$ with $n_1 < n_2 < \ldots < n_p$, then $R_1 = K[\![\Gamma_1]\!]$, where $\Gamma_1 = v(R_1)$ is generated by $n_1, n_2 - n_1, \ldots, n_p - n_1$; by repeating this argument we get the result for R_j, $j \ge 2$.

4.1. Example

Let $e, r, r' \in \mathbb{N}$ with $e \ge 3$, $1 \le r$, $1 \le r'$, $r + r' \le e - 1$ and let Γ be the semigroup generated by the sequence $e, e+r, e+r+r', e+r+r'+1, \ldots, 2e+r+r'-1$. We consider the four cases (i) $r' = r = 1$; (ii) $r' = 1$, $r \ge 2$; (iii) $1 < r' \le r$; (iv) $r < r'$ separately.

(a) We first compute the type sequence of R.

Case (i): $(r',r)=(1,1)$: This case is considered in 3.11-(1) ($p=1$). In this case $\mathrm{t}_1=e-1$ is the type sequence of R.

Case (ii): $r'=1$ and $r\geq 2$: In this case $c=e+r$ and $\Gamma\setminus\mathbb{N}_c=\{0,e\}$. Therefore $n=2$ and $\mathrm{v}_1=e$. Further, $\Gamma(1)\setminus\Gamma(0)=T(\Gamma)=[r,e-1]\cup[e+1,e+r-1]$ and $\Gamma(2)\setminus\Gamma(1)=[1,r-1]$. Therefore $\mathrm{t}_1=\tau_R=e-1,\mathrm{t}_2=r-1$ and the type sequence of Γ is $e-1,r-1$. Therefore, R is almost Gorenstein if and only if $r=2$.

Case (iii): $1<r'\leq r$: In this case $c=e+r+r'$ and $\Gamma\setminus\mathbb{N}_c=\{0,e,e+r\}$. Therefore $n=3$ and $\mathrm{v}_1=e$, $\mathrm{v}_2=e+r$. Further, we have

$$\Gamma(1)\setminus\Gamma(0)=T(\Gamma)=\{r\}\cup[r+r',e+r+r'-1]\setminus\{e,e+r\},$$
$$\Gamma(2)\setminus\Gamma(1)=\begin{cases}[r+1,r+r'-1], & \text{if } r=r',\\ [r',r+r'-1]\setminus\{r\}, & \text{if } r'<r,\end{cases}$$

and

$$\Gamma(3)\setminus\Gamma(2)=\begin{cases}[1,r-1], & \text{if } r'=r,\\ [1,r'-1], & \text{if } r'<r.\end{cases}$$

Therefore

$$\mathrm{t}_1=\tau_R=e-1,\qquad \mathrm{t}_2=\begin{cases}r'-1, & \text{if } r'=r,\\ r-1, & \text{if } r'<r,\end{cases}\qquad \mathrm{t}_3=\begin{cases}r-1, & \text{if } r'=r,\\ r'-1, & \text{if } r'<r,\end{cases}$$

and the type sequence of Γ is

$$\begin{cases}e-1,r'-1,r-1, & \text{if } r'=r,\\ e-1,r-1,r'-1, & \text{if } r'<r.\end{cases}$$

Therefore, R is almost Gorenstein if and only if $(r',r)=(2,2)$.

Case (iv): $r<r'$: In this case $c=e+r+r'$ and $\Gamma\setminus\mathbb{N}_c=\{0,e,e+r\}$. Therefore $n=3$ and $\mathrm{v}_1=e$, $\mathrm{v}_2=e+r$. Further, we have $\Gamma(1)\setminus\Gamma(0)=T(\Gamma)=[r+r',e+r+r'-1]\setminus\{e,e+r\}$, $\Gamma(2)\setminus\Gamma(1)=[r',r+r'-1]$ and $\Gamma(3)\setminus\Gamma(2)=[1,r'-1]$. Therefore $\mathrm{t}_1=\tau_R=e-2$, $\mathrm{t}_2=r$, $\mathrm{t}_3=r'-1$ and the type sequence of Γ is $e-2,r,r'-1$. Therefore, R is almost Gorenstein if and only if $(r,r')=(1,2)$.

(b) Now we shall show that R is an Arf ring in cases (i), (ii), (iii) and R is not Arf in case (iv).

Case (i): $(r',r)=(1,1)$: in this case R is an Arf ring (see 3.11-(1) ($p=1$)).

Case (ii): $r'=1$ and $r\geq 2$: In this case, let $\Gamma_0:=\Gamma$, Γ_1 be the numerical semigroup generated by $[r,2r-1]$, $\Gamma_2:=\mathbb{N}$ and let $R_j:=K[\![\Gamma_j]\!]$ for $j=0,1,2$. Then it is easy to see that $e(R_0)=e=\operatorname{embdim}(R_0)$, $e(R_1)=r=\operatorname{embdim}(R_1)$, $e(R_2)=1=\operatorname{embdim}(R_2)$, $\Gamma=\Gamma_0\subsetneq\Gamma_1\subsetneq\Gamma_2=\mathbb{N}$ and $R=R_0\subsetneq R_1\subsetneq R_2=\overline{R}$ is the branch sequence of R. Therefore R is an Arf ring by 3.5.

Case (iii): $1 < r' \leq r$: In this case, let $\Gamma_0 := \Gamma$, Γ_1 be the numerical semigroup generated by $\{r\} \cup [r + r', 2r + r' - 1]$ (note that Γ_1 is minimally generated by $\{r\} \cup ([r + r', 2r + r' - 1] \setminus \{2r\})$), Γ_2 be the numerical semigroup generated by $[r', 2r' - 1]$, $\Gamma_3 := \mathbb{N}$ and let $R_j := K[\![\Gamma_j]\!]$ for $j = 0, 1, 2, 3$. Then it is easy to see that $e(R_0) = e = \operatorname{embdim}(R_0)$, $e(R_1) = r = \operatorname{embdim}(R_1)$, $e(R_2) = r' = \operatorname{embdim}(R_2)$, $e(R_3) = 1 = \operatorname{embdim}(R_3)$, $\Gamma = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \Gamma_3 = \mathbb{N}$ and $R = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = \overline{R}$ is the branch sequence of R. Therefore R is an Arf ring by 3.5.

Case (iv): $1 < r' \leq r$: $r < r'$: In this case, since $e(R) = ne > e - 1 = \operatorname{embdim}(R)$, R is not an Arf ring by 3.5.

4.2. Example

Let $m, d, p \in \mathbb{N}$, $m \geq 2$, $p \geq 1$, $d \geq 1$, $\gcd(m, d) = 1$, Γ be the semigroup generated by an arithmetic sequence $m, m + d, \ldots, m + pd$ and let $R = K[\![\Gamma]\!]$. Let B be the blowing-up of R along the maximal ideal of R. Then (see 3.1) $\mathrm{B} = K[\![\Gamma']\!]$, where Γ' is the semigroup generated by m, d, and so $\operatorname{embdim}(\mathrm{B}) = 2$. Further, by 3.5:

(i) If $d = 1$, then R is Arf if and only if $\operatorname{embdim}(R) = m$ (in fact, in this case, $\mathrm{B} = K[\![T]\!]$). The case $d = 1$ is also contained in Proposition 4.4 of the article [3].

(ii) If $d = 2$ or $m = 2$, then for every $j \geq 2$ the j-th term in the branch sequence of R is $R_j = K[\![\Gamma_j]\!]$, where Γ_j is the semigroup generated by $2, 2n + 1$ for some integer $n \geq 1$ and so $\operatorname{embdim}(R_j) = e(R_j)$ for every $j \geq 1$. Therefore, R is an Arf ring if and only if $\operatorname{embdim}(R) = m$; in particular, if $m = 2$, then R is an Arf ring.

(iii) If $d \geq 3$ and $m \geq 3$, then $e(\mathrm{B}) \geq 3$, $\operatorname{embdim}(\mathrm{B}) < e(\mathrm{B})$ and hence R is not an Arf ring.

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