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Complementary means and double sequences

Abstract. We look after the complementary means with respect to a weighted geometric mean of Stolarsky means in the family of Gini means and in the family of Stolarsky means.

1. Means

Usually, the means are given by the following

Definition 1
A *mean* (on the interval J) is a function $M\colon J^2 \longrightarrow J$, which has the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \qquad \forall\, a,b \in J.$$

Each mean is *reflexive*, that is

$$M(a,a) = a, \qquad \forall\, a \in J,$$

which will be used also as a definition of $M(a,a)$ if it is necessary.
A mean can have additional properties.

Definition 2
The mean M is called:

a) *symmetric* if
$$M(a,b) = M(b,a), \qquad \forall\, a,b \in J;$$

b) *strict at the left* if
$$M(a,b) = a \implies a = b,$$

strict at the right if
$$M(a,b) = b \implies a = b,$$

and *strict* if is strict at the left and strict at the right.

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We can compose three means M, N and P on J to define another mean $P(M,N)$ by

$$P(M,N)(a,b) = P(M(a,b),N(a,b)), \qquad a,b \in J.$$

Most of the usual means are defined on $\mathbb{R}_+$. So are the *Stolarsky* (or *extended*) means given by

$$\mathcal{E}_{r,s}(a,b) = \left(\frac{s}{r}\cdot\frac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}, \qquad r\cdot s\cdot(r-s)\neq 0,\ a\neq b$$

and the weighted Gini means defined by

$$\mathcal{B}_{r,s;\lambda}(a,b) = \left[\frac{\lambda\cdot a^r+(1-\lambda)\cdot b^r}{\lambda\cdot a^s+(1-\lambda)\cdot b^s}\right]^{\frac{1}{r-s}}, \qquad r\neq s\ ,$$

with $\lambda \in [0,1]$ fixed. Weighted Lehmer means, $\mathcal{C}_{r;\lambda} = \mathcal{B}_{r,r-1;\lambda}$ and weighted power means $\mathcal{P}_{r;\lambda} = \mathcal{B}_{r,0;\lambda}$ $(r\neq 0)$ are also used. We can remark that $\mathcal{P}_{0;\lambda} = \mathcal{G}_\lambda = \mathcal{B}_{r,-r;\lambda}$ is the weighted geometric mean. Also

$$\mathcal{B}_{r,s;0}= \mathcal{C}_{r;0} = \mathcal{P}_{r;0} = \Pi_2 \quad \text{and} \quad \mathcal{B}_{r,s;1}= \mathcal{C}_{r;1} = \mathcal{P}_{r;1} = \Pi_1\ ,$$

where we denote by Π_1 and Π_2 respectively the first and the second projection defined by

$$\Pi_1(a,b) = a, \quad \Pi_2(a,b) = b, \qquad \forall\, a,b \geq 0.$$

2. Gaussian double sequences

The well known arithmetic-geometric process of Gauss was generalized for arbitrary means as follows. Consider two means M and N defined on the interval J and two initial values $a,b\in J$.

Definition 3
The pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by

$$a_{n+1} = M(a_n,b_n) \quad \text{and} \quad b_{n+1} = N(a_n,b_n)\,, \qquad n\geq 0\,, \tag{1}$$

where $a_0 = a$, $b_0 = b$, is called a *Gaussian double sequence.*

Definition 4
The mean M is *compoundable in the sense of Gauss (or G -compoundable) with the mean* N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by (1) are convergent to a common limit $M\otimes N(a,b)$ for each $a,b\in J$.

The function $M\otimes N$ defines a mean which is called *Gaussian compound mean* (or *G-compound mean*).

The study of convergence is quite complicated. A general result was proved in [6]. If the means M and N are continuous and strict at the left then M and N are G-compoudable. There is also a variant for means which are strict at the right. The result is not valid if we assume one mean to be strict at the left and the other strict at the right. For example the means Π_1 and Π_2 are not G-compoudable (in any order). But, as we proved in [11], we can G-compose a continuous strict mean with any mean. A similar result was given recently in [5].

Some G-compound means can be determined using a characterization based on the following result, proved in [2] (and rediscovered in [5]).

Theorem 5 (Invariance Principle)
Suppose that $M \otimes N$ exists and is continuous. Then $M \otimes N$ is the unique mean P which is (M, N)-invariant, that is

$$P(M, N) = P. \tag{2}$$

In fact, this is the way in which Gauss proved that the arithmetic-geometric G-compound mean can be represented by

$$\mathcal{A} \otimes \mathcal{G}(a, b) = \frac{\pi}{2} \cdot \left[\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \right]^{-1}.$$

As usual $\mathcal{A} = \mathcal{P}_{1;\frac{1}{2}}$ and $\mathcal{G} = \mathcal{G}_{\frac{1}{2}}$.

3. Complementary means

As we can see in the last example, the product of two simply means, like $\mathcal{A}$ and $\mathcal{G}$, can be very complicated. So, to obtain some results, we change the point of view from the Invariance Principle. The determination of an invariant mean is very difficult. To simplify the search, we start with the following definitions given in [7]: the means $2\mathcal{A} - M$ and $\frac{\mathcal{G}^2}{M}$ are called complementary of M and inverse of M, respectively. We proposed in [11] a more general definition (that was given again in [10]).

Definition 6
The mean N is called *P-complementary to M* (or *complementary with respect to P of M*) if it satisfies (2).

If the P-complementary of M exists and is unique, we denote it by M^P. It is easy to verify that $M^{\mathcal{A}} = 2\mathcal{A} - M$ and $M^{\mathcal{G}} = \frac{\mathcal{G}^2}{M}$, thus the definitions given in [7] are indeed special cases. An existence theorem of complementary means for symmetric means, was proved in [10]. It is easy to verify the following results.

PROPOSITION 7
For every continuous strict mean M we have

$$M^M = M, \quad \Pi_1^M = \Pi_2, \quad M^{\Pi_2} = \Pi_2$$

and if M is moreover symmetric then

$$\Pi_2^M = \Pi_1 \,.$$

We shall call these results as trivial cases *of complementariness.*

Many non trivial examples can be found in [12]. In fact, for the ten Greek means (or neo-Pytagorean means, as they are called in [2]), we determined the ninety complementaries of a mean with respect to another. They are done by direct computation. To make other determinations of complementaries, we use series expansions. We try to identify the complementary of a mean from a given family of means in an other family of means.

To illustrate this method, we study the complementariness with respect to the weighted geometric mean $\mathcal{G}_\lambda$. Denote the $\mathcal{G}_\lambda$-complementary of M by $M^{\mathcal{G}(\lambda)}$ and we call it *generalized inverse of M*. For example, in [3] one was looking after the generalized inverse of a weighted Gini mean in the same family. Here we determine the generalized inverse of a weighted Gini mean in the family of Stolarsky means and converse, the generalized inverse of a Stolarsky mean in the family of weighted Gini means.

We omit anywhere to write λ if it is equal to $\frac{1}{2}$. In this case the generalized inverse of M is simply the inverse of M. For instance, in [1] they are determined the inverses of Stolarsky means in the same family of means.

4. Series expansion of means

For the study of some problems related to means, in [9] the power series expansion is used. Usually, for a mean M the series of the normalized function $M(1, 1-x)$, $x \in (0,1)$ is considered.

For example, in [8] it is proved that the extended mean $\mathcal{E}_{r,s}$ has the following first terms of the power series expansion

$$\begin{aligned}
&\mathcal{E}_{r,s}(1, 1-x) \\
&\quad = 1 - \frac{x}{2} + (r+s-3) \cdot \frac{x^2}{24} + (r+s-3) \cdot \frac{x^3}{48} \\
&\qquad - \big[2(r^3 + r^2 s + r s^2 + s^3) - 5(r+s)^2 - 70(r+s) + 225\big] \cdot \frac{x^4}{5760} \\
&\qquad - \big[2(r^3 + r^2 s + r s^2 + s^3) - 5(r+s)^2 - 30(r+s) + 105\big] \cdot \frac{x^5}{3840} + \cdots .
\end{aligned}$$

Also in [4] it is given the series expansion of the weighted Gini mean, for $r \neq 0$,

$$
\begin{aligned}
&\mathcal{B}_{q,q-r;\nu}(1,1-x)\\
&\quad= 1-(1-\nu)\cdot x+\nu\,(1-\nu)\,(2q-r-1)\cdot\frac{x^2}{2!}\\
&\qquad-\nu\,(1-\nu)\cdot\{\nu[6q^2-6q\,(r+1)+(r+1)\,(2r+1)]\\
&\qquad\qquad-3q\,(q-r)-(r-1)\,(r+1)\}\cdot\frac{x^3}{3!}\\
&\qquad-\nu\,(1-\nu)\cdot\{\nu^2[-24q^3+36q^2\,(r+1)-12q\,(r+1)\,(2r+1)\\
&\qquad\qquad+(r+1)\,(2r+1)\,(3r+1)]\\
&\qquad\qquad+\nu[24q^3-12q^2\,(3r+1)+12q\,(r+1)\,(2r-1)\\
&\qquad\qquad-3\,(r+1)\,(2r+1)\,(r-1)]\\
&\qquad\qquad-4q^3+6q^2\,(r-1)-2q\left(2r^2-3r-1\right)\\
&\qquad\qquad+(r-2)\,(r-1)\,(r+1)\}\cdot\frac{x^4}{4!}\\
&\qquad-\nu\,(1-\nu)\cdot\left\{\nu^3\left[120q^4-240q^3\,(r+1)+120q^2(r+1)(2r+1)\right.\right.\\
&\qquad\qquad-20q(r+1)(2r+1)(3r+1)\\
&\qquad\qquad\left.+(r+1)(2r+1)(3r+1)(4r+1)\right]\\
&\qquad\qquad+\nu^2\left[-180q^4+180q^3(2r+1)-90q^2(r+1)(4r-1)\right.\\
&\qquad\qquad+30q(r+1)(2r+1)(3r-2)\\
&\qquad\qquad\left.-6(r-1)(r+1)(2r+1)(3r+1)\right]\\
&\qquad\qquad+\nu\left[70q^4-20q^3(7r-2)+10q^2(14r^2-6r-9)\right.\\
&\qquad\qquad-10q(r+1)(7r^2-12r+3)\\
&\qquad\qquad\left.+(r-1)(2r+1)(7r-11)(r+1)\right]\\
&\qquad\qquad-5q^4+10q^3(r-2)-5q^2(2r^2-6r+3)\\
&\qquad\qquad+5q(r-2)(r^2-2r-1)\\
&\qquad\qquad\left.-(r+1)(r-1)(r-2)(r-3)\right\}\cdot\frac{x^5}{5!}+\cdots.
\end{aligned}
$$

In the special case $r=1$ we get the series expansion of the weighted Lehmer means, while for $q=r\neq 0$ we get the series expansion of the weighted power means.

5. Generalized inverses

In [3] the series expansion of the generalized inverse of $\mathcal{B}_{p,p-r;\mu}$ was given.

Theorem 8
The first terms of the series expansion of the generalized inverse of $\mathcal{B}_{p,p-q;\mu}$ are

$$
\begin{aligned}
&\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-q;\mu}(1,1-x)\\
&= 1-(\alpha\mu-\alpha+1)\cdot x-\alpha(1-\mu)\left[(\alpha+2p-q)\mu-(\alpha-1)\right]\cdot\frac{x^2}{2!}\\
&\quad+\alpha(1-\mu)\big\{\big[6p^2+6(\alpha-q)p+(\alpha-q)(\alpha-2q)\big]\ \mu^2\\
&\qquad\qquad-\big[3p^2-3p(q-2\alpha)+(2\alpha-q)\ (\alpha-q)\big]\mu\\
&\qquad\qquad+\ (\alpha-1)(\alpha+1)\big\}\cdot\frac{x^3}{3!}\\
&\quad-\alpha(1-\mu)\big\{\big[24p^3+36p^2(\alpha-q)\ +12(\alpha-q)(\alpha-2q)p\\
&\qquad\qquad+(\alpha-q)(\alpha-2q)\ (\alpha-3q)\big]\mu^3\\
&\qquad\qquad+\big[-24p^3+12p^2(3q-4\alpha-1)\\
&\qquad\qquad\quad-12(2\alpha-2q+1)(\alpha-q)p\\
&\qquad\qquad\quad-(\alpha-2q)(\alpha-q)\ (3\alpha+2-3q)\big]\mu^2\\
&\qquad\qquad+\big[4p^3+6(2\alpha-q+1)p^2\\
&\qquad\qquad\quad+2\big(6\alpha(2\alpha-2q+1)-3q+2q^2-1\big)p\\
&\qquad\qquad\quad+(\alpha-q)(3\alpha^2+4\alpha-3q\alpha-2q+q^2-1)\big]\mu\\
&\qquad\qquad-\ (\alpha-1)(\alpha+1)(\alpha+2)\big\}\cdot\frac{x^4}{4!}\\
&\quad+\alpha(1-\mu)\big\{\big[120p^4+240p^3(\alpha-q)+120(\alpha-q)(\alpha-2q)p^2\\
&\qquad\qquad+20(\alpha-q)(\alpha-2q)(\alpha-3q)p\\
&\qquad\qquad+(\alpha-q)\cdot(\alpha-2q)(\alpha-3q)\ (\alpha-4q)\big]\mu^4\\
&\qquad\qquad+\big[-180p^4+60(6q-7\alpha-2)p^3\\
&\qquad\qquad\quad-90p^2(3\alpha-4q+2)(\alpha-q)\\
&\qquad\qquad\quad-30(\alpha-q)(\alpha-2q)(2\alpha+2-3q)p\\
&\qquad\qquad\quad-(\alpha-q)(\alpha-2q)(\alpha-3q)\ (4\alpha+5-6q)\big]\mu^3\\
&\qquad\qquad+\big[70p^4+20(10\alpha-7q+6)p^3\\
&\qquad\qquad\quad+10(-30q\alpha+18\alpha^2+24\alpha+3+14q^2-18q)p^2\\
&\qquad\qquad\quad+10(\alpha-q)\big(6\alpha^2+12\alpha-12q\alpha+7q^2-12q+3\big)p\\
&\qquad\qquad\quad+\big(6\alpha^2-12q\alpha+15\alpha+5+7q^2-15q)(\alpha-2q)\ (\alpha-q)\big]\mu^2\\
&\qquad\qquad+\big[-5p^4+10(q-2-2\alpha)p^3\\
&\qquad\qquad\quad+\big(30q\alpha-30\alpha^2-60\alpha-15\ -10q^2+30q\big)p^2\\
&\qquad\qquad\quad-52\alpha+(2-q)\big(2\alpha^2-2q\alpha+4\alpha-2q+q^2-1)p\\
&\qquad\qquad\quad-(\alpha-q)\big(4\alpha^3-6q\alpha^2+15\alpha^2-15q\alpha+10\alpha+4q^2\alpha\\
&\qquad\qquad\qquad\qquad-5+5q^2-q^3-5q)\big]\mu
\end{aligned}
$$

$$+ (\alpha - 1)(\alpha + 1)(\alpha + 2)(\alpha + 3)\} \cdot \frac{x^5}{5!} + \cdots,$$

where $\alpha = \frac{\lambda}{1-\lambda}$.

Using it, we can prove the following result.

THEOREM 9
The relation

$$\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-q;\mu} = \mathcal{E}_{r,s}$$

holds if and only if we are in one of the following three cases:

$$\text{(i)} \quad \mathcal{B}^{\mathcal{G}(\frac{1}{3})}_{p,p-q;0} = \mathcal{E}_{r,-r}\,;$$

$$\text{(ii)} \quad \mathcal{B}^{\mathcal{G}}_{p,-p} = \mathcal{E}_{r,-r}\,;$$

$$\text{(iii)} \quad \mathcal{B}^{\mathcal{G}}_{p,0} = \mathcal{E}_{-p,-2p}\,,$$

or in equivalent cases, taking into account that $\mathcal{B}_{s,r;\nu} = \mathcal{B}_{r,s;\nu}$ *and* $\mathcal{E}_{r,s} = \mathcal{E}_{s,r}$.

Proof. Equating the coefficients of x , in $\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-q;\mu}(1, 1-x)$ and in $\mathcal{E}_{r,s}(1, 1-x)$, we have the condition

$$2\alpha\mu = 2\alpha - 1. \tag{3}$$

Then the coefficients of x^2 give the condition

$$(r + s)\,\alpha = 3\,(2\alpha - 1)\,(q - 2p)\,, \tag{4}$$

and the coefficients of x^3 are equal if, moreover,

$$\mu\,(2\mu - 1)\left(3p^2 - 3pq + q^2\right) = 0. \tag{5}$$

This gives the cases:

1) $\mu = 0$; thus from (3), $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$ and from (4), $s = -r$, so (i) and

2) $\mu = \frac{1}{2}$ which implies, from (3) and (4), $\alpha = 1$ and

$$r + s = 3\,(q - 2p)\,. \tag{6}$$

Equating also the coefficients of x^4, we obtain in this case:

$$(2p - q)\left(2q^2 - 3qr + r^2 - 13pq + 6pr + 13p^2\right) = 0.$$

So, the case 2) splits into the cases:

2.1) $2p = q$, giving from (4), $r = -s$ and leading to (ii)

and

2.2)

$$(q - r)(2q - r) = p(13q - 6r - 13p). \qquad (7)$$

The coefficients of x^5 are equal in this case. Equating also the coefficients of x^6, we obtain

$$p(p - q)(2p - q)\left(11p^2 - 11pq + 3q^2\right) = 0. \qquad (8)$$

Thus, we get a new splitting in the cases:

2.2.1) $p = 0$, $r = q$ (from (7)) and $s = 2q$ (from (6)), so we have (iii);

2.2.2) $p = 0$, $r = 2q$ (from (7)) and $s = q$ (from (6)), so we have again (iii);

2.2.3) $p = q$, $r = -q$ (from (7)) and $s = -2q$ (from (6)), so we have (iii);

2.2.4) $p = q$, $r = -2q$ (again from (7)) and $s = -q$ (from (6)), so we have also (iii);

2.2.5) $q = 2p$, $r = \pm p\sqrt{5}$ (from (7)) and $s = -r$ (from (6)), so we have a special case of (ii).

We have no other possibilities in (8) or (5). By direct computation, we verify that the four obtained cases are valid. In fact they reduce to the following results:

$$\text{(i)}\ \ \Pi_1^{\mathcal{G}(\frac{1}{3})} = \mathcal{G}; \qquad \text{(ii)}\ \ \mathcal{G}^{\mathcal{G}} = \mathcal{G}; \qquad \text{(iii)}\ \ \mathcal{P}_p^{\mathcal{G}} = \mathcal{P}_{-p}\,.$$

Corollary 10

The relation

$$\mathcal{B}_{p,p-q;\mu}^{\mathcal{G}(\lambda)} = \mathcal{E}_{r,s}$$

holds only in the following nontrivial cases:

$$\text{(i)} \qquad \mathcal{B}_{p,p-q;0}^{\mathcal{G}(\frac{1}{3})} = \mathcal{E}_{r,-r}\,;$$

$$\text{(ii)} \qquad \mathcal{B}_{p,0}^{\mathcal{G}} = \mathcal{E}_{-p,-2p}\,,$$

or in equivalent cases, taking into account that $\mathcal{B}_{s,r;\nu} = \mathcal{B}_{r,s;\nu}$ *and* $\mathcal{E}_{r,s} = \mathcal{E}_{s,r}$ *.*

As it is done in [3], we can give the series expansion of the generalized inverse of $\mathcal{E}_{r,s}$, using Euler's formula: if the function f has the Taylor series expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$, p is a real number, and $[f(x)]^p = \sum_{n=0}^{\infty} b_n x^n$, then we have the recurrence relation

$$\sum_{k=0}^{n} [k(p + 1) - n]\, a_k b_{n-k} = 0, \qquad n \geq 0.$$

Theorem 11

The first terms of the series expansion of the generalized inverse of $\mathcal{E}_{r,s}$ *are*

$$
\begin{aligned}
&\mathcal{E}_{r,s}^{\mathcal{G}(\lambda)}(1,1-x)\\
&\quad= 1+\frac{1}{2}(\alpha-2)\cdot x-\frac{\alpha}{24}\left[r+s-3(\alpha-2)\right]\cdot x^2\\
&\qquad-\frac{\alpha}{48}\left[\alpha(r+s)-\left(\alpha^2-4\right)\right]\cdot x^3\\
&\qquad+\frac{\alpha}{5760}\left[2\left(r^3+s^3\right)+5\alpha(r+s)^2+2rs(r+s)\right.\\
&\qquad\qquad\left.+10(r+s)\left(2-6\alpha-3\alpha^2\right)+15\left(\alpha^2-4\right)(\alpha+4)\right]\cdot x^4\\
&\qquad+\frac{\alpha(\alpha+2)}{11520}\left[2\left(r^3+s^3\right)+5\alpha(r+s)^2+20rs(r+s)\right.\\
&\qquad\qquad+10(r+s)\left(2-4\alpha-\alpha^2\right)\\
&\qquad\qquad\left.+3(\alpha-2)(\alpha+4)(\alpha+6)\right]\cdot x^5+\cdots,
\end{aligned}
$$

where $\alpha=\frac{\lambda}{1-\lambda}$.

Using it we can prove some new results.

THEOREM 12
The relation

$$\mathcal{E}_{r,s}^{\mathcal{G}(\lambda)}=\mathcal{B}_{p,p-q;\mu}$$

holds if and only if we are in one of the following cases:

$$
\begin{aligned}
&\text{(i)} && \mathcal{E}_{r,s}^{\mathcal{G}(0)}=\mathcal{B}_{p,p-q;0}\,;\\
&\text{(ii)} && \mathcal{E}_{r,-r}^{\mathcal{G}(\frac{2}{3})}=\mathcal{B}_{p,p-q;1}\,;\\
&\text{(iii)} && \mathcal{E}_{r,-r}^{\mathcal{G}}=\mathcal{B}_{p,-p}\,;\\
&\text{(iv)} && \mathcal{E}_{2s,s}^{\mathcal{G}}=\mathcal{B}_{0,-s}
\end{aligned}
$$

or in the equivalent cases, taking into account the properties $\mathcal{B}_{s,r;\nu}=\mathcal{B}_{r,s;\nu}$ *and* $\mathcal{E}_{r,s}=\mathcal{E}_{s,r}$ *.*

Proof. Equating the coefficients of x in $\mathcal{E}_{r,s}^{\mathcal{G}(\lambda)}(1{,}1-x)$ and in $\mathcal{B}_{p,p-q;\mu}(1{,}1-x)$ we have the condition

$$\alpha=2\mu. \tag{9}$$

Then, the equality of the coefficients of x^2 gives the condition

$$\alpha\left[r+s+(1-\mu)(2p-q)\right]=0.$$

We have thus

1) $\alpha=0$, which gives $\mu=0$ and so the equality (i)

or

2)

$$r + s + (1 - \mu)(2p - q) = 0. \tag{10}$$

Replacing (9) and (10) into the coefficients of x^3, we get the condition

$$\alpha(\alpha - 1)(\alpha - 2)\left(3p^2 - 3pq + q^2\right) = 0$$

The last factor cannot be zero for $q \neq 0$.

So, we have only the following possibilities:

2.1) $\alpha = 2$, so, by (9) and (10), $\mu = 1$ and $r = -s$, thus (ii)

or

2.2) $\alpha = 1$, for which we have to consider (9), (10) and also the equality of the coefficients of x^4, giving

$$(q - 2p)\left(2q^2 - 3qs - 13pq + 13p^2 + s^2 + 6ps\right) = 0.$$

This also splits into:

2.2.1) $q = 2p$, so $r = -s$, giving (iii);

and

2.2.2)

$$2q^2 - 3qs - 13pq + 13p^2 + s^2 + 6ps = 0. \tag{11}$$

In this case, the coefficients of x^5 are equal, while the coefficients of x^6 give

$$p(p - q)(2p - q)\left(11p^2 - 11pq + 3q^2\right) = 0.$$

We have so the possibilities:

2.2.2.1) $p = 0$, for which (11) becomes

$$(q - s)(2q - s) = 0,$$

obtaining two variants of (iv);

2.2.2.2) $p = q$, for which (11) becomes

$$(q + s)(2q + s) = 0,$$

obtaining again two variants of (iv);

2.2.2.3) $p = 2q$, for which (6) becomes

$$28q^2 + 9qs + s^2 = 0,$$

which has no solution $q \neq 0$.

The validity of the cases (i)-(iv) can be verified directly. In fact they can be rewritten in order as:

(i) $\mathcal{B}^{\Pi_2}_{p,p-q,\mu} = \Pi_2$; (ii) $\mathcal{G}^{\mathcal{G}(2/3)} = \Pi_1$; (iii) $\mathcal{G}^{\mathcal{G}} = \mathcal{G}$; (iv) $\mathcal{P}^{\mathcal{G}}_s = \mathcal{P}_{-s}$.

Corollary 13
The relation

$$\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s} = \mathcal{B}_{p,q;\mu}$$

holds only in the following two nontrivial cases:

$$\text{(i)} \quad \mathcal{E}^{\mathcal{G}(\frac{2}{3})}_{r,-r} = \mathcal{B}_{p,q;1}\,;$$

$$\text{(ii)} \quad \mathcal{E}^{\mathcal{G}}_{2s,s} = \mathcal{B}_{0,-s}\,.$$

Theorem 14
The relation

$$\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s} = \mathcal{E}_{p,q}$$

holds if and only if we are in one of the following cases:

$$\text{(i)} \quad \mathcal{E}^{\mathcal{G}}_{r,-r} = \mathcal{E}_{p,-p}\,;$$

$$\text{(ii)} \quad \mathcal{E}^{\mathcal{G}}_{r,s} = \mathcal{E}_{-r,-s}\,.$$

or in an equivalent case, taking into account the property $\mathcal{E}_{r,s} = \mathcal{E}_{s,r}$.

Proof. Equating the coefficients of x, in $\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s}(1, 1-x)$ and in $\mathcal{E}_{p,q}(1, 1-x)$ we have the condition $\alpha = 1$, thus $\lambda = \frac{1}{2}$. Then, the equality of the coefficients of x^2 gives the condition

$$r + s + p + q = 0.$$

In these conditions, the coefficients of x^3 are allways equal, but those of x^4 are equal only if

$$(r + q)\,(s + q)\,(r + s) = 0.$$

This implies:

1) $r = -q$, $s = -p$ giving a variant of (ii);
2) $s = -q$, $r = -p$ giving (ii);
3) $s = -r$, $q = -p$ which implies (i).

Corollary 15
The relation

$$\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s} = \mathcal{E}_{p,q}$$

holds only in the following nontrivial case:

$$\mathcal{E}^{\mathcal{G}}_{r,s} = \mathcal{E}_{-r,-s}\,.$$

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