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A note on some elementary properties and applications of certain operators to certain functions analytic in the unit disk

Abstract. This scientific note relates to introducing certain elementary operators defined in the unit disk in the complex plane, then determining various applications (specified by those operators) to certain analytic functions, and also revealing a number of possible implications of them.

1. Introduction and preliminary information

First of all, here and in parallel to this investigation, let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ denote the set of all complex numbers, the set of all real numbers and the set of all positive integers, respectively.

Let $\mathcal{A}(n)$ also denote the family of functions $f(z)$, which are normalized with the following conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1$$

and are then of the form

$$f(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \cdots , \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}.$$

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Moreover, let $\mathcal{H}_n(a)$ denote the family of analytic functions $g(z)$ given by the following Taylor-Maclaurin series

$$g(z) = e + e_n z^n + e_{n+1} z^{n+1} + \cdots , \qquad n \in \mathbb{N},\ z \in \mathbb{U}. \tag{2}$$

In particular, let $\mathcal{A} := \mathcal{A}(1)$, $\mathcal{H}_n := \mathcal{H}_n(1)$, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The *fractional derivative operator* (of order μ) for an analytic function $\kappa(z)$ is usually denoted by

$$\mathcal{D}_z^{\mu}[\kappa(z)], \qquad 0 \leq \mu < 1$$

and is also defined by the definition given in [1, 2, 3, 7, 11, 12, 13, 14, 16, 20].

Let $\kappa(z)$ be an analytic function in a simply-connected region of the complex plane containing the origin. Then, its fractional derivative of order μ is defined by

$$\mathcal{D}_z^{\mu}[\kappa(z)] = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{\kappa(\xi)}{(z-\xi)^{\mu}}\, d\xi, \qquad 0 \leq \mu < 1, \tag{3}$$

where the multiplicity of $(z-\xi)^{-\mu}$ above is removed requiring $\log(z-\xi)$ to be real when $z-\xi > 0$ and Γ is the well-known gamma function.

As it is given by an application in [7, 20, 22], by means of the well-known definition in (3), for a function $\kappa(z)$, the fractional derivative, which is called as Srivastava-Owa derivative of order $k+\mu$, is then presented by

$$\mathcal{D}_z^{k+\mu}[\kappa(z)] = \frac{d^k}{dz^k}(\mathcal{D}_z^{\mu}[\kappa(z)]),$$

which also yields

$$\mathcal{D}_z^{0+\mu}[\kappa(z)] = \mathcal{D}_z^{\mu}[\kappa(z)]$$

and

$$\mathcal{D}_z^{1+\mu}[\kappa(z)] = \frac{d}{dz}(\mathcal{D}_z^{\mu}[\kappa(z)]),$$

where $k \in \mathbb{N}_0$ and $0 \leq \mu < 1$.

Through the instrument of the definition in (3), one of the operators is also the Tremblay operator, which was defined in the domain of the complex plane and whose properties in several spaces were discussed systematically by several researchers for their researches. For the details one may refer to [1, 2, 3, 7, 8, 14, 15, 16, 18, 19, 20, 22].

For an analytic function $f(z) \in \mathcal{A}(n)$ being of form (1), and as it is also used in [5] and [8], the Tremblay operator is often denoted by $\mathcal{T}_{\tau,\mu}[f](z)$, $\mathcal{T}_{\tau,\mu}[f(z)]$ or $\mathcal{T}_{\tau,\mu}[f]$ and is also defined as

$$\mathcal{T}_{\tau,\mu}[f](z) := \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \mathcal{D}_z^{\tau-\mu}[z^{\tau-1} f(z)], \tag{4}$$

where $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$ and $z \in \mathbb{U}$. Note that the operator $\mathcal{D}_z^{\tau-\mu}[\cdot]$ is also equivalent to the operator, which is Srivastava-Owa operator of fractional derivative(s) of order $\tau - \mu$, given by (3).

The following-elementary results are only two of several applications of the operator, defined as in (4), which will be needed for proving our main results relating to the analytic functions of the form (1). In the same breath, for their details and some of their extensive applications, it can be also checked the papers in [5], and see also the recent works in [11, 12, 13, 16] as certain examples.

Lemma 1.1
Let $f(z) \in \mathcal{A}(n)$, $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$ and $z \in \mathbb{U}$. Then,

$$\mathcal{T}_{\tau,\mu}[f(z)] = \frac{\tau}{\mu} z + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\tau)\Gamma(\mu)}{\Gamma(k+\mu)\Gamma(\tau)} a_k z^k. \tag{5}$$

Lemma 1.2
Let $f(z) \in \mathcal{A}(n)$, $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$ and $z \in \mathbb{U}$. Then,

$$\frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f(z)]) = \frac{\tau}{\mu} + \sum_{k=n+1}^{\infty} \frac{k\Gamma(k+\tau)\Gamma(\mu)}{\Gamma(k+\mu)\Gamma(\tau)} a_k z^{k-1}. \tag{6}$$

In view of the information mentioned in (1)–(6), and, of course, for a function $f(z) \in \mathcal{A}(n)$ and also admissible values of the related parameters μ and τ, it is clear that

$$\frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f(z)]) \in \mathcal{H}_n(\tau/\mu), \tag{7}$$

and, more exactly,

$$\frac{d}{dz}(\mathcal{T}_{1,1}[f(z)]) = f'(z) \in \mathcal{H}_n \tag{8}$$

and

$$\mathcal{T}_{1,1}[f(z)] = f(z) \in \mathcal{A}(n), \tag{9}$$

where $z \in \mathbb{U}$.

Under the conditions of the admissible values of the related parameters:

$$0 \leq \theta \leq 1, \quad 0 \leq \lambda \leq 1, \quad 0 < \mu \leq 1, \quad 0 < \tau \leq 1 \quad \text{and} \quad 0 \leq \tau - \mu < 1 \tag{10}$$

for a function $f \equiv f(z) \in \mathcal{A}(n)$, there is a need to introduce the following operators:

$$\mathbb{T}_f(z) \equiv \mathbb{T}^{\lambda}_{\tau,\mu}[f(z)] \quad \text{and} \quad \mathbf{T}_f(z) \equiv \mathbf{T}^{\theta}_{\tau,\mu}[f(z)], \tag{11}$$

defined by

$$\mathbb{T}_f(z) := \lambda \mathcal{T}_{\tau,\mu}[f(z)] + (1-\lambda) z \frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f(z)]) \tag{12}$$

and

$$\mathbf{T}_f(z) := \theta\ \mathbb{T}_f(z) + (1-\theta) z \frac{d}{dz}(\mathbb{T}_f(z)), \tag{13}$$

respectively, where $z \in \mathbb{U}$. We remark here that, for the contents of the two operators just above, it can be focused on certain earlier results relating to certain analytic functions considered in [11] and [12]. In the same time, certain relations between operators in (11)–(13) and some of their implications can be also compared with some of the results determined by those operators in [13] and [16].

Through the instrumentality of the series expansions given in (1), (5) and (6), under the conditions of the admissible values of the parameters indicated in (10) and also for a function $f := f(z) \in \mathcal{A}(n)$, the operators in (12) and (13), are then equal respectively to the series expansions:

$$\mathbb{T}_f(z) := \frac{\tau}{\mu}\Big(z + \sum_{k=n+1}^{\infty}[k(1-\lambda)+\lambda]\frac{\mu\Gamma(k+\tau)\Gamma(\mu)}{\tau\Gamma(k+\mu)\Gamma(\tau)}a_k z^k\Big) \tag{14}$$

and

$$\mathbf{T}_f(z) := \frac{\tau}{\mu}\Big(z + \sum_{k=n+1}^{\infty}[k(1-\theta)+\theta][k(1-\lambda)+\lambda]\frac{\mu\Gamma(k+\tau)\Gamma(\mu)}{\tau\Gamma(k+\mu)\Gamma(\tau)}a_k z^k\Big). \tag{15}$$

When considering the function classes in (1) and (2), it is clear that

$$\frac{\mu}{\tau}\mathbb{T}_f(z) \in \mathcal{A}(n) \quad \text{and} \quad \frac{d}{dz}(\mathbb{T}_f(z)) \in \mathcal{H}_n(\tau/\mu), \tag{16}$$

and

$$\frac{\mu}{\tau}\mathbf{T}_f(z) \in \mathcal{A}(n) \quad \text{and} \quad \frac{d}{dz}(\mathbf{T}_f(z)) \in \mathcal{H}_n(\tau/\mu) \tag{17}$$

and, under the conditions accentuated in (10), the following-comprehensive relations between the related operators, which were also indicated in [11, 12, 13, 16], can be easily emphasized by the equivalent statements:

$$\begin{cases} \theta := 1 \Rightarrow \mathbf{T}_f(z) \equiv \mathbb{T}_f(z), \\ \theta := 0 \Rightarrow \mathbf{T}_f(z) \equiv \frac{d}{dz}(\mathbb{T}_f(z)), \\ \theta := 1 =: \lambda \Rightarrow \mathbf{T}_f(z) \equiv \mathbb{T}_f(z) \equiv \mathcal{T}_{\tau,\mu}[f](z), \\ \theta := 1 =: \lambda - 1 \Rightarrow \mathbf{T}_f(z) \equiv \mathbb{T}_f(z)z \equiv \frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f](z)), \\ \theta := 0 =: \lambda \Rightarrow \mathbf{T}_f(z) \equiv z\frac{d}{dz}(\mathbb{T}_f(z)) \equiv z\frac{d}{dz}\Big[z\frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f](z))\Big], \\ \theta := 0 =: \lambda - 1 \Rightarrow \mathbf{T}_f(z) \equiv z\frac{d}{dz}(\mathbb{T}_f(z)) \equiv \frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f](z)). \end{cases} \tag{18}$$

2. A set of main results and related implications

The well-known assertion (Lemma 2.1, below), proven by Nunokawa [19], will be needed to state and also prove main results of this investigation. For both its details and some implications of the earlier results demonstrated by the help of Lemma 2.1, one may refer to [17] and [18], and, as example, also [9, 10, 21].

Lemma 2.1 ([12])
Let

$$b \geq \frac{1}{2}, \quad a \in \mathbb{R}^*, \quad p(z) \in \mathcal{H}_n \quad \textit{and} \quad z \in \mathbb{U}. \tag{19}$$

At this stage, if there exists a point z_0 in the open disk $\mathbb{U}$ such that

$$\Re e(p(z)) > 0 \ \ \textit{for}\ |z| < |z_0| < 1, \quad \Re e(p(z_0)) = 0 \quad \textit{and} \quad p(z_0) \neq 0, \tag{20}$$

then

$$p(z_0) = ia \quad \textit{and} \quad z_0 p'(z_0) = ib\Big(a + \frac{1}{a}\Big)p(z_0). \tag{21}$$

By making use of the lemma above, we can now prove our main results which consist of various comprehensive implications relating to the functions of the forms (1) and (2).

Theorem 2.1
Let $f \equiv f(z) \in \mathcal{A}(n)$ and also let $\boldsymbol{T}_f(z)$ be given by (13). *For the admissible values of the parameters restricted by the conditions in* (10), *if*

$$\left|\operatorname{Arg}\left\{z\frac{d^2}{dz^2}(\boldsymbol{T}_f(z))\right\}\right| \neq \pi, \tag{22}$$

then

$$\Re e\left\{\frac{d}{dz}(\boldsymbol{T}_f(z))\right\} > \frac{\tau}{\mu}\alpha \tag{23}$$

where $0 \leq \alpha < 1$ *and* $z \in \mathbb{U}$.

Proof. Firstly, for a function $f := f(z) \in \mathcal{A}(n)$, let $\mathbf{T}_f(z)$ be of the form (13). Under conditions (10) and with the help of the result in (15), let us consider an implicit function $p(z)$,

$$\frac{d}{dz}(\mathbf{T}_f(z)) = \frac{\tau}{\mu}[\alpha + (1-\alpha)p(z)], \tag{24}$$

where $0 \leq \alpha < 1$ and $z \in \mathbb{U}$. It is obvious that $p(z) \in \mathcal{H}_n$ is an analytic function in $\mathbb{U}$ and also satisfies $p(0) = 1$ in Lemma 2.1. By (24), it follows that

$$z\frac{d^2}{dz^2}(\mathbf{T}_f(z)) = (1-\alpha)zp'(z), \qquad 0 \leq \alpha < 1,\ z \in \mathbb{U}. \tag{25}$$

For the desired proof, there is a need now to assume that there exists a point $z_0 \in \mathbb{U}$ satisfying one of the conditions given in (20), which is

$$\Re e(p(z_0)) = 0, \qquad z_0 \in \mathbb{U},\ p(z_0) \neq 0. \tag{26}$$

Under the conditions given in (19) and (20), and in view of (25), by applying the results given in (21) of Lemma 2.1, we get

$$\begin{aligned} z\frac{d^2}{dz^2}(\mathbf{T}_f(z))\Big|_{z:=z_0} &= (1-\alpha)\frac{\tau}{\mu}zp'(z)\Big|_{z:=z_0} \\ &= (1-\alpha)\frac{\tau}{\mu}ib\Big(a+\frac{1}{a}\Big)p(z_0) \\ &= -(1-\alpha)\frac{\tau}{\mu}b(1+a^2). \end{aligned} \tag{27}$$

Thus, in consideration of the admissible values of the parameters presented in (10) and (19), from (27) it follows that

$$\left|\operatorname{Arg}\Big(z\frac{d^2}{dz^2}(\mathbf{T}_f(z))\Big|_{z:=z_0}\Big)\right| = \left|\operatorname{Arg}\Big(-(1-\alpha)\frac{\tau}{\mu}\frac{b}{2}(1+a^2)\Big)\right| = |\pm\pi| = \pi,$$

which contradicts the inequality given by (22). This means that there is no any point $z_0 \in \mathbb{U}$ satisfying the condition supposed in (26). In this case, it have to satisfy the inequality given by

$$\Re e(p(z)) > 0 \qquad \text{for all } z \in \mathbb{U}.$$

So that the mentioned expression, stated in (24), promptly requires the following inequality

$$\Re e\Big\{\frac{d}{dz}(\mathbf{T}_f(z))\Big\} = \Re e\Big\{\frac{\mu}{\tau}[\alpha + (1-\alpha)p(z)]\Big\} > 0, \qquad z \in \mathbb{U},$$

which immediately implies the provision (of Theorem 2.1) given in (23). This also finishes the desired proof.

For an analytic function $f(z) \in \mathcal{A}(n)$, by taking account of the definitions in (11), (13) and (15), for each one of the proofs of the following theorems, which are Theorems 2.2–2.4 below, it is enough to take into consideration the same definition of the function $p(z)$ established by (24), also follow all the similar steps followed in the proof of Theorem 2.1, and then use the same assumption given in (26) together with the other assertions of Lemma 2.1. Therefore, the similar details of their proofs are omitted and we hope that everyone can easily complete those.

Theorem 2.2
Let $f \equiv f(z) \in \mathcal{A}(n)$ and also let $\boldsymbol{T}_f(z)$ be of the form in (13). For the admissible values of the parameters restricted by (10), if the inequality

$$\Big|z\frac{d^2}{dz^2}(\boldsymbol{T}_f(z))\Big| < \frac{\tau(1-\alpha)}{2\mu}$$

is satisfied, then the inequality in (23) is also satisfied, where $0 \leq \alpha < 1$ and $z \in \mathbb{U}$.

Theorem 2.3
Let $f \equiv f(z) \in \mathcal{A}(n)$ and also let $\boldsymbol{T}_f(z)$ be of the form in (13). For the admissible values of the parameters restricted by the conditions in (10), if the condition:

$$\Re e\Big\{z\frac{d^2}{dz^2}(\boldsymbol{T}_f(z))\Big\} > \frac{\tau(1-\alpha)}{2\mu}$$

is provided, then the condition in (23) is also provided, where $0 \leq \alpha < 1$ and $z \in \mathbb{U}$.

Theorem 2.4
Let $f \equiv f(z) \in \mathcal{A}(n)$ and also let $\boldsymbol{T}_f(z)$ be of the form in (13). For the admissible values of the parameters restricted by the conditions in (10), if the expression:

$$\Im m\left\{z\frac{d^2}{dz^2}\Big(\boldsymbol{T}_f(z)\Big)\right\} \neq 0$$

holds true, then the expression in (23) also holds true, where $0 \leq \alpha < 1$ and $z \in \mathbb{U}$.

As we constituted above in this section, namely, after some information associating with four comprehensive operators, defined as (3), (4) and (11)–(13), have been firstly introduced for applying to the analytic functions belonging to the class $\mathcal{A}(n)$, various special applications of them, given as (5)–(9) and (14)–(17), have been then presented. By focusing on the main results above, it can be easily seen that there is a great number of special results. For this reason, we think here that there is a need to point out some extra information as certain conclusions and some recommendations for the related researchers. When considering admissible values of the related parameters limited by the conditions given in (10) together with all theorems (Theorem 2.1–Theorem 2.4), too much special results relating to applications of the operators given by (3), (4), (12) and (13) can be then revealed. In fact, as we have indicated before, some of those can be compared with some special results given by [16]–[13]. In the light of the new information indicated as above, of course, it is impossible to present all of them but, for you, we want to constitute only three of those implications as examples. To reveal (or, redetermine) the others omitted in this note, there also needs extra searches of the researchers who have been working on the topics of this investigation.

Firstly, by means of the relations given by (18) and (12), taking the value of the parameter θ as $\theta := 1$ in Theorem 2.1, one of the special results can be stated as:

Proposition 2.5
Let $f \equiv f(z) \in \mathcal{A}(n)$ and $z \in \mathbb{U}$. For the admissible values of the parameters limited by the conditions in

$$0 \leq \alpha < 1, \quad 0 \leq \lambda \leq 1, \quad 0 < \mu \leq 1, \quad 0 < \tau \leq 1 \quad \text{and} \quad 0 \leq \tau - \mu < 1,$$

the following statement is true

$$\Big|\mathrm{Arg}\Big\{z\Big[\lambda\frac{d^2}{dz^2}(\mathcal{T}_{\tau,\mu}[f]) + (1-\lambda)z\frac{d^3}{dz^3}(\mathcal{T}_{\tau,\mu}[f])\Big]\Big\}\Big| \neq \pi$$
$$\Longrightarrow \Re e\Big\{\lambda\frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f]) + (1-\lambda)z\frac{d^2}{dz^2}(\mathcal{T}_{\tau,\mu}[f])\Big\} > \frac{\tau}{\mu}\alpha.$$

Secondly, by taking the value of the parameter λ as $\lambda := 1$ in Proposition 2.5 (or, by setting $\theta := 1$ and $\lambda := 1$ in Theorem 2.1), one of the special results can be also constituted as:

Proposition 2.6
Let $f \equiv f(z) \in \mathcal{A}(n)$ and $z \in \mathbb{U}$. For the admissible values of the parameters restricted by

$$0 \leq \alpha < 1, \quad 0 < \mu \leq 1, \quad 0 < \tau \leq 1 \quad \text{and} \quad 0 \leq \tau - \mu < 1,$$

the following statement holds true

$$\Big|\mathrm{Arg}\Big\{z\frac{d^2}{dz^2}(\mathcal{T}_{\tau,\mu}[f])\Big\}\Big| \neq \pi \Longrightarrow \Re e\Big\{\frac{d}{dz}(\mathcal{T}_{\tau,\mu}[f])\Big\} > \frac{\tau}{\mu}\alpha.$$

Lastly, by making use of the relationships given by (18), (12), (8) and (9), and by setting $\theta := 1$, $\tau := 1$ and $\mu := 1$ in Theorem 2.1, a number of extensive results, which also consist of several special results and are closely associated with (Analytic and) Geometric Function Theory (see, [4] and [6]), can be then composed by the following proposition.

PROPOSITION 2.7
Let $f(z) \in \mathcal{A}(n)$ and $z \in \mathbb{U}$, and let also $0 \leq \alpha < 1$. Then

$$|\mathrm{Arg}\{z[\lambda f''(z) + (1-\lambda)zf'''(z)]\}| \neq \pi \Longrightarrow \Re e\{\lambda f'(z) + (1-\lambda)zf''(z)\} > \alpha.$$

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