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Partial difference equations arising from the Cauchy-Riemann equations

Abstract. We consider some functional equations arising from the Cauchy–Riemann equations, and certain related functional equations. First we propose a new functional equation (E.1) below, over a 2-divisible Abelian group, which is a discrete version of the Cauchy-Riemann equations, and give the general solutions of (E.1). Next we study a functional equation which is equivalent to (E.1). Further we propose and solve partial difference-differential functional equations and nonsymmetric partial difference equations which are also arising from the Cauchy–Riemann equations.

1. Introduction

Let $(G,+)$ be an additive Abelian group in which it is possible to divide by 2. Let $\mathbb{C}$ be the field of complex numbers. The main aim of this note is to determine the general solution of the following new functional equation

$$f(x+t,y)-f(x-t,y)=-i[f(x,y+t)-f(x,y-t)] \tag{E.1}$$

for all $x,y,t\in G$, where $f\colon G\times G\longrightarrow\mathbb{C}$ and i is the imaginary unit.

Let $\mathbb{R}$ be the field of real numbers. For a function $f\colon\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$ we define the divided partial difference operators $\triangle_{x,t}$ and $\triangle_{y,t}$ by

$$(\triangle_{x,t}f)(x,y)=\frac{f(x+t,y)-f(x,y)}{t}$$

and

$$(\triangle_{y,t}f)(x,y)=\frac{f(x,y+t)-f(x,y)}{t},$$

respectively. Then the partial difference equation

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$$\triangle_{x,t} f = -i \triangle_{y,t} f$$

may be considered as a discrete analogue of the Cauchy–Riemann equation

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

This equation may be rewritten in the form

$$f(x+t,y) - f(x,y) = -i[f(x,y+t) - f(x,y)] \tag{E.2}$$

for all $x, y, t \in \mathbb{R}$, which has a simple geometric interpretation on the plane.

Equation (E.2) is considered in the papers of J. Aczél and S. Haruki 1981 [3], and S. Haruki 1986 [6]. The authors show, among others, that (E.2) does not lead essentially beyond a linear function in the case when $\mathbb{R}$ is replaced by an arbitrary monoid M. Further, in the paper of S. Haruki and C.T. Ng 1994 [7] the general solution of (E.2) is obtained in more general algebraic structures than M and $\mathbb{C}$.

It is natural to ask what happens if, instead of operators $\triangle_{x,t}$ and $\triangle_{y,t}$, we impose the divided partial mean difference operators $\bigtriangledown_{x,t}$ and $\bigtriangledown_{y,t}$ defined by

$$(\bigtriangledown_{x,t} f)(x,y) = \frac{f(x+t,y) - f(x-t,y)}{2t}$$

and

$$(\bigtriangledown_{y,t} f)(x,y) = \frac{f(x,y+t) - f(x,y-t)}{2t}.$$

In this case we have the partial difference equation

$$\bigtriangledown_{x,t} f = -i \bigtriangledown_{y,t} f,$$

which is also a discrete analogue of the Cauchy–Riemann equation. This leads to the above functional equation (E.1) which also has a simple geometric interpretation on the plane. As a main result of this note we show that equation (E.1) for $f\colon G \times G \longrightarrow \mathbb{C}$ does not lead essentially beyond a quadratic function.

In Section 2 we determine the general and the regular solutions (when G is replaced by $\mathbb{R}$) of equation (E.1).

We also show in Section 3 that similar results hold for certain related functional equations. In Section 3.1 we consider the functional equation

$$f(x+t,y+t) - f(x-t,y-t) = -i[f(x-t,y+t) - f(x+t,y-t)]. \tag{E.3}$$

In Section 3.2 we study the partial difference-differential equations

$$\frac{\partial f(x,y)}{\partial x} = -i \left[\frac{f(x,y+t) - f(x,y-t)}{2t} \right],$$

$$\frac{f(x+t,y) - f(x-t,y)}{2t} = -i \frac{\partial f(x,y)}{\partial y}.$$

Finally in Section 3.3 we propose and solve several functional equations of nonsymmetric type, which are also analogous to the Cauchy–Riemann equation.

2. Functional equation (E.1)

2.1. The general solution

A function $A^1\colon G \longrightarrow \mathbb{C}$ is said to be *additive* if A^1 satisfies

$$A^1(x+y) = A^1(x) + A^1(y) \qquad \text{for all } x, y \in G.$$

A function $A_2\colon G\times G \longrightarrow \mathbb{C}$ is said to be *bi-additive* if A_2 satisfies both equations

$$A_2(x+y,z) = A_2(x,z) + A_2(y,z)$$

and

$$A_2(x,y+z) = A_2(x,y) + A_2(x,z)$$

for all $x, y, z \in G$. A function $A^2\colon G \longrightarrow \mathbb{C}$ is *the diagonalization of the* A_2 if

$$A^2(x) = A_2(x,x),$$

whenever $A_2\colon G \times G \longrightarrow \mathbb{C}$ is symmetric and additive in each argument.

Our main result of this note is as follows.

Theorem 2.1
A function $f\colon G \times G \longrightarrow \mathbb{C}$ satisfies equation (E.1) *for all $x, y, z \in G$ if and only if there exist*

(i) *a complex constant A^0,*

(ii) *an additive function $A^1\colon G \longrightarrow \mathbb{C}$,*

(iii) *a symmetric bi-additive function $A_2\colon G \times G \longrightarrow \mathbb{C}$*

such that

$$f(x,y) = A^0 + A^1(x) + iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x,y) \tag{S.1}$$

for all $x, y \in G$, where $A^2\colon G \longrightarrow \mathbb{C}$ is the diagonalization of A_2.

We impose the following notations. Define the shift operators X^t and Y^t by

$$(X^t f)(x,y) = f(x+t,y) \quad \text{and} \quad (Y^t f)(x,y) = f(x,y+t) \qquad \text{for all } x, y, t \in G.$$

In particular $1 = X^0 = Y^0$ denote the identity operator. Further, define the partial mean difference operators $\delta_{x,t}$ and $\delta_{y,t}$ by

$$\delta_{x,t} = X^t - X^{-t} \quad \text{and} \quad \delta_{y,t} = Y^t - Y^{-t} \qquad \text{for all } x, y, t \in G.$$

Notice that the ring of operators generated by this family of operators is commutative and distributive.

In order to prove Theorem 2.1 we need the following two lemmas. One of them is:

Lemma 2.1
If a function $f: G \times G \longrightarrow \mathbb{C}$ *satisfies equation* (E.1) *for all* $x, y, t \in G$, *then* f *also satisfies each one of the following three functional equations*

$$(\delta^3_{x,t} f)(x,y) = 0 \qquad \textit{and} \qquad (\delta^3_{y,t} f)(x,y) = 0 \tag{2.1}$$

$$((\delta^2_{x,t} + \delta^2_{y,t})f)(x,y) = 0 \tag{2.2}$$

or as the expanded form (2t replaced by t)

$$f(x+t,y) + f(x-t,y) + f(x,y+t) + f(x,y-t) = 4f(x,y) \tag{2.3}$$

for all $x, y \in G$.

The above Lemma 2.1 shows that equation (E.1) yields equation (2.3). On the other hand, J. Aczél, H. Haruki, M.A. McKiernan and G.N. Sakovič 1968 [2, p. 43, Lemma 3] proved that equation (2.3) is equivalent to the Haruki functional equation (M.A. McKiernan [11], H. Światak [13], among others)

$$f(x+t,y+t)+f(x+t,y-t)+f(x-t,y+t)+f(x-t,y-t) = 4f(x,y). \tag{2.4}$$

Hence, if we directly apply a general theorem of M.A. McKiernan 1970 [12, p.32, Theorem 2] to equation (2.4), then we obtain

$$(\delta^k_{x,t} f)(x,y) = 0 \qquad \text{and} \qquad (\delta^k_{y,t} f)(x,y) = 0 \tag{2.5}$$

with $k = 11$. On the other hand, it is also known, cf. [2, p. 43, Lemma 3], that if an arbitrary f satisfies (2.4), then f also satisfies difference equations (2.5) for $k = 4$. However, equations (2.1), that is, (2.5) serve as a better tool to prove the *'only if'* part of Theorem 2.1, since if $k > 3$ in (2.5), then the solution of (2.1) contains more symmetric multiadditive functions of higher order (cf. S. Mazur and W. Orlicz 1934 [9], and M.A. McKiernan 1967 [10], among others).

The other is a lemma which is a particular case of Lemma 6 in [2, p. 49-50]. We note that if we replace $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $f: G \times G \longrightarrow \mathbb{C}$ in Lemma 6 of [2], then it follows from a general theorem of S. Mazur and W. Orlicz [9] that the result of Lemma 6 in [2] still holds for the case $\delta_{x,t} = X^t - X^{-t}$ and $\delta_{y,t} = Y^t - Y^{-t}$ instead of $\Delta_{x,t} = X^{\frac{t}{2}} - X^{-\frac{t}{2}}$ and $\Delta_{y,t} = Y^{\frac{t}{2}} - Y^{-\frac{t}{2}}$ defined in [2, p. 43].

Lemma 2.2
A function $f: G \times G \longrightarrow \mathbb{C}$ *satisfies both equations* (2.1) *for all* $x, y, t \in G$ *if and only if* f *is given by*

$$\begin{aligned} f(x,y) = A^0 + A^1(x) + A^2(x) + B^1(y) + B^2(y) \\ + A^{1,1}(x;y) + A^{2,1}(x;y) + A^{1,2}(x;y) + A^{2,2}(x;y) \end{aligned} \tag{2.6}$$

for all $x, y \in G$, where $A^0, A^1, A^2\colon G \longrightarrow \mathbb{C}$ are defined in Theorem 2.1, $B^1\colon G \longrightarrow \mathbb{C}$ is an additive function, and $B^2\colon G \longrightarrow \mathbb{C}$ is the diagonalization of a symmetric bi-additive function. Further, the functions $A^{1,1}, A^{2,1}, A^{1,2}, A^{2,2}$: $G \times G \longrightarrow \mathbb{C}$ are defined as follows:

$$\begin{aligned} A^{1,1}(x;y) &= A_{1,1}(x;y), & A^{2,1}(x;y) &= A_{2,1}(x,x;y),\\ A^{1,2}(x;y) &= A_{1,2}(x;y,y), & A^{2,2}(x;y) &= A_{2,2}(x,x;y,y), \end{aligned}$$

where $A_{i,j}$, $i, j = 1, 2$ are additive functions in each of their variables.

By Lemma 6 in [2, p. 49-50] we have

$$f(x,y) = \sum_{n,m=0}^{2} A^{n,m}(x;y),$$

that is, $f(x, y)$ is a generalized quadratic polynomial in x and y and can be written as expression (2.6).

Proof of Lemma 2.1. We multiply (E.1) by i and then write equation (E.1) in the operator form

$$[(iX^t - iX^{-t})f](x,y) = [(Y^t - Y^{-t})f](x,y)$$

which may be written briefly as

$$iX^t - iX^{-t} = Y^t - Y^{-t} \tag{2.7}$$

for f. We will omit the f whenever no confusion can rise. Now, cube the operators on both sides of (2.7) to obtain

$$-iX^{3t} + 3iX^t - 3iX^{-t} + iX^{-3t} = Y^{3t} - 3Y^t + 3Y^{-t} - Y^{-3t}. \tag{2.8}$$

By multiplying (2.7) by 3

$$3iX^t - 3iX^{-t} = 3Y^t - 3Y^{-t}, \tag{2.9}$$

while by multiplying the both sides by -1 and by replacing t by $3t$ in (2.7) we have

$$-iX^{3t} + iX^{-3t} = -Y^{3t} + Y^{-3t}. \tag{2.10}$$

If we substitute (2.9) and (2.10) in (2.8) in order to eliminate the operators $3iX^t$, $-3X^{-t}$, $-X^{3t}$, and iX^{-3t} from (2.8), then

$$Y^{3t} - 3Y^t = Y^{-3t} - 3Y^{-t}, \qquad \text{and} \qquad (Y^t - Y^{-t})^3 = 0, \tag{2.11}$$

which is the second equation of (2.1).

Similarly, substitute (2.9) and (2.10) in (2.8) to eliminate the operators Y^{3t}, $-3Y^t$, $3Y^{-t}$, and $-Y^{-3t}$ from (2.8). Then we obtain the first equation of (2.1).

Next, square both sides of (2.7) to obtain (2.2), while replace $2t$ by t in (2.2) to obtain (2.3). This completes the proof of Lemma 2.1.

Proof of Theorem 2.1. Since f satisfies (2.1), by Lemma 2.2 f is given by (2.6). If we substitute (2.6) into equation (2.3), then

$$\begin{aligned} A_2(t,t) + A_{2,1}(t,t;y) + B_2(t;t) + A_{1,2}(x;t,t) \\ + A_{2,2}(x,x;t,t) + A_{2,2}(t,t;y,y) \\ = 0. \end{aligned} \tag{2.12}$$

Set $x = y = 0$ in (2.12) to obtain

$$B^2(t) = -A^2(t), \tag{2.13}$$

which, with (2.12), implies

$$A_{2,1}(t,t;y) + A_{1,2}(x;t,t) + A_{2,2}(x,x;t,t) + A_{2,2}(t,t;y,y) = 0. \tag{2.14}$$

Further, set $x = 0$, $y = 0$, respectively, in (2.14). Then we have

$$A^{2,1}(t;y) + A^{2,2}(t;y) = 0 \qquad \text{and} \qquad A^{1,2}(x;t) + A^{2,2}(x;t) = 0$$

for all $x, y, t \in G$, which show that

$$A^{2,1}(x;y) + A^{2,2}(x;y) = 0 \tag{2.15}$$

and

$$A^{1,2}(x;y) + A^{2,2}(x;y) = 0 \tag{2.16}$$

for all $x, y \in G$. Subtract (2.16) from (2.15) to obtain

$$A^{2,1}(x;y) - A^{1,2}(x;y) = 0. \tag{2.17}$$

Thus it follows from (2.6), (2.13), and (2.15) that

$$f(x,y) = A^0 + A^1(x) + A^2(x) + B^1(y) - A^2(y) + A^{1,1}(x;y) + A^{1,2}(x;y). \tag{2.18}$$

Next, substitute (2.18) into equation (2.7), that is,

$$i[f(x+t,y) - f(x-t,y)] = f(x,y+t) - f(x,y-t)],$$

to obtain

$$\begin{aligned} iA^1(t) + 2iA_2(x,t) + iA^{1,1}(t,y) + iA_{1,2}(t;y,y) \\ = B^1(t) + A^{1,1}(x,t) - 2A_2(y,t) + 2A_{1,2}(x;y,t). \end{aligned} \tag{2.19}$$

Set $x = 0$, $y = 0$, and $x = y = 0$, respectively in (2.19). Then we have the following three equations

$$iA^1(t) + iA^{1,1}(t,y) + iA_{1,2}(t;y,y) = B^1(t) - 2A_2(y,t) \tag{2.20}$$

$$iA^1(t) + 2iA_2(x,t) = B^1(t) + A^{1,1}(x,t) \tag{2.21}$$

$$iA^1(t) = B^1(t). \tag{2.22}$$

Equations (2.20) and (2.22) yield

$$iA^{1,1}(t;y) + iA_{1,2}(t;y,y) = -2A_2(y,t)$$

which can be rewritten in the form

$$iA^{1,1}(x,y) + iA_{1,2}(x;y,y) = -2A_2(y,x) \tag{2.23}$$

while (2.21) and (2.22) imply $2iA_2(x,t) = A^{1,1}(x,t)$ and

$$2iA_2(x,y) = A^{1,1}(x;y). \tag{2.24}$$

Further, it follows from (2.24) and (2.23) that $-2A_2(x,y) + iA_{1,2}(x;y,y) = -2A_2(y,x)$, and, since A_2 is symmetric,

$$A^{1,2}(x;y) = 0. \tag{2.25}$$

Thus equation (2.18) with (2.22), (2.24) and (2.25) implies (S.1) for all $x, y \in G$.

Conversely, (S.1) satisfies equation (E.1). This completes the proof of Theorem 2.1.

2.2. Regular solutions

In addition, as soon as some suitable regularity assumptions are imposed on f for the case $G = \mathbb{R}$ in the above Theorem 2.1, it can be readily shown by the following lemma that f is an ordinary complex polynomial of degree at most two. The following lemma is a consequence of Theorem 2.1.

Lemma 2.3

Let $(F,+)$ be an additive group. If $f\colon F \times F \longrightarrow \mathbb{C}$ is given by (S.1) *for all $x, y \in F$, then all functions $A^1, A^2\colon F \longrightarrow \mathbb{C}$ and $A_2\colon F \times F \longrightarrow \mathbb{C}$ can be represented in terms of f and a constant A^0 by*

$$A^1(x) = \frac{f(x,y) - f(-x,-y) - f(-x,y) + f(x,-y)}{4}, \tag{2.26}$$

$$A^1(y) = \frac{f(x,y) - f(-x,-y) + f(-x,y) - f(x,-y)}{4i}, \tag{2.27}$$

$$A_2(x,y) = \frac{f(x,y) + f(-x,-y) - f(-x,y) - f(x,-y)}{8i}, \tag{2.28}$$

$$A^2(x) = f(x,0) - A^0 - \frac{f(x,y) - f(-x,-y) - f(-x,y) + f(x,-y)}{4}, \tag{2.29}$$

$$A^2(y) = -f(0,y) + A^0 + \frac{f(x,y) - f(-x,-y) + f(-x,y) - f(x,-y)}{4} \tag{2.30}$$

for all $x, y \in F$.

Proof. It follows from

$$f(x,y) = A^0 + A^1(x) + iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x,y) \tag{S.1}$$

that

$$f(-x,-y) = A^0 - A^1(x) - iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x,y), \tag{2.31}$$

$$f(-x,y) = A^0 - A^1(x) + iA^1(y) + A^2(x) - A^2(y) - 2iA_2(x,y), \tag{2.32}$$

$$f(x,-y) = A^0 + A^1(x) - iA^1(y) + A^2(x) - A^2(y) - 2iA_2(x,y). \tag{2.33}$$

Subtract (2.31) from (S.1) and (2.33) from (2.32), respectively, to obtain

$$f(x,y) - f(-x,-y) = 2A^1(x) + 2iA^1(y), \tag{2.34}$$

$$f(-x,y) - f(x,-y) = -2A^1(x) + 2iA^1(y). \tag{2.35}$$

Add (2.34) and (2.35) to obtain (2.27). Subtract (2.35) from (2.34) to obtain (2.26). Next by adding (S.1) and (2.31) we have

$$f(x,y) + f(-x,-y) = 2A^0 + 2A^2(x) - 2A^2(y) + 4iA_2(x,y). \tag{2.36}$$

Further, add (2.32) and (2.33) to obtain

$$f(-x,y) + f(x,-y) = 2A^0 + 2A^2(x) - 2A^2(y) - 4iA_2(x,y). \tag{2.37}$$

If we subtract (2.37) from (2.36), then we have (2.28). By setting $y = 0$ in (S.1) and then by using (2.26) we have (2.29). Set $x = 0$ in (S.1) and then use (2.27) to obtain (2.30). This completes the proof of Lemma 2.3.

If we assume that, for example, $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ is continuous everywhere, then by applying the above Lemma 2.3 we have the following result.

Theorem 2.2
A continuous function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies (E.1) *for all $x, y, t \in \mathbb{R}$ if and only if f is given by*

$$f(x,y) = a(x^2 - y^2 + 2ixy) + b(x + iy) + c \tag{2.38}$$

for all $x, y \in \mathbb{R}$, where a, b, and c are complex constants.

Proof. Lemma 2.3 clearly holds for the case $F = \mathbb{R}$. If f is continuous everywhere, then it readily follows from (2.26) and (2.28) of Lemma 2.3 that $A^1(x)$ and $A_2(x,y)$ are also continuous for all $x, y \in \mathbb{R}$. It is well-known that a continuous additive function $A^1(x): \mathbb{R} \longrightarrow \mathbb{C}$ is given by $A^1(x) = bx$ [1, p. 36] for all $x \in \mathbb{R}$, where b is a complex constant. It readily follows from this result that a continuous symmetric bi-additive function $A_2: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ is given by $A_2(x,y) = axy$ for all $x, y \in \mathbb{R}$, where a is a complex constant. Hence, (2.38) follows from (S.1) with $A^0 = c$. Conversely, (2.38) satisfies (E.1). This completes the proof of Theorem 2.2.

Equation (E.1) can also be rewritten in the complex form

$$f(z+t)-f(z-t)=\,-\,i[f(z+it)-f(z-it)] \tag{2.39}$$

for all $z\in\mathbb{C}$ and $t\in\mathbb{R}$, where $f(z):=f(x,y)$ for all $x,y\in\mathbb{R}$ and $f\colon\mathbb{C}\longrightarrow\mathbb{C}$. In this case the continuous solution (2.38) is given by a complex polynomial of degree at most two such that

$$f(z)=az^2+bz+c$$

for all $z\in\mathbb{C}$.

3. Certain related functional equations

3.1. Equations (E.3) and its variations

Here we mainly consider the functional equation

$$f(x+t,y+t)-f(x-t,y-t)=\,-\,i[f(x-t,y+t)-f(x+t,y-t)] \tag{E.3}$$

for all $x,y,t\in G$, where $f\colon G\times G\longrightarrow\mathbb{C}$, and determine the general and regular solutions of (E.3).

One of applications of functional equations is that to a geometric characterization of complex polynomials from the standpoint of conformal mapping properties. In particular, H. Haruki 1971 [4] obtains the functional equation

$$f(z+te^{\frac{\pi i}{4}})-f(z-te^{\frac{\pi i}{4}})=i[f(z+te^{-\frac{\pi i}{4}})-f(z-te^{-\frac{\pi i}{4}})] \tag{3.1}$$

for all $z\in\mathbb{C}$ and $t\in\mathbb{R}$, where $f\colon\mathbb{C}\longrightarrow\mathbb{C}$, from two geometric properties on f. Equation (3.1) yields (E.3) for all $x,y,t\in\mathbb{R}$, where $f(x,y):=f(z)$ for $z=x+iy$ and $f\colon\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$. The continuous solution $f\colon\mathbb{C}\longrightarrow\mathbb{C}$ of equation (3.1) is obtained in [4] by using the regularity of solutions of Haruki's functional equation (2.4). We note that the continuity assumption in order to consider equation (3.1) is natural from the point of view of geometric properties of f yielding (3.1) in [4]. Further, it is possible to obtain the general solution of equations (E.3) and (3.1) for $f\colon\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$ and $f\colon\mathbb{C}\longrightarrow\mathbb{C}$ when no regularity assumptions are imposed on f, since it is shown in [4] that (3.1) implies (2.4), and the general solution of (2.4) is obtained in [2, p. 50-51, Theorem 5]. However, in this section we first show that equation (E.3) is equivalent to (E.1) when no regularity assumptions are imposed on f so that by Theorem 2.1 in §2.1 we immediately obtain the general solution of equation (E.3) under no regularity assumptions on f. We emphasize that we do not apply the general solution of equation (2.4). We will be able to generalize our results from $\mathbb{R}$ to G.

Theorem 3.1
The function $f : G \times G \longrightarrow \mathbb{C}$ *satisfies equation* (E.1) *for all* $x, y, t \in G$*, if and only if it satisfies equation* (E.3) *for all* $x, y, t \in G$.

Proof. As before equation (E.1) can be rewritten in the simple operator form

$$X^t - X^{-t} + iY^t - iY^{-t} = 0 \qquad \text{or} \qquad iX^t - iX^{-t} - Y^t + Y^{-t} = 0. \tag{3.2}$$

Multiply the second equation of (3.2) by the operator $(1-i)(X^t + X^{-t} + Y^t + Y^{-t})$ to obtain

$$\begin{aligned} 2(iX^tY^t - iX^{-t}Y^{-t} - X^{-t}Y^t + X^tY^{-t}) & \\ + (X^{2t} - X^{-2t} + iY^{2t} - iY^{-2t}) & \\ + (iX^{2t} - iX^{-2t} - Y^{2t} + Y^{-2t}) & \\ = 0. & \end{aligned}$$

By replacing t by $2t$ in (3.2) we have

$$X^{2t} - X^{-2t} + iY^{2t} - iY^{-2t} = 0 \qquad \text{or} \qquad iX^{2t} - iX^{-2t} - Y^{2t} + Y^{-2t} = 0.$$

Hence, it follows from these operator equations that

$$iX^tY^t - iX^{-t}Y^{-t} - X^{-t}Y^t + X^tY^{-t} = 0 \tag{3.3}$$

and

$$iX^tY^t - iX^{-t}Y^{-t} = X^{-t}Y^t - X^tY^{-t}.$$

Further, multiply the both sides of this equation by $-i$ to obtain

$$X^tY^t - X^{-t}Y^{-t} = -i(X^{-t}Y^t - X^tY^{-t}) \tag{3.4}$$

which is the operator form of equation (E.3). Thus equation (E.1) implies (E.3).

Conversely, multiply (3.4) by i to obtain equation (3.3). Next, by multiplying (3.3) by the operator $(1+i)(X^tY^t + X^{-t}Y^{-t} + X^{-t}Y^t + X^tY^{-t})$ we have

$$\begin{aligned} (iX^{2t}Y^{2t} - iX^{-2t}Y^{-2t} - X^{-2t}Y^{2t} + X^{2t}Y^{-2t}) & \\ + (-X^{2t}Y^{2t} + X^{-2t}Y^{-2t} - iX^{-2t}Y^{2t} + iX^{2t}Y^{-2t}) & \\ + 2(iX^{2t} - iX^{-2t} - Y^{2t} + Y^{-2t}) & \\ = 0. & \end{aligned}$$

It follows from (3.3) and (3.4) that

$$iX^{2t}Y^{2t} - iX^{-2t}Y^{-2t} - X^{-2t}Y^{2t} + X^{2t}Y^{-2t} = 0$$

and

$$-X^{2t}Y^{2t} + X^{-2t}Y^{-2t} - iX^{-2t}Y^{2t} + iX^{2t}Y^{-2t} = 0.$$

Therefore we obtain

$$iX^{2t} - iX^{-2t} = Y^{2t} - Y^{-2t}. \tag{3.5}$$

If we multiply both sides of (3.5) by $-i$ and replace $2t$ by t, then we obtain equation (E.1). Hence, equation (E.3) implies equation (E.1). Therefore (E.1) and (E.3) are equivalent.

The following result is an immediate consequence of Theorems 2.1 and 3.1.

Corollary 3.1
A function $f: G \times G \longrightarrow \mathbb{C}$ satisfies equation (E.3) *for all $x, y, t \in G$ if and only if f is given by expression* (S.1) *for all $x, y \in G$.*

We also readily obtain from Theorems 3.1 and 2.2 in the case of $G = \mathbb{R}$ that a continuous function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies (E.3) for all $x, y, t \in \mathbb{R}$ if and only if f is given by (2.38) for all $x, y \in \mathbb{R}$.

H. Haruki in [4, p. 37] proved the following theorem on regular solutions of functional equation (2.4).

Theorem 3.2
A continuous function $f: \mathbb{C} \longrightarrow \mathbb{C}$ satisfies equation (3.1) *for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$ if and only if f is given by a quadratic polynomial of z.*

The following proof of Theorem 3.2 is an alternative one, without applying a regularity of functional equation (2.4).

Proof. Equation (3.1) yields the functional equation

$$f(z+t+it) - f(z-t-it) = -i[f(z-t+it) - f(z+t-it)] \tag{3.6}$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Define $f(z) = f(x, y)$ for $z = x + iy$. Then equation (3.6) implies equation (E.3). Hence, by Theorem 2.2 and Theorem 3.1 f is a quadratic polynomial. Conversely, a quadratic polynomial satisfies (3.6). This proves Theorem 3.2.

Theorem 3.3
Assume that $f: G \times G \longrightarrow \mathbb{C}$ satisfies one of the following three functional equations

$$f(x+t, y) - f(x-t, y) = -i[f(x, y+t) - f(x, y-t)] \tag{E.36}$$

$$f(x+t, y) - f(x, y) = -i[f(x, y+t) - f(x, y)] \tag{E.37}$$

$$f(x+t, y+t) - f(x-t, y-t) = -i[f(x-t, y+t) - f(x+t, y-t)] \tag{E.38}$$

for all $x, y, t \in G$. Then f also satisfies the Haruki functional equation

$$f(x+t, y+t) + f(x-t, y+t) + f(x+t, y-t) + f(x-t, y-t) = 4f(x, y) \quad (2.4)$$

for all $x, y, t \in G$.

Proof. Replace t by $-t$ in (E.2) and then substract the result from (E.2) to obtain equation (E.1). Theorem 3.1 shows that (E.1) is equivalent to (E.3). By Lemma 2.1, (E.1) implies (2.3) which is equivalent to (2.4) ([2, Lemma 3, p. 43]). We note that if we replace $\mathbb{R}$ by G in Lemma 3 of [2, p. 43], then the proof of equivalency of (2.3) and (2.4) still holds.

3.2. Differential-difference equations

If one side of the Cauchy–Riemann equation $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ is replaced by the operators $\bigtriangledown_{x,t} f$ or $\bigtriangledown_{y,t} f$ defined in §1, then we have the following two partial difference-differential equations

$$\frac{\partial f(x,y)}{\partial x} = -i\left[\frac{f(x, y+t) - f(x, y-t)}{2t}\right] \quad (3.7)$$

$$\frac{f(x+t, y) - f(x-t, y)}{2t} = -i\frac{\partial f(x,y)}{\partial y}. \quad (3.8)$$

We determine the general solutions of equations (3.7) and (3.8).

THEOREM 3.4
A function $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ satisfies equation (3.7) for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$ if and only if f is given by

$$f(x,y) = \frac{1}{2}a(y^2 - x^2 - 2ixy) + b(y - ix) + c, \quad (3.9)$$

where $a, b,$ and c are complex constants.

Proof. We replace (3.7) by

$$\frac{f(x, y+t) - f(x, y-t)}{2t} = \phi(x,y), \quad (3.10)$$

where $\phi(x,y) := i\frac{\partial f(x,y)}{\partial x}$, or, since x is the same parameter in each term of (3.10), by

$$\frac{g(y+t) - g(y-t)}{2t} = \psi(y). \quad (3.11)$$

It follows from [5, p. 577] that the general solution of (3.11) is given by

$$\psi(y) = \alpha y + \beta, \qquad g(y) = \frac{1}{2}\alpha y^2 + \beta y + \gamma,$$

where α, β, and γ are complex constants. Hence, ϕ and f are represented by

$$\phi(x,y) = i\frac{\partial f(x,y)}{\partial x} = \alpha(x)y + \beta(x), \tag{3.12}$$

$$f(x,y) = \frac{1}{2}\alpha(x)y^2 + \beta(x)y + \gamma(x), \tag{3.13}$$

where $\alpha, \beta, \gamma \colon \mathbb{R} \longrightarrow \mathbb{C}$. Now, substitute (3.13) into (3.12) to obtain

$$i\left[\frac{1}{2}\alpha'(x)y^2 + \beta'(x)y + \gamma'(x)\right] = \alpha(x)y + \beta(x).$$

Therefore $\alpha'(x) = 0$, $i\beta'(x) = \alpha(x)$, and

$$\beta(x) = \gamma'(x)i. \tag{3.14}$$

Consequently, $\alpha(x) = a$, where a is a complex constant, $\beta'(x) = -ai$. Therefore $\beta(x) = -axi + b$, which with (3.14) implies $\gamma'(x) = -ax - bi$ and $\gamma(x) = -\frac{1}{2}ax^2 - bxi + c$, where b and c are complex constants. These $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ with (3.13) yield (3.9). Conversely, (3.9) satisfies differential functional equation (3.7). This completes the proof of Theorem 3.4.

In view of the similarity of equations (3.7) and (3.8) the following theorem readily follows from a proof similar to the above proof of Theorem 3.4.

Theorem 3.5
A function $f \colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ satisfies equation (3.8) for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}\setminus\{0\}$ if and only if f is given by

$$f(x,y) = \frac{1}{2}a(x^2 - y^2 + 2ixy) + b(x+iy) + c,$$

where a, b, and c are complex constants.

3.3. Nonsymmetric partial difference equations

In Sections 2 and 3.1, as well as in the papers [3], [6], [7] as a discrete analogue of the Cauchy–Riemann equation $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$, the symmetric equations

$$\triangle_{x,t} f = -\,i\,\triangle_{y,t}\, f \qquad \text{and} \qquad \triangledown_{x,t}\, f = -\,i\,\triangledown_{y,t}\, f$$

were considered, and the general solution of functional equations (E.1) and (E.2) was determined when no regularity assumptions are imposed on f.

In this final subsection, we determine the general solutions of the following two nonsymmetric partial difference equations which are also discrete analogue of the Cauchy–Riemann equations:

$$\triangle_{x,t} f = -i \triangle_{y,s} f \qquad \text{and} \qquad \triangledown_{x,t} f = -i \triangledown_{y,s} f.$$

These equations are given by

$$\frac{f(x+t,y)-f(x,y)}{t} = -i\left[\frac{f(x,y+s)-f(x,y)}{s}\right] \tag{3.15}$$

$$\frac{f(x+t,y)-f(x-t,y)}{2t} = -i\left[\frac{f(x,y+s)-f(x,y-s)}{2s}\right] \tag{3.16}$$

for all $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R}\backslash\{0\}$. By applying the general solutions of functional equations (E.1) and (E.2) we obtain the general solutions of nonsymmetric functional equations (3.15) and (3.16) under no regularity assumptions.

As a closely related to (E.3) we can also derive the following functional equation, and we obtain its general solution without any regularity assumptions as well:

$$\begin{aligned} &\frac{f(x+t,y+t)-f(x-t,y-t)}{t} \\ &\quad = -i\left[\frac{f(x-s,y+s)-f(x+s,y-s)}{s}\right] \end{aligned} \tag{3.17}$$

for all $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R} \setminus \{0\}$.

Theorem 3.6
A function $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ satisfies equation (3.15) *for all $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R} \setminus \{0\}$ if and only if f is given by*

$$f(x,y) = a(x+iy) + b \tag{3.18}$$

for all $x, y \in \mathbb{R}$, where a and b are complex constants.

Proof. Set $t = s$ in (3.15) to obtain equation (E.2). From [3, Theorem 2, p. 99] it is known that the general solution of (E.2) is given by

$$f(x,y) = A(x) + iA(y) + b, \tag{3.19}$$

where $A\colon \mathbb{R} \longrightarrow \mathbb{C}$ is additive and b is a complex constant. Substitute (3.19) into (3.15). Then we have $\frac{A(t)}{t} = \frac{A(s)}{s}$ which implies $A(t) = at$ for fixed $s = s_0 \neq 0$ where $a = \frac{A(s_0)}{s_0}$ is a complex constant. Hence, we obtain (3.18) from (3.19). Conversely, (3.18) satisfies equation (3.15).

THEOREM 3.7
A function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.16) *for all* $x, y \in \mathbb{R}$ *and* $s, t \in \mathbb{R} \setminus \{0\}$ *if and only if* f *is given by* (2.38) *for all* $x, y \in \mathbb{R}$*, where* a*,* b*, and* c *are complex constants.*

Proof. Set $t = s$ in equation (3.16) to obtain equation (E.1). Hence, it follows from Theorem 2.1 that f is given by (S.1). Substitute (S.1) into equation (3.16) to obtain

$$\begin{aligned}\frac{2A^1(t) + 4A_2(x,t) + 4iA_2(t,y)}{2t}\\ = -\frac{[2iA^1(s) - 4A_2(y,s) + 4iA_2(x,s)]i}{2s}.\end{aligned} \tag{3.20}$$

Now, set $x = y = 0$ in (3.20). Then we have $\frac{A^1(t)}{t} = \frac{A^1(s)}{s}$, since $A_2(0,t) = A_2(t,0) = 0$, which implies

$$A^1(t) = bt \tag{3.21}$$

where $b = \frac{A^1(s_0)}{s_0}$, $s_0 \neq 0$, is a complex constant. Next, by setting $y = 0$ and $t = 1$ in (3.20) with (3.21) we obtain $A_2(x,s) = sA_2(x,1) = sA(x)$, where $A(x) := A_2(x,1)$ is additive for all $x \in \mathbb{R}$. But A_2 is symmetric. Hence, we obtain $sA(x) = xA(s)$ and $A(x) = ax$ where $a = \frac{A(s_0)}{s_0}$, $s_0 \neq 0$, is a complex constant. Hence, we obtain

$$A_2(x,y) = axy. \tag{3.22}$$

Then it follows from (3.21), (3,22), and (S.1) with $A^0 = c$ that f is given by (2.38). Conversely, (2.38) satisfies equation (3.16).

THEOREM 3.8
A function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.17) *for all* $x, y \in \mathbb{R}$ *and* $s, t \in \mathbb{R} \setminus \{0\}$ *if and only if* f *is given by* (2.38) *for all* $x, y \in \mathbb{R}$*, where* a*,* b*, and* c *are complex constants.*

Proof. Set $t = s$ in equation (3.17). Then we have equation (E.3). Hence, by Corollary 3.1, f is given by (S.1). Substitute (S.1) in equation (3.17). Then we have

$$\begin{aligned}\frac{A^1(t) + iA^1(t) + 2A_2(x,t) - 2A_2(t,y) + 2iA_2(t,y) + 2iA_2(x,t)}{t}\\ = \frac{iA^1(s) + A^1(s) + 2iA_2(s,x) + 2iA_2(s,y) + 2A_2(x,s) - 2A_2(s,y)}{s}.\end{aligned} \tag{3.23}$$

Next, set $x = y = 0$ in (3.23) to obtain

$$\frac{A^1(t) + iA^1(t)}{t} = \frac{iA^1(s) + A^1(s)}{s}, \tag{3.24}$$

since $A_2(0,x) = A_2(x,0) = 0$ for all $x \in \mathbb{R}$. So, it follows from (3.23) and (3.24) that

$$\begin{aligned}&\frac{A_2(x,t) - A_2(y,t) + iA_2(x,t) + iA_2(y,t)}{t}\\ &\quad = \frac{iA_2(x,s) + iA_2(y,s) + A_2(x,s) - A_2(y,s)}{s}.\end{aligned} \tag{3.25}$$

Further, on putting $x = 0$ and $y = 0$ in (3.25) independently, we have the following two equations

$$\frac{-A_2(y,t) + iA_2(y,t)}{t} = \frac{iA_2(y,s) - A_2(y,s)}{s} \tag{3.26}$$

$$\frac{A_2(x,t) + iA_2(x,t)}{t} = \frac{iA_2(x,s) + A_2(x,s)}{s}. \tag{3.27}$$

Replace x by y in (3.27) to obtain

$$\frac{A_2(y,t) + iA_2(y,t)}{t} = \frac{iA_2(y,s) + A_2(y,s)}{s}. \tag{3.28}$$

Add both sides of (3.26) and (3.28) to obtain $\frac{A_2(y,t)}{t} = \frac{A_2(y,s)}{s}$, which yields $A_2(y,t) = tA(y)$, where $A(y) = \frac{A_2(y,s_0)}{s_0}$ for fixed $s = s_0 \neq 0$ is additive, and $A_2(x,y) = xA(y)$. Since A_2 is symmetric, as before,

$$A_2(x,y) = axy \tag{3.29}$$

where a is a complex constant. On the other hand, (3.24) implies $\frac{A^1(t)}{t} = \frac{A^1(s)}{s}$. Hence

$$A^1(t) = bt \tag{3.30}$$

where $b = \frac{A^1(s_0)}{s_0}$, $s_0 \neq 0$, is a complex constant. Hence, it follows from (3.29), (3.30), and (S.1) with $A^0 = c$ that f is given by (2.38). Conversely, (2.38) satisfies (3.17). This completes the proof of Theorem 3.8.

Functional equations (3.15), (3.16), and (3.17) can also be rewritten in the complex forms

$$\frac{f(z+t) - f(z)}{t} = -i\left[\frac{f(z+is) - f(z)}{s}\right] \tag{3.31}$$

$$\frac{f(z+t) - f(z-t)}{2t} = -i\left[\frac{f(z+is) - f(z-is)}{2s}\right] \tag{3.32}$$

$$\frac{f(z+t+it) - f(z-t-it)}{t} = -i\left[\frac{f(z-s+is) - f(z+s-is)}{s}\right] \tag{3.33}$$

for all $z \in \mathbb{C}$ and $s, t \in \mathbb{R} \setminus \{0\}$, where $f(z) := f(x, y)$ for all $x, y \in \mathbb{R}$ and $f: \mathbb{C} \longrightarrow \mathbb{C}$. In this case, it follows from Theorem 3.6 and (3.18) that the general solution of (3.31) is given by $f(z) = az + b$. On the other hand, by Theorems 3.7 and 3.8 and (2.38) the general solution of (3.32) and (3.33) are given by $f(z) = az^2 + bz + c$. Thus, it is remarkable that the only solutions of the above three functional equations are certain complex polynomials of bounded degree when no regularity assumptions are imposed on f.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic Press, New York and London, 1966.

[2] J. Aczél, H. Haruki, M.A. McKiernan, G.N. Sakovič, *General and regular solutions of functional equations characterizing harmonic polynomials*, Aequationes Math. **1** (1968), 37-53.

[3] J. Aczél, S. Haruki, *Partial difference equations analogous to the Cauchy–Riemann equations*, Funkcial. Ekvac. **24** (1981), 95-102.

[4] H. Haruki, *On an application of the square functional equation to a geometric characterization of quadratic functions from the standpoint of conformal-mapping properties*, Aequationes Math. **6** (1971), 36-38.

[5] S. Haruki, *A property of quadratic polynomials*, Amer. Math. Monthly **86** (1979), 577-579.

[6] S. Haruki, *Partial difference equations analogous to the Cauchy–Riemann equations II*, Funkcial. Ekvac. **29** (1986), 237-241.

[7] S. Haruki, C.T. Ng, *Partial difference equations analogous to the Cauchy–Riemann equations and related functional equations on rings and fields*, Results in Math. **26** (1994), 316-323.

[8] J.H.B. Kemperman, *A general functional equation*, Trans. Amer. Math. Soc. **86** (1957), 28-56.

[9] S. Mazur, W. Orlicz, *Grundlegende Eigenschaften der polynomischen Operationen*, Studia Math. **5** (1934), 50-68.

[10] M.A. McKiernan, *On vanishing n-th ordered differences and Hamel basis*, Ann. Polon. Math. **19** (1967), 331-336.

[11] M.A. McKiernan, *On Haruki's functional equation*, Aequationes Math. **1** (1968), 143.

[12] M.A. McKiernan, *Boundedness on a set of positive measure and the mean-value property characterizes polynomials on a space V^n*, Aequationes Math. **4** (1970), 31-36.

[13] H. Światak, *A generalization of the Haruki functional equation*, Ann. Polon. Math. **22** (1970), 370-376.

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