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Existence results of ψ -Hilfer integro-differential equations with fractional order in Banach space

Abstract. In this article we present the existence and uniqueness results for fractional integro-differential equations with ψ -Hilfer fractional derivative. The reasoning is mainly based upon different types of classical fixed point theory such as the Mönch fixed point theorem and the Banach fixed point theorem. Furthermore, we discuss E_α -Ulam-Hyers stability of the presented problem. Also, we use the generalized Gronwall inequality with singularity to establish continuous dependence and uniqueness of the δ -approximate solution.

1. Introduction

In this article, we consider the boundary value problem of ψ -Hilfer fractional derivative of the form

$$\begin{cases} {}^H D_{a^+}^{\alpha, \beta; \psi} y(t) = f(t, y(t), \int_a^t k(t, s)y(s)ds), & t \in J := (a, b], \\ I_{a^+}^{1-\gamma, \psi} [py(a^+) + qy(b^-)] = c, & (\gamma = \alpha + \beta - \alpha\beta), \end{cases} \quad (1)$$

where ${}^H D_{a^+}^{\alpha, \beta; \psi}(\cdot)$ is the generalized Hilfer fractional derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$ and $I_{a^+}^{1-\gamma, \psi}(\cdot)$ is the generalized fractional integral in the sense of Riemann-Liouville of order $1 - \gamma$, $\gamma = \alpha + \beta - \alpha\beta$ and $f: J \times E \times E \rightarrow E$ is a continuous function in Banach space E , E is an abstract Banach space,

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$p, q \in \mathbb{R}$, $c \in E$, $p + q \neq 0$ and $\int_a^t k(t, s)y(s)ds$ is a linear integral operator with $\eta = \max\{\int_a^t |k(t, s)|ds : (t, s) \in J \times J\}$, $k: J \times J \rightarrow \mathbb{R}$.

The fractional calculus has been given proper attention by many researchers in the last few decades. This branch of mathematics was founded by Leibniz and Newton in seventeenth century. In the eighteenth century some notable definitions about fractional derivatives were given by some famous mathematicians like Riemann, Liouville, Grönwal, Letnikove, Hadamard and many others, for more detail see [10, 12, 16, 17]. In the last few decades significant work has been done on various aspects of fractional calculus due to the fact that, the modelling of various phenomenons in the fields of science and engineering is done more precisely using fractional differential equations as compared to ordinary differential equations. Since a boundary value problem of differential equations represent an important class of applied analysis, the area mentioned was given more importance, see [1, 2, 8, 16, 23] and the references therein. An important characteristic is that engineers and scientists have developed some new models that involve fractional differential equations. These models have been applied successfully, for instance in theory of viscoelasticity and viscoplasticity, modelling of polymers and proteins, transmission of ultrasound waves, modelling of human tissue under mechanical loads, etc. There have been extensive consideration in the last decades of the existence theory of boundary value problems including fractional differential equation, see [5, 4, 19, 7, 3, 18].

This paper is organized as follows. In Section 2, we introduce some notations, definitions, and preliminary facts, which are use throughout this paper. By using measure of noncompactness and Mönch fixed point theorem we present the existence result of our problem in Section 3. We discuss E_α -Ulam-Hyers stability of problem (1) in Section 4. Finally, in Section 5, by using generalized Gronwall inequality with singularity we establish continuous dependence and uniqueness of δ -approximate solution of problem (1).

2. Preliminaries

Let $J := (a, b]$, $(-\infty < a < b < \infty)$ be a finite interval and let $C[J, E]$ be the Banach space of continuous functions on J into E with the norm $\|y\|_{C[J, E]} = \sup\{\|y(t)\| : t \in J\}$, $\psi: J \rightarrow \mathbb{R}$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in J$. For $0 \leq \gamma < 1$ and $n \in \mathbb{N}$, the weighted spaces $C_{1-\gamma; \psi}[J, E]$, $C_{1-\gamma; \psi}^n[J, E]$ of continuous function $f: (a, b] \rightarrow E$ are defined by

$$C_{1-\gamma; \psi}[J, E] = \{f: (a, b] \rightarrow E : (\psi(\cdot) - \psi(a))^{1-\gamma} f(\cdot) \in C[J, E]\},$$

$$C_{1-\gamma; \psi}^n[J, E] = \{f: (a, b] \rightarrow E : f \in C^{n-1}[J, E], f^{(n)} \in C_{1-\gamma; \psi}[J, E]\}.$$

Obviously $C_{1-\gamma; \psi}[J, E]$, $C_{1-\gamma; \psi}^n[J, E]$ are the Banach spaces with the norms

$$\|f\|_{C_{1-\gamma; \psi}[J, E]} = \max_{t \in [a, b]} \|(\psi(t) - \psi(a))^{1-\gamma} f(t)\|,$$

$$\|f\|_{C_{1-\gamma; \psi}^n[J, E]} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{C[J, E]} + \|f^{(n)}\|_{C_{1-\gamma; \psi}[J, E]},$$

respectively. For $n = 0$, we have $C_{1-\gamma;\psi}^0[J, E] = C_{1-\gamma;\psi}[J, E]$.

DEFINITION 2.1 ([20])

Let $\alpha > 0$ and ψ be a positive and increasing function, having a continuous derivative ψ' on the interval (a, b) . Then the left-sided ψ -Riemann-Liouville fractional integral of a function $f: [a, \infty) \rightarrow \mathbb{R}$ of order α is defined by

$$I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_{\psi}^{\alpha-1}(t, s) f(s) ds,$$

where $\mathcal{N}_{\psi}^{\alpha-1}(t, s) = \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}$ and Γ is a gamma function.

DEFINITION 2.2 ([20])

Let $n - 1 < \alpha < n$ with $n \in \mathbb{N}$, let $[a, b]$ be an interval $(-\infty < a < b < \infty)$ and $f, \psi \in C^n[a, b]$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a, b]$. The left-sided ψ -Hilfer fractional derivative of function f of order α and type $0 \leq \beta \leq 1$ is defined by

$${}^H D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t).$$

THEOREM 2.1 ([20])

Let $f \in C^1[a, b]$, $0 < \alpha < 1$, and $0 \leq \beta \leq 1$. Then

$${}^H D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} f(t) = f(t).$$

LEMMA 2.1 ([12])

Let $\alpha, \gamma > 0$, then

$$I_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} (\psi(t) - \psi(a))^{\alpha+\gamma-1}$$

and

$$D_{a^+}^{\gamma;\psi} (\psi(t) - \psi(a))^{\gamma-1} = 0,$$

where

$$D_{a^+}^{\gamma;\psi} y(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) I_{a^+}^{(1-\beta)(1-\alpha);\psi} y(t).$$

LEMMA 2.2 ([20])

If $f \in C^1[J, E]$, $0 < \alpha < 1$, and $0 \leq \beta \leq 1$, then

$$I_{a^+}^{\alpha;\psi} {}^H D_{a^+}^{\alpha,\beta;\psi} f(t) = f(t) - \frac{I_{a^+}^{(1-\beta)(1-\alpha);\psi} f(a)}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1}.$$

Now, we give definitions of fundamental spaces. For $\gamma = \alpha + \beta - \alpha\beta$ and $0 < \alpha, \beta, \gamma < 1$, $0 \leq \mu < 1$, we define

$$C_{1-\gamma,\psi}^{\alpha,\beta}[J, E] = \{f \in C_{1-\gamma,\psi}[J, E] : {}^H D_{a^+}^{\alpha,\beta;\psi} f \in C_{1-\gamma,\psi}[J, E]\},$$

$$C_{1-\gamma,\psi}^{\gamma}[J, E] = \{f \in C_{1-\gamma,\psi}[J, E] : D_{a^+}^{\gamma;\psi} f \in C_{1-\gamma,\psi}[J, E]\}.$$

It is clear that $C_{1-\gamma,\psi}^\gamma[J, E] \subset C_{1-\gamma,\psi}^{\alpha,\beta}[J, E]$.

Next, we introduce the Hausdorff measure of noncompactness $\Phi(\cdot)$ on each bounded subset $K \subset E$ by

$$\Phi(K) = \inf\{r > 0 : \text{for which } K \text{ has a finite } r\text{-net in } E\}.$$

In the following Lemmas, we recall some basic properties of $\Phi(\cdot)$.

LEMMA 2.3 ([6])

Let A_1, A_2 be a nonempty subsets of a Banach space E . The measure of noncompactness $\Phi(\cdot)$ satisfies:

- (1) $\Phi(A) = 0$ if and only if A is precompact in E ;
- (2) for all bounded subsets A_1, A_2 of E , $A_1 \subseteq A_2$ implies $\Phi(A_1) \leq \Phi(A_2)$;
- (3) $\Phi(\{x\} \cup A) = \Phi(A)$ for every $x \in E$ and every nonempty subset $A \subseteq E$;
- (4) $\Phi(A) = \Phi(\bar{A}) = \Phi(\text{conv } A)$, where \bar{A} is the closure of A and $\text{conv } A$ is the convex hull of A ;
- (5) $\Phi(A_1 + A_2) \leq \Phi(A_1) + \Phi(A_2)$, where $A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$;
- (6) $\Phi(A_1 \cup A_2) = \max\{\Phi(A_1), \Phi(A_2)\}$;
- (7) $\Phi(\lambda A) \leq |\lambda|\Phi(A)$ for any $\lambda \in \mathbb{R}$.

For any $\mathcal{V} \subset C[J, E]$, we define

$$\int_a^t \mathcal{V}(s)ds = \left\{ \int_a^t u(s)ds : u \in \mathcal{V} \right\} \quad \text{for } t \in J,$$

where $\mathcal{V}(s) = \{u(s) \in E : u \in \mathcal{V}\}$.

LEMMA 2.4 ([9])

If $\mathcal{V} \subset C[J, E]$ is bounded and equicontinuous, then $t \rightarrow \Phi(\mathcal{V}(t))$ is continuous on J , and

$$\Phi(\mathcal{V}(J)) = \max_{t \in J} \Phi(\mathcal{V}(t)), \quad \left(\Phi\left(\int_a^t \mathcal{V}(s)ds \right) \leq \int_a^t \Phi(\mathcal{V}(s))ds \right) \quad \text{for } t \in J,$$

where $\mathcal{V}(J) = \{u(s) : u \in \mathcal{V}, s \in J\}$.

LEMMA 2.5 ([9])

Let $h \in L^1(J, \mathbb{R}^+)$ and $\{u_n\}_{n=1}^\infty$ be a sequence of Bochner integrable functions from J into E with $\|u_n(t)\| \leq h(t)$ for almost all $t \in J$ and every $n \geq 1$, then the function $\mathcal{W}(t) = \Phi(\{u_n(t)\}_{n=1}^\infty)$ belongs to $L^1(J, \mathbb{R}^+)$ and satisfies

$$\Phi\left(\int_a^t u_n(s)ds : n \geq 1 \right) \leq 2 \int_a^t \mathcal{W}(s)ds \quad \text{for } t \in J.$$

LEMMA 2.6 ([14])

Let \mathcal{U} be a closed convex and nonempty subset of a Banach space E with $0 \in \mathcal{U}$. Suppose that $G: \mathcal{U} \rightarrow E$ is a continuous map satisfying the Mönch's condition (i.e. if set $M \subseteq \mathcal{U}$ is countable and $M \subseteq \text{conv}(\{0\} \cup G(M))$, then M is relatively compact), then G has a fixed point in \mathcal{U} .

LEMMA 2.7 ([21])

Let $\theta: (0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then the ψ -Hilfer problem

$$\begin{aligned} {}^H D_{0+}^{\alpha, \beta; \psi} u(t) &= \theta(t, u(t)), & t \in (0, b], \\ I_{0+}^{1-\gamma; \psi} u(0) &= u_0, \end{aligned}$$

is equivalent to the integral equation

$$u(t) = \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \theta(s, u(s)) ds.$$

THEOREM 2.2

Let $f: J \times E \times E \rightarrow E$ be a continuous function such that $f \in C_{1-\gamma, \psi}[J, E]$ for all $y \in C_{1-\gamma, \psi}[J, E]$. Then the problem (1) is equivalent to the following integral equation

$$\begin{aligned} y(t) &= \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \left\{ c - \frac{q}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_{\psi}^{\alpha-\gamma}(b, s) f(s, y(s), (By)(s)) ds \right\} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_{\psi}^{\alpha-1}(t, s) f(s, y(s), (By)(s)) ds, \end{aligned}$$

where $(By)(t) := \int_a^t k(t, s) y(s) ds$ and $\mathcal{N}_{\psi}^{\alpha-1}(t, s) := \psi'(s) (\psi(t) - \psi(s))^{\alpha-1}$.

Proof. In view of lemma 2.7, a solution of the first equation of (1) can be expressed by

$$u(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a+}^{1-\gamma; \psi} y(a^+) + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_{\psi}^{\alpha}(t, s) f(s, y(s), (By)(s)) ds.$$

Now, by using condition $I_{a+}^{1-\gamma, \psi}[py(a^+) + qy(b^-)] = c$, we get

$$\begin{aligned} y(t) &= \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \left\{ c - \frac{q}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_{\psi}^{\alpha-\gamma}(b, s) f(s, y(s), (By)(s)) ds \right\} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_{\psi}^{\alpha-1}(t, s) f(s, y(s), (By)(s)) ds. \end{aligned}$$

For more details, see [4, 19].

3. Existence and uniqueness of solution

To obtain our results, the following conditions must be satisfied.

(H₁) The function $f : J \times E \times E \rightarrow E$ satisfies a Carathéodory condition.

(H₂) There exists a function $\mu \in C_{1-\gamma, \psi}[J, E]$ such that

$$\|f(t, x, y)\| \leq \|\mu(t)\|, \quad t \in J, \quad x, y \in E.$$

(H₃) There exist constant numbers $L, M > 0$ such that

$$\|f(t, x_1, By_1) - f(t, x_2, By_2)\| \leq L\|x_1 - x_2\| + M\|By_1 - By_2\|,$$

for each $t \in J$ and $x_1, x_2, y_1, y_2 \in E$ and

$$\|By_1 - By_2\| \leq \eta\|y_1 - y_2\|.$$

(H₄) There exist constants $\widehat{L}, \widehat{M} > 0$ such that

$$\Phi(f(t, y_1, y_2)) \leq \widehat{L}\Phi(y_1) + \widehat{M}\Phi(y_2), \quad t \in J,$$

where y_1, y_2 are bounded subsets of E .

Now, by using the Mönch fixed point theorem, we present the existence result for the problem (1).

THEOREM 3.1

Assume that $f : J \times E \times E \rightarrow E$ is a function such that $f(\cdot, y(\cdot), (By)(\cdot)) \in C_{1-\gamma, \psi}^{\beta(1-\alpha)}[J, E]$ for all $y \in C_{1-\gamma, \psi}[J, E]$ and satisfies the conditions (H₁)–(H₄). If

$$\Theta := \left[\left(\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right) (\psi(b) - \psi(a))^\alpha \right] < 1.$$

Then the problem (1) has at least one solution in $C_{1-\gamma, \psi}^\gamma[J, E]$.

Proof. Consider the operator $\mathcal{G} : C_{1-\gamma, \psi}[J, E] \rightarrow C_{1-\gamma, \psi}[J, E]$ defined by

$$\begin{aligned} \mathcal{G}y(t) &= \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \\ &\quad \times \left\{ c - \frac{q}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) f(s, y(s), (By)(s)) ds \right\} \quad (2) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, y(s), (By)(s)) ds. \end{aligned}$$

Clearly, \mathcal{G} is well defined and the fixed point of the operator \mathcal{G} is a solution of the problem (1). Define a bounded, closed and convex set

$$\mathbb{k}_\xi = \{y \in C_{1-\gamma, \psi}[J, E] : \|y\|_{C_{1-\gamma, \psi}} \leq \xi\}$$

of Banach space $C_{1-\gamma,\psi}[J, E]$ with $\xi \geq \frac{\omega}{1-\Theta}$, where

$$\omega := \left[\frac{\|c\|}{|p+q|} \frac{1}{\Gamma(\gamma)} \right].$$

Claim(1). The operator \mathcal{G} maps the set \mathbb{k}_ξ in to itself ($\mathcal{G}\mathbb{k}_\xi \subset \mathbb{k}_\xi$).

For any $y \in \mathbb{k}_\xi$, $t \in J$, we have

$$\begin{aligned} & \|\mathcal{G}y(t)(\psi(t) - \psi(a))^{1-\gamma}\| \\ & \leq \left| \frac{c}{p+q} \right| \frac{1}{\Gamma(\gamma)} + \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)\Gamma(\alpha-\gamma+1)} \\ & \quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) \|f(s, y(s), (By)(s))\| ds \\ & \quad + \frac{(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) \|f(s, y(s), (By)(s))\| ds \\ & \leq \left| \frac{c}{p+q} \right| \frac{1}{\Gamma(\gamma)} + \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)\Gamma(\alpha-\gamma+1)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) \|\mu(t)\| ds \\ & \quad + \frac{(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) \|\mu(t)\| ds \\ & \leq \left| \frac{c}{p+q} \right| \frac{1}{\Gamma(\gamma)} + \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)\Gamma(\alpha-\gamma+1)} \\ & \quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) (\psi(s) - \psi(a))^{\gamma-1} \|\mu\|_{C_{1-\gamma,\psi}[J,E]} ds \\ & \quad + \frac{(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) (\psi(s) - \psi(a))^{\gamma-1} \|\mu\|_{C_{1-\gamma,\psi}[J,E]} ds \\ & \leq \left| \frac{c}{p+q} \right| \frac{1}{\Gamma(\gamma)} + \left| \frac{q}{p+q} \right| \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)} \|\mu\|_{C_{1-\gamma,\psi}[J,E]} \\ & \quad + \frac{\Gamma(\gamma)(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha+\gamma)} \|\mu\|_{C_{1-\gamma,\psi}[J,E]} \\ & \leq \left| \frac{c}{p+q} \right| \frac{1}{\Gamma(\gamma)} + \left(\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right) \\ & \quad \times (\psi(b) - \psi(a))^\alpha \|\mu\|_{C_{1-\gamma,\psi}[J,E]} \\ & \leq \left[\left| \frac{c}{p+q} \right| \frac{1}{\Gamma(\gamma)} \right] + \left[\left(\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right) (\psi(b) - \psi(a))^\alpha \right] \xi \\ & \leq \omega + \Theta\xi. \end{aligned}$$

It follows that $\|\mathcal{G}y\|_{C_{1-\gamma,\psi}} \leq \xi$. Thus $\mathcal{G}\mathbb{k}_\xi \subset \mathbb{k}_\xi$.

Claim(2). The operator \mathcal{G} is continuous on \mathbb{K}_ξ .

Let $\{y_n\}_{n=1}^\infty$ be a sequence such that $y_n \rightarrow y$ in \mathbb{K}_ξ as $n \rightarrow \infty$, then for each $y \in \mathbb{K}_\xi$, $t \in J$, we have

$$\begin{aligned} & \|(\mathcal{G}y_n(t) - \mathcal{G}y(t))(\psi(t) - \psi(a))^{1-\gamma}\| \\ & \leq \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)\Gamma(\alpha+1-\gamma)} \\ & \quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b,s)(\psi(s) - \psi(a))^{\gamma-1} \|f_{y_n} - f_y\|_{C_{1-\gamma,\psi}} ds \\ & \quad + \frac{(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t,s)(\psi(s) - \psi(a))^{\gamma-1} \|f_{y_n} - f_y\|_{C_{1-\gamma,\psi}} ds \\ & \leq \left\{ \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right\} (\psi(b) - \psi(s))^\alpha \|f_{y_n} - f_y\|_{C_{1-\gamma,\psi}}, \end{aligned}$$

where $f_{y_n} = f(s, y_n(s), (By_n)(s))$ and $f_y = f(s, y(s), (By)(s))$. By Lebesgue convergence theorem, we conclude that

$$\|\mathcal{G}y_n - \mathcal{G}y\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence the operator \mathcal{G} is continuous on \mathbb{K}_ξ .

Claim(3). The operator \mathcal{G} is equicontinuous on \mathbb{K}_ξ .

For any $t_1, t_2 \in J$ such that $a < t_1 < t_2 < b, y \in \mathbb{K}_\xi$, we have

$$\begin{aligned} & \|\mathcal{G}y(t_2)(\psi(t_2) - \psi(a))^{1-\gamma} - \mathcal{G}y(t_1)(\psi(t_1) - \psi(a))^{1-\gamma}\| \\ & \leq \left\| (\psi(t_2) - \psi(a))^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_a^{t_2} \mathcal{N}_\psi^{\alpha-1}(t_2,s) f(s, y(s), (By)(s)) ds \right. \\ & \quad \left. - (\psi(t_1) - \psi(a))^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \mathcal{N}_\psi^{\alpha-1}(t_1,s) f(s, y(s), (By)(s)) ds \right\| \\ & \leq \left\| (\psi(t_2) - \psi(a))^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \mathcal{N}_\psi^{\alpha-1}(t_2,s) f(s, y(s), (By)(s)) ds \right. \\ & \quad \left. + (\psi(t_2) - \psi(a))^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \mathcal{N}_\psi^{\alpha-1}(t_2,s) f(s, y(s), (By)(s)) ds \right. \\ & \quad \left. - (\psi(t_1) - \psi(a))^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \mathcal{N}_\psi^{\alpha-1}(t_1,s) f(s, y(s), (By)(s)) ds \right\| \\ & \leq \frac{\|f_y\|_{C_{1-\gamma,\psi}[J,E]}}{\Gamma(\alpha)} \left\{ \int_a^{t_1} (\mathcal{N}_\psi^{\alpha-1}(t_2,s)(\psi(t_2) - \psi(a))^{1-\gamma} \right. \\ & \quad \left. - \mathcal{N}_\psi^{\alpha-1}(t_1,s)(\psi(t_1) - \psi(a))^{1-\gamma})(\psi(s) - \psi(a))^{\gamma-1} ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}(\psi(t_2) - \psi(a))^{1-\gamma} \right. \\ & \quad \left. \times (\psi(s) - \psi(a))^{\gamma-1} ds \right\} \rightarrow 0 \quad \text{as } t_2 \rightarrow t_1. \end{aligned}$$

Thus, $\mathcal{G}(\mathbb{k}_\xi)$ is equicontinuous, that is $\text{mod}_c(\mathcal{G}(\mathbb{k}_\xi)) = 0$, where $\text{mod}_c(\mathcal{G}(\mathbb{k}_\xi))$ is the modulus of equicontinuity of $\mathcal{G}(\mathbb{k}_\xi)$.

Claim(4). The Mönch condition is satisfied.

For brevity, let K be a bounded subset of a Banach space $C[J, E]$ and Ω be the measure of noncompactness in the Banach space $C[J, E]$ which is defined by

$$\Omega(K) = \max_{Z \in \nabla(K)} (\sigma(Z), \text{mod}_c(Z)),$$

where $\nabla(K)$ is the collection of all countable subsets of K , and σ is the real measure of noncompactness defined by

$$\sigma(Z) = \sup_{t \in [a, b]} e^{-Lt} \Phi(Z(t))$$

such that $Z(t) = \{y(t) : y \in Z\}$, $t \in J$, L is the suitably constant and $\text{mod}_c(Z)$ is the modulus of equicontinuity of Z given by

$$\text{mod}_c(Z) = \lim_c \sup_{\sigma \rightarrow 0} \max_{y \in Z} \max_{|t_2 - t_1| \leq \sigma} \|y(t_2) - y(t_1)\|.$$

Observe that Ω is well defined [19, 11] and is a monotone, nonsingular and regular measure of noncompactness. Let $\mathcal{U} \subset \mathbb{k}_\xi$ be a countable set such that $\mathcal{U} \subset \text{conv}(\mathcal{G}(\mathcal{U}) \cup \{0\})$. Now we need to show that \mathcal{U} is precompact. Let $\{x_n\}_{n=1}^\infty \subseteq \mathcal{G}(\mathcal{U})$ be a countable set. Then there exists a set $\{y_n\}_{n=1}^\infty$ such that $x_n(t) = (\mathcal{G}y_n)(t)$ for all $t \in J$, $n \geq 1$. Using (H₄) and Lemmas 2.3, 2.4, we get

$$\begin{aligned} & \Phi(\{x_n(t)\}_{n=1}^\infty) \\ &= \Phi(\{(\mathcal{G}y_n)(t)\}_{n=1}^\infty) \\ &\leq \left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \\ &\quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) \Phi(f(s, \{y_n(s)\}_{n=1}^\infty, (B\{y_n(s)\}_{n=1}^\infty))) ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) \Phi(f(s, \{y_n(s)\}_{n=1}^\infty, (B\{y_n(s)\}_{n=1}^\infty))) ds \\ &\leq \left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \\ &\quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) (\widehat{L} + 2\widehat{M}\eta) \sup_{s \in [a, b]} \Phi(\{y_n(s)\}_{n=1}^\infty) ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) (\widehat{L} + 2\widehat{M}\eta) \sup_{s \in [a, b]} \Phi(\{y_n(s)\}_{n=1}^\infty) ds \\ &\leq \left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \\ &\quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) \sup_{t \in [a, b]} e^{-Ls} \Phi(\{y_n(s)\}_{n=1}^\infty) ds \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) \sup_{s \in [a, b]} e^{-Ls} \Phi(\{y_n(s)\}_{n=1}^\infty) ds \\
& \leq \left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} \frac{\sigma(\{y_n\}_{n=1}^\infty)}{\Gamma(\alpha+1-\gamma)} \\
& \quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \\
& \quad + \frac{2\sigma(\{y_n\}_{n=1}^\infty)}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \\
& \leq \left[\left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \right. \\
& \quad \left. + \frac{2}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \right] \sigma(\{y_n\}_{n=1}^\infty).
\end{aligned}$$

It follows

$$\begin{aligned}
\sigma(\{x_n\}_{n=1}^\infty) & \leq \sup_{t \in [a, b]} e^{-Lt} \left[\left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \right. \\
& \quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \\
& \quad \left. + \frac{2}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \right] \sigma(\{y_n\}_{n=1}^\infty) \\
& \leq \mathcal{R} \sigma(\{y_n\}_{n=1}^\infty),
\end{aligned}$$

where $\mathcal{R} \in (0, 1)$ is the suitable constant, such that

$$\begin{aligned}
\mathcal{R} & = \sup_{t \in [a, b]} e^{-Lt} \left[\left| \frac{q}{p+q} \right| \frac{2}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \right. \\
& \quad \left. + \frac{2}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) e^{Ls} (\widehat{L} + 2\widehat{M}\eta) ds \right].
\end{aligned}$$

Notice that

$$\sigma(\{y_n\}_{n=1}^\infty) \leq \sigma(\mathcal{U}) \leq \sigma(\text{conv}(\mathcal{G}(\mathcal{U}) \cup \{0\})) = \sigma(\{x_n\}_{n=1}^\infty) \leq \mathcal{R} \sigma(\{y_n\}_{n=1}^\infty),$$

which implies that $\sigma(\{y_n\}_{n=1}^\infty) = 0$ and hence $\sigma(\{x_n\}_{n=1}^\infty) = 0$. Now, by step 3, we have an equicontinuous set $\{x_n\}_{n=1}^\infty$ on J . Hence $\Omega(\mathcal{U}) \leq \Omega(\text{conv}(\mathcal{G}(\mathcal{U}) \cup \{0\})) \leq \Omega(\mathcal{G}(\mathcal{U}))$, where $\Omega(\mathcal{G}(\mathcal{U})) = \Omega(\{x_n\}_{n=1}^\infty) = 0$. Thus \mathcal{U} is precompact. Hence, by Lemma 2.6, the operator \mathcal{G} has a fixed point y^* , which is a solution of the problem (1) in $C_{1-\gamma, \psi}[J, E]$. Finally, we need to show that such a fixed point $y^* \in C_{1-\gamma, \psi}[J, E]$ is in $C_{1-\gamma, \psi}^\gamma[J, E]$. Since y^* is a fixed point of operator \mathcal{G} in $C_{1-\gamma, \psi}[J, E]$, then, for each $t \in J$, we have

$$\begin{aligned}
y^*(t) &= \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \left\{ c - \frac{q}{\Gamma(\alpha+1-\gamma)} \right. \\
&\quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b,s) f(s, y^*(s), (By^*)(s)) ds \Big\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t,s) f(s, y^*(s), (By^*)(s)) ds.
\end{aligned}$$

Applying $D_{a^+}^\gamma$ on both sides and using Lemma 2.1, we get

$$\begin{aligned}
D_{a^+}^{\gamma,\psi} y^*(t) &= D_{a^+}^{\gamma,\psi} I_{a^+}^\alpha f(t, y^*(t), (By^*)(t)) \\
&= D_{a^+}^{\beta(1-\alpha),\psi} f(t, y^*(t), (By^*)(t)).
\end{aligned}$$

Since $\gamma \geq \alpha$ and $f \in C_{1-\gamma,\psi}^{\beta(1-\alpha)}[J, E]$, then the right hand side is in $C_{1-\gamma,\psi}[J, E]$ and hence $D_{a^+}^{\gamma,\psi} y^*(t) \in C_{1-\gamma,\psi}[J, E]$, which implies that $y^* \in C_{1-\gamma,\psi}^\gamma[J, E]$. As a consequence of the above steps, we conclude that the problem (1) has at least one solution in $C_{1-\gamma,\psi}^\gamma[J, E]$.

In the forthcoming theorem, by using Banach contraction principle, we present the uniqueness of solution for the problem (1).

THEOREM 3.2

Assume that (H_1) and (H_3) hold. If

$$\left[\left(\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right) (L + \eta M) (\psi(b) - \psi(a))^\alpha \right] < 1, \quad (3)$$

then the problem (1) has a unique solution in $C_{1-\gamma,\psi}^\gamma[J, E] \subset C_{1-\gamma,\psi}[J, E]$.

Proof. By using the Banach contraction principle we shall show that the operator \mathcal{G} , defined by (2), has a unique fixed point, which is a unique solution of the problem (1) in $C_{1-\gamma,\psi}[J, E]$. Let $y, v \in C_{1-\gamma,\psi}[J, E]$ and $t \in J$, then, by our hypotheses, we have

$$\begin{aligned}
&\|(\psi(t) - \psi(a))^{1-\gamma} [\mathcal{G}y(t) - \mathcal{G}v(t)]\| \\
&\leq \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)\Gamma(\alpha+1-\gamma)} \\
&\quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b,s) \|f(s, y(s), (By)(s)) - f(s, v(s), (Bv)(s))\| ds \\
&\quad + \frac{(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \\
&\quad \times \int_a^t \mathcal{N}_\psi^{\alpha-1}(t,s) \|f(s, y(s), (By)(s)) - f(s, v(s), (Bv)(s))\| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \frac{q}{p+q} \right| \frac{(L+\eta M)}{\Gamma(\gamma)\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b,s) \|y(s) - v(s)\| ds \\
&\quad + \frac{(L+\eta M)(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t,s) \|y(s) - v(s)\| ds \\
&\leq \left| \frac{q}{p+q} \right| \frac{(L+\eta M)}{\Gamma(\gamma)\Gamma(\alpha+1-\gamma)} \\
&\quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b,s) (\psi(s) - \psi(a))^{\gamma-1} \|y - v\|_{C_{1-\gamma,\psi}} ds \\
&\quad + \frac{(L+\eta M)(\psi(t) - \psi(a))^{1-\gamma}}{\Gamma(\alpha)} \\
&\quad \times \int_a^t \mathcal{N}_\psi^{\alpha-1}(t,s) (\psi(s) - \psi(a))^{\gamma-1} \|y - v\|_{C_{1-\gamma,\psi}} ds \\
&\leq \left[\left| \frac{q}{p+q} \right| \frac{(L+\eta M)}{\Gamma(\alpha+1)} (\psi(b) - \psi(a))^\alpha \right. \\
&\quad \left. + \frac{(L+\eta M)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} (\psi(t) - \psi(a))^\alpha \right] \|y - v\|_{C_{1-\gamma,\psi}} \\
&\leq \left[\left(\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right) \right. \\
&\quad \left. \times (L+\eta M)(\psi(b) - \psi(a))^\alpha \right] \|y - v\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

By (3), the operator $\mathcal{G}: C_{1-\gamma,\psi}[J, E] \rightarrow C_{1-\gamma,\psi}[J, E]$ is a contracting mapping. According to the Banach contraction principle we conclude that the operator \mathcal{G} has a unique fixed point y^* in $C_{1-\gamma,\psi}[J, E]$ which is a unique solution of (1).

4. E_α -Ulam-Hyers stability

In this part, we discuss the E_α -Ulam-Hyers stability of problem (1). The following observations are taken from [13, 15, 22].

LEMMA 4.1 ([21])

Let $\alpha > 0$ and x, y be two nonnegative function locally integrable on $[a, b]$. Assume that g is a continuous, nonnegative and nondecreasing function, and let $\psi \in C^1[a, b]$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in [a, b]$. If

$$x(t) \leq y(t) + g(t) \int_a^t \mathcal{N}_\psi^{\alpha-1}(t,s) x(s) ds, \quad t \in [a, b].$$

Then

$$x(t) \leq y(t) + \int_a^t \sum_{n=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \mathcal{N}_\psi^{n\alpha-1}(t,s) y(s) ds, \quad t \in [a, b].$$

If y be a nondecreasing function on $[a, b]$. Then we have

$$x(t) \leq y(t) E_\alpha(g(t)\Gamma(\alpha)[\psi(t) - \psi(a)]^\alpha), \quad t \in [a, b],$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function defined by

$$E_\alpha(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + 1)}, \quad y \in \mathbb{C}, \Re(\alpha) > 0.$$

REMARK 4.1

A function $z \in C_{1-\gamma, \psi}^\gamma[J, E]$ is a solution of the inequality

$$\|{}^H D_{a^+}^{\alpha, \beta, \psi} z(t) - f(t, z(t), (Bz)(t))\| \leq \varepsilon E_\alpha(\psi(t) - \psi(a))^\alpha, \quad t \in J, \quad (4)$$

if and only if there exists a function $g \in C_{1-\gamma, \psi}^\gamma[J, E]$ such that

- (i) $\|g(t)\| \leq \varepsilon E_\alpha((\psi(t) - \psi(a))^\alpha), \quad t \in J;$
- (ii) ${}^H D_{a^+}^{\alpha, \beta, \psi} z(t) = f(t, z(t), (Bz)(t)) + g(t), \quad t \in J.$

DEFINITION 4.1

The problem (1) is E_α -Ulam-Hyers stable with respect to $E_\alpha((\psi(t) - \psi(a))^\alpha)$ if there exists a real number $C_E > 0$ such that, for each $\varepsilon > 0$ and each $z \in C_{1-\gamma, \psi}^\gamma[J, E]$, which satisfies (4), there exists a solution $u \in C_{1-\gamma, \psi}^\gamma[J, E]$ of (1) such that

$$\|z(t) - u(t)\| \leq C_E \varepsilon E_\alpha((\psi(t) - \psi(a))^\alpha), \quad t \in J.$$

LEMMA 4.2

Let $\gamma = \alpha + \beta - \alpha\beta$ be such that $\alpha \in (0, 1)$, $\beta \in [0, 1]$. If a function $z \in C_{1-\gamma, \psi}^\gamma[J, E]$ satisfies (4), then z satisfies the following integral inequality

$$\begin{aligned} & \left\| z(t) - \mathcal{T}_z - \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, y(s), (By)(s)) ds \right\| \\ & \leq \varepsilon \left[\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} E_{\alpha, 2-\gamma}(\psi(t) - \psi(a))^\alpha + E_\alpha(\psi(t) - \psi(a))^\alpha \right], \end{aligned}$$

where

$$\mathcal{T}_z := \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \left\{ c - \frac{q}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) f(s, z(s), (Bz)(s)) ds \right\}$$

and $E_{\alpha, 2-\gamma}$ is the Mittag-Liffler function defined as $E_{\alpha, 2-\gamma} z = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+2-\gamma)}$.

Proof. According to Theorem 2.2 and Remark 4.1, the following equation

$${}^H D_{0^+}^{\alpha, \beta, \psi} z(t) = f(t, z(t), (Bz)(t)) + g(t), \quad t \in J$$

with condition $I_{a^+}^{1-\gamma, \psi} [pz(a^+) + qz(b^-)] = c$ is equivalent to the integral equation

$$\begin{aligned} z(t) = & \mathcal{T}_z - \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \frac{q}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) g(s) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, z(s), (Bz)(s)) ds + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) g(s) ds. \end{aligned}$$

The above implies that

$$\begin{aligned}
& \left\| z(t) - \mathcal{T}_z - \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, z(s), (Bz)(s)) ds \right\| \\
& \leq \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) \|g(s)\| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^t \mathcal{N}_\psi^{\alpha-1}(t, s) \|g(s)\| ds \\
& \leq \left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} \frac{(\psi(t) - \psi(a))^{\gamma-1} \varepsilon}{\Gamma(\alpha+1-\gamma)} \\
& \quad \times \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) E_\alpha((\psi(s) - \psi(a))^\alpha) ds \\
& \quad + \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \mathcal{N}_\psi^{\alpha-1}(t, s) E_\alpha((\psi(s) - \psi(a))^\alpha) ds \\
& \leq \left| \frac{q}{p+q} \right| \frac{\varepsilon}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(\psi(b) - \psi(a)^\alpha)^n}{\Gamma(n\alpha+2-\gamma)} + \varepsilon \sum_{n=0}^{\infty} \frac{(\psi(t) - \psi(a)^\alpha)^n}{\Gamma(n\alpha+1)} \\
& = \varepsilon \left[\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} E_{\alpha, 2-\gamma}(\psi(b) - \psi(a)^\alpha) + E_\alpha(\psi(t) - \psi(a)^\alpha) \right].
\end{aligned} \tag{5}$$

Now, in the following theorem we prove the E_α -Ulam-Hyers stability result for ψ -Hilfer problem (1).

THEOREM 4.1

Assume that (H_1) and (H_3) are satisfied. Then equation

$${}^H D_{0^+}^{\alpha, \beta; \psi} y(t) = f(t, y(t), (By)(t)), \quad t \in J \tag{6}$$

is E_α -Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$ and let $z \in C_{1-\gamma, \psi}^\gamma[J, E]$ be a solution of the following inequality

$$\| {}^H D_{0^+}^{\alpha, \beta; \psi} z(t) - f(t, z(t), (Bz)(t)) \| \leq \varepsilon E_\alpha(\psi(t) - \psi(a)^\alpha), \quad t \in J.$$

Let $u \in C_{1-\gamma, \psi}^\gamma[J, E]$ be a unique solution of the problem

$$\begin{aligned}
{}^H D_{a^+}^{\alpha, \beta; \psi} y(t) &= f\left(t, y(t), \int_a^t k(t, s) y(s) ds\right), \quad t \in J := (a, b], \\
I_{a^+}^{1-\gamma; \psi} [py(a^+) + qy(b^-)] &= I_{a^+}^{1-\gamma; \psi} [pz(a^+) + qz(b^-)],
\end{aligned}$$

Now, by Theorem 2.2, we have

$$u(t) = \mathcal{T}_u + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, u(s), (Bu)(s)) ds, \quad t \in J,$$

where

$$\mathcal{T}_u := \frac{(\psi(t) - \psi(a))^{\gamma-1}}{(p+q)\Gamma(\gamma)} \left\{ c - \frac{q}{\Gamma(\alpha+1-\gamma)} \int_a^b \mathcal{N}_\psi^{\alpha-\gamma}(b, s) f(s, u(s), (Bu)(s)) ds \right\}.$$

Since $I_{a^+}^{1-\gamma; \psi}[pu(a^+) + qu(b^-)] = I_{a^+}^{1-\gamma; \psi}[pz(a^+) + qz(b^-)]$, we can easily find that $\mathcal{T}_u = \mathcal{T}_z$. Hence using (5), we get for each $t \in J$,

$$\begin{aligned} \|z(t) - u(t)\| &\leq \left\| z(t) - \mathcal{T}_z - \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, z(s), (Bz)(s)) ds \right\| \\ &\quad + \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, z(s), (Bz)(s)) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{N}_\psi^{\alpha-1}(t, s) f(s, u(s), (Bu)(s)) ds \right\| \\ &\leq \varepsilon \left[\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} E_{\alpha, 2-\gamma}(\psi(t) - \psi(a))^\alpha + E_\alpha(\psi(t) - \psi(a))^\alpha \right] \\ &\quad + \frac{L + \eta M}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \|z(s) - u(s)\| ds. \end{aligned}$$

Using Lemma 4.1, we obtain

$$\begin{aligned} \|z(t) - u(t)\| &\leq \varepsilon \left[\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} E_{\alpha, 2-\gamma}(\psi(t) - \psi(a))^\alpha + E_\alpha(\psi(t) - \psi(a))^\alpha \right] \\ &\quad \times E_\alpha((L + \eta M)[\psi(t) - \psi(a)]^\alpha) \\ &\leq \varepsilon \left[\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} E_{\alpha, 2-\gamma}(\psi(b) - \psi(a))^\alpha + E_\alpha(\psi(b) - \psi(a))^\alpha \right] \\ &\quad \times E_\alpha((L + \eta M)[\psi(t) - \psi(a)]^\alpha) \\ &\leq C_E \varepsilon E_\alpha((L + \eta M)[\psi(t) - \psi(a)]^\alpha), \end{aligned}$$

where

$$C_E = \left[\left| \frac{q}{p+q} \right| \frac{1}{\Gamma(\gamma)} E_{\alpha, 2-\gamma}(\psi(b) - \psi(a))^\alpha + E_\alpha(\psi(b) - \psi(a))^\alpha \right].$$

Thus (6) is E_α -Ulam-Hyers stable.

5. δ -approximation solution

DEFINITION 5.1 ([19])

A function $y \in C_{1-\gamma, \psi}^\gamma[J, E]$ satisfying the ψ -Hilfer fractional integro-differential inequality

$$\| {}^H D_{a^+}^{\alpha, \beta, \psi} y(t) - f(t, y(t), (By)(t)) \| \leq \delta, \quad t \in J$$

and

$$I_{a^+}^{1-\gamma; \psi}[py(a^+) + qy(b^-)] = c$$

is called an δ -approximate solutions of the problem (1).

THEOREM 5.1

Assume that (H₁) and (H₃) hold. Let $y_i \in C_{1-\gamma,\psi}^\gamma[J, E]$, $i = 1, 2$, be an δ -approximation solution of the following problem

$$\begin{aligned} {}^H D_{a^+}^{\alpha,\beta,\psi} y_i(t) &= f(t, y_i(t), (By_i)(t)), \quad t \in J := (a, b], \\ I_{a^+}^{1-\gamma,\psi} [py_i(a^+) + qy_i(b^-)] &= \widehat{c}_i. \end{aligned} \quad (7)$$

Then

$$\begin{aligned} \|y_1 - y_2\|_{C_{1-\gamma,\psi}} &\leq \frac{1}{\Upsilon} \left\{ (\delta_1 + \delta_2) \left(\frac{(\psi(t) - \psi(a))^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} \right. \right. \\ &\quad + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma((n+1)\alpha+1)} (\psi(t) - \psi(a))^{(n+1)\alpha-\gamma+1} \\ &\quad \left. \left. + \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p} \left(\frac{1}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha} \right) \right\}, \end{aligned}$$

where

$$\Upsilon := 1 - \frac{q}{p} \left(1 + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha} \right).$$

Proof. Let $y_i \in C_{1-\gamma,\psi}^\gamma[J, E]$, $i = 1, 2$ be an δ -approximation solution of (7). Then by Definition 5.1 we have

$$\|{}^H D_{0^+}^{\alpha,\beta,\psi} y_i(t) - f(t, y_i(t), (By_i)(t))\| \leq \delta_i \quad (8)$$

and

$$I_{a^+}^{1-\gamma,\psi} [py_i(a^+) + qy_i(b^-)] = \widehat{c}_i.$$

Applying $I_{a^+}^{\alpha,\psi}$ on both sides of (8), we get

$$\begin{aligned} &\frac{(\psi(t) - \psi(a))^\alpha \delta_i}{\Gamma(\alpha+1)} \\ &\geq \left\| y_i(t) - \frac{(\psi(t) - \psi(a))^{\gamma-1} \widehat{c}_i}{p\Gamma(\gamma)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} I_{a^+}^{1-\gamma,\psi} y_i(b^-) \right. \\ &\quad \left. - I_{a^+}^{\alpha,\psi} f(t, y_i(t), (By_i)(t)) \right\|. \end{aligned} \quad (9)$$

From the fact $|x| - |y| \leq |x - y| \leq |x| + |y|$, we obtain

$$\begin{aligned} &\frac{(\psi(t) - \psi(a))^\alpha (\delta_1 + \delta_2)}{\Gamma(\alpha+1)} \\ &\geq \left\| y_1(t) - \frac{(\psi(t) - \psi(a))^{\gamma-1} \widehat{c}_1}{p\Gamma(\gamma)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} I_{a^+}^{1-\gamma,\psi} y_1(b^-) \right. \\ &\quad \left. - I_{a^+}^{\alpha,\psi} f(t, y_1(t), (By_1)(t)) \right\| \end{aligned}$$

$$\begin{aligned}
& + \left\| y_2(t) - \frac{(\psi(t) - \psi(a))^{\gamma-1} \widehat{c}_2}{p\Gamma(\gamma)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} I_{a^+}^{1-\gamma, \psi} y_2(b^-) \right. \\
& \left. - I_{a^+}^{\alpha, \psi} f(t, y_2(t), (By_2)(t)) \right\| \\
\geq & \left\| \left[y_1(t) - \frac{(\psi(t) - \psi(a))^{\gamma-1} \widehat{c}_1}{p\Gamma(\gamma)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} I_{a^+}^{1-\gamma, \psi} y_1(b^-) \right. \right. \\
& \left. \left. - I_{a^+}^{\alpha, \psi} f(t, y_1(t), (By_1)(t)) \right] - \left[y_2(t) - \frac{(\psi(t) - \psi(a))^{\gamma-1} \widehat{c}_2}{p\Gamma(\gamma)} \right. \right. \\
& \left. \left. + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} I_{a^+}^{1-\gamma, \psi} y_1(b^-) - I_{a^+}^{1-\gamma, \psi} y_2(b^-) \right] \right\| \\
\geq & \left\| [y_1(t) - y_2(t)] - \frac{(\psi(t) - \psi(a))^{\gamma-1} (\widehat{c}_1 - \widehat{c}_2)}{p\Gamma(\gamma)} \right. \\
& \left. + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} [I_{a^+}^{1-\gamma, \psi} y_1(b^-) - I_{a^+}^{1-\gamma, \psi} y_2(b^-)] \right. \\
& \left. - [I_{a^+}^{\alpha, \psi} f(t, y_1(t), (By_1)(t)) - I_{a^+}^{\alpha, \psi} f(t, y_2(t), (By_2)(t))] \right\| \\
\geq & \|y_1(t) - y_2(t)\| - \left| \frac{(\psi(t) - \psi(a))^{\gamma-1} (\widehat{c}_1 - \widehat{c}_2)}{p\Gamma(\gamma)} \right| \\
& - \left\| \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} [I_{a^+}^{1-\gamma, \psi} y_1(b^-) - I_{a^+}^{1-\gamma, \psi} y_2(b^-)] \right\| \\
& - \| [I_{a^+}^{\alpha, \psi} f(t, y_1(t), (By_1)(t)) - I_{a^+}^{\alpha, \psi} f(t, y_2(t), (By_2)(t))] \| .
\end{aligned}$$

In consequence, we have

$$\begin{aligned}
& \|y_1(t) - y_2(t)\| \\
& \leq \frac{(\psi(t) - \psi(a))^\alpha (\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} \|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)} \\
& \quad + \left\| \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} [I_{a^+}^{1-\gamma, \psi} y_1(b^-) - I_{a^+}^{1-\gamma, \psi} y_2(b^-)] \right\| \\
& \quad + \| [I_{a^+}^{\alpha, \psi} f(t, y_1(t), (By_1)(t)) - I_{a^+}^{\alpha, \psi} f(t, y_2(t), (By_2)(t))] \| \\
& \leq \frac{(\psi(t) - \psi(a))^\alpha (\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} \|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)} \\
& \quad + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} \| [I_{a^+}^{1-\gamma, \psi} y_1(b^-) - I_{a^+}^{1-\gamma, \psi} y_2(b^-)] \| \\
& \quad + \| [I_{a^+}^{\alpha, \psi} f(t, y_1(t), (By_1)(t)) - I_{a^+}^{\alpha, \psi} f(t, y_2(t), (By_2)(t))] \| \\
& \leq \frac{(\psi(t) - \psi(a))^\alpha (\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} \|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p\Gamma(\gamma)} I_{a^+}^{1-\gamma, \psi} \|y_1(b) - y_2(b)\| \\
& + (L + \eta M) I_{a^+}^{\alpha, \psi} \|y_1(t) - y_2(t)\| \\
\leq & \frac{(\psi(t) - \psi(a))^\alpha (\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} \|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)} \\
& + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p} \|y_1 - y_2\|_{C_{1-\gamma, \psi}} \\
& + (L + \eta M) I_{a^+}^{\alpha, \psi} \|y_1(t) - y_2(t)\| \\
\leq & \Theta(t) + (L + \eta M) I_{a^+}^{\alpha, \psi} \|y_1(t) - y_2(t)\|,
\end{aligned}$$

where

$$\begin{aligned}
\Theta(t) = & \frac{(\psi(t) - \psi(a))^\alpha (\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} + \frac{(\psi(t) - \psi(a))^{\gamma-1} \|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)} \\
& + \frac{(\psi(t) - \psi(a))^{\gamma-1} q}{p} \|y_1 - y_2\|_{C_{1-\gamma, \psi}}.
\end{aligned}$$

Using Lemma 4.1, we obtain

$$\begin{aligned}
& \|y_1(t) - y_2(t)\| \\
\leq & \Theta(t) + \sum_{n=1}^{\infty} (L + \eta M)^n I_{a^+}^{n\alpha, \psi} \Theta(t) \\
\leq & \Theta(t) + \frac{(\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} \sum_{n=1}^{\infty} (L + \eta M)^n I_{a^+}^{n\alpha, \psi} (\psi(t) - \psi(a))^\alpha \\
& + \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)} \sum_{n=1}^{\infty} (L + \eta M)^n I_{a^+}^{n\alpha, \psi} (\psi(t) - \psi(a))^{\gamma-1} \\
& + \frac{q}{p\Gamma(\gamma)} \|y_1 - y_2\|_{C_{1-\gamma, \psi}} \sum_{n=1}^{\infty} (L + \eta M)^n I_{a^+}^{n\alpha, \psi} (\psi(t) - \psi(a))^{\gamma-1} \\
\leq & \Theta(t) + \frac{(\delta_1 + \delta_2)}{\Gamma(\alpha + 1)} \sum_{n=1}^{\infty} (L + \eta M)^n \frac{\Gamma(\alpha + 1)}{\Gamma((n + 1)\alpha + 1)} (\psi(t) - \psi(a))^{(n+1)\alpha} \\
& + \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p\Gamma(\gamma)} \sum_{n=1}^{\infty} (L + \eta M)^n \frac{\Gamma(\gamma)}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha + \gamma - 1} \\
& + \frac{q}{p\Gamma(\gamma)} \|y_1 - y_2\|_{C_{1-\gamma, \psi}} \sum_{n=1}^{\infty} (L + \eta M)^n \frac{\Gamma(\gamma)}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha + \gamma - 1} \\
= & (\delta_1 + \delta_2) \left(\frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma((n + 1)\alpha + 1)} (\psi(t) - \psi(a))^{(n+1)\alpha} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p} \left(\frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha + \gamma - 1} \right) \\
& + \frac{q}{p} \|y_1 - y_2\|_{C_{1-\gamma, \psi}} ((\psi(t) - \psi(a))^{\gamma-1}) \\
& + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha + \gamma - 1}.
\end{aligned}$$

Hence for each $t \in [a, b]$ we have

$$\begin{aligned}
& \|(\psi(t) - \psi(a))^{1-\gamma} [y_1(t) - y_2(t)]\| \\
& \leq (\delta_1 + \delta_2) \left(\frac{(\psi(t) - \psi(a))^{\alpha - \gamma + 1}}{\Gamma(\alpha + 1)} \right. \\
& \quad + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma((n+1)\alpha + 1)} (\psi(t) - \psi(a))^{(n+1)\alpha - \gamma + 1} \Big) \\
& \quad + \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p} \left(\frac{1}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha} \right) \\
& \quad + \frac{q}{p} \|y_1 - y_2\|_{C_{1-\gamma, \psi}} \left(1 + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha} \right).
\end{aligned}$$

Thus

$$\begin{aligned}
\|y_1 - y_2\|_{C_{1-\gamma, \psi}} & \leq \frac{1}{\Upsilon} \left\{ (\delta_1 + \delta_2) \left(\frac{(\psi(t) - \psi(a))^{\alpha - \gamma + 1}}{\Gamma(\alpha + 1)} \right. \right. \\
& \quad + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma((n+1)\alpha + 1)} (\psi(t) - \psi(a))^{(n+1)\alpha - \gamma + 1} \Big) \\
& \quad \left. \left. + \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p} \left(\frac{1}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(t) - \psi(a))^{n\alpha} \right) \right\}. \tag{10}
\end{aligned}$$

REMARK 5.1

If $\delta_1 = \delta_2 = 0$ in the inequality (10), then we have

$$\|y_1 - y_2\|_{C_{1-\gamma, \psi}} \leq \frac{1}{\Upsilon} \frac{\|\widehat{c}_1 - \widehat{c}_2\|}{p} \left(\frac{1}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} \frac{(L + \eta M)^n}{\Gamma(n\alpha + \gamma)} (\psi(b) - \psi(a))^{n\alpha} \right),$$

which provides continuous dependence on the problem (1). In addition, if $\widehat{c}_1 = \widehat{c}_2$, then we have

$$\|y_1 - y_2\|_{C_{1-\gamma, \psi}} = 0,$$

which proves the uniqueness of solutions of the problem (1).

REMARK 5.2

- (1) If $p = 1$ and $q = 0$, then (1) reduces to the initial value problem of ψ -Hilfer fractional derivative.
- (2) If $p = 0$ and $q = 1$, then (1) reduces to the terminal value problem of ψ -Hilfer fractional derivative.
- (3) If $p = q = 1$ and $c = 0$, then (1) reduces to the anti-periodic value problem of ψ -Hilfer fractional derivative.

6. An example

Consider the following fractional differential equation with the boundary condition

$${}^H D_{0^+}^{\frac{1}{2}, \frac{1}{3}; 2^t} y(t) = \frac{1}{4} \left[\tan^{-1} y(t) + \int_0^t \sin \frac{s}{2} y(s) ds \right], \quad t \in (0, 1], \quad (11)$$

$$I_{0^+}^{\frac{1}{3}; 2^t} [y(0^+) + 2y(1^-)] = 3,$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \alpha + \beta - \alpha\beta = \frac{2}{3}$, $p = 1$, $q = 2$, $c = 3$, $J = (0, b] = (0, 1]$, $\psi(t) = 2^t$ and $\eta = \max_{t \in (0, 1]} \left\{ \int_0^t \sin \frac{s}{2} ds \right\}$. Define $f: (0, 1] \times E \times E \rightarrow E$ by

$$f(t, y(t), (By)(t)) = \frac{1}{4} \left[\tan^{-1} y(t) + \int_0^t \sin \frac{s}{2} y(s) ds \right], \quad t \in (0, 1].$$

Then, for any $t \in (0, 1]$ and $y, y^*: (0, 1] \rightarrow \mathbb{R}_+$, we have

$$\|f(t, y(t), (By)(t)) - f(t, y^*(t), (By^*)(t))\| \leq \frac{1}{4} (\|y - y^*\| + \eta \|y - y^*\|),$$

which implies that (H₁) and (H₃) are satisfied with $\eta = \max\{\int_0^t \sin \frac{s}{2} ds : t \in J\} \simeq 0.24$ and $M = L = \frac{1}{4}$. Further, by some simple calculations, we see that

$$\left[\left(\frac{q}{(p+q)\Gamma(\alpha+1)} + \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \right) (L + \eta M) (\psi(b) - \psi(a))^\alpha \right] \simeq 0.68 < 1.$$

Thus, (3) is satisfied. Now all the hypotheses in Theorem 3.2 are fulfilled. So the ψ -Hilfer problem (11) has a unique solution on $(0, 1]$. Let us notice that the following inequality

$$|{}^H D_{0^+}^{\frac{1}{2}, \frac{1}{3}; 2^t} z(t) - f(t, z(t), (Bz)(t))| \leq \varepsilon E_{\frac{1}{2}} ((2^t - 1)^{\frac{1}{2}})$$

is satisfied. Thus the equation (6) is E_α -Ulam-Hyers stable with

$$|z(t) - u(t)| \leq C_E \varepsilon E_{\frac{1}{2}} ((2^t - 1)^{\frac{1}{2}}), \quad t \in (0, 1],$$

where $C_E = [\frac{2}{3} E_{\frac{1}{2}, \frac{4}{3}}(1) + E_{\frac{1}{2}}(1)] > 0$.

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