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On continuous and residual spectra of operators connected with iterative functional equations

Abstract. The sum type operator F, given by

$$F[\varphi](x) := \sum_{\nu=0}^{\infty} 2^{-\nu}\varphi(2^{\nu}x),$$

will be considered on the space D of bounded real functions, equipped with the supremum norm and on its three proper closed subspaces. All the according restrictions are Banach space automorphisms. In their spectral theory some iterative functional equations arise in a natural way. We determine in all four cases the resolvent set, the point spectrum, the continuous spectrum and the residual spectrum.

1. Introduction

All the sets

$$D_{11} := \left\{\varphi\colon \mathbb{R} \longrightarrow \mathbb{R};\ \sum_{\nu=0}^{\infty} 2^{-\nu}\varphi(2^{\nu}x) \text{ converges for every } x \in \mathbb{R}\right\},$$
$$D_{21} := \{\varphi \in D_{11};\ \varphi \text{ bounded in a vicinity of } -\infty \text{ and of } +\infty\},$$
$$D_{31} := \{\varphi \in D_{11};\ \varphi \text{ bounded}\} = D,$$
$$D_{41} := \{\varphi \in D_{11};\ \varphi \text{ bounded and continuous}\},$$

as well as

$$D_{k2} := \{\varphi \in D_{k1};\ \varphi \text{ 1-periodic}\},\ 1 \le k \le 4,$$
$$D_{k3} := \{\varphi \in D_{k1};\ \varphi \text{ even}\},\ 1 \le k \le 4,$$
$$D_{k4} := \{\varphi \in D_{k1};\ \varphi \text{ 1-periodic and even}\},\ 1 \le k \le 4,$$

are real vector spaces and, in particular, the sets D_{3m} and D_{4m}, $1 \le m \le 4$, are real Banach spaces (equipped with the supremum norm).

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The sum type operators $F_{km}\colon D_{km} \longrightarrow F_{km}(D_{km})$, given by

$$F_{km}[\varphi](x) := \sum_{\nu=0}^{\infty} 2^{-\nu}\varphi(2^{\nu}x), \tag{1}$$

are vector space isomorphisms and, in particular, the operators F_{3m} and F_{4m}, $1 \leq m \leq 4$, are Banach space automorphisms.

Motivations to study the operators F_{km} and further references are given in [1], [2] and [3]. The structure of the basic domain D_{11} is described in [3]. A first connection of our operators to iterative functional equations is given in

Proposition 1
Assume that $\varphi \in D_{km}$. Then $f = F_{km}[\varphi]$ satisfies

$$\forall\, x \in \mathbb{R}:\ f(x) - \frac{1}{2}f(2x) = \varphi(x). \tag{2}$$

A second connection appears when describing the eigenspace $E(F_{km}, \lambda)$ of F_{km} with respect to the eigenvalue $\lambda \in \sigma_p(F_{km})$:

Proposition 2
$\varphi \in E(F_{km}, \lambda)$ iff $\varphi \in D_{km}$ and

$$\forall\, x \in \mathbb{R}:\ \varphi(x) = \gamma\varphi(2x), \quad \gamma := \frac{1}{2}\frac{\lambda}{\lambda - 1}. \tag{3}$$

A proof of the de Rham type equation (2) for $F_{km}[\varphi]$ is straightforward, a proof of Proposition 2 is given in [1]. The Schröder equation (3) and more iterative functional equations will appear later again when we consider the surjectivity of the operator $\lambda I - F_{km}$, $I = \mathrm{id}\,|_{D_{km}}$.

The point spectra $\sigma_p(F_{k1})$, $1 \leq k \leq 4$, and the corresponding eigenspaces can be found in [2], as well as the continuous spectra $\sigma_c(F_{31})$, $\sigma_c(F_{41})$ and the residual spectra $\sigma_r(F_{31})$, $\sigma_r(F_{41})$. The full set of point spectra $\sigma_p(F_{km})$, $1 \leq k$, $m \leq 4$, and a description of the corresponding eigenspaces is given in the recent paper [1].

In Chapter 2. we shall describe the resolvent $\rho(F_{3m})$ and the continuous and the residual spectra of F_{3m} for the remaining values $2 \leq m \leq 4$.

So this note can be considered as an extension of [2] and as well as an extension of [1].

2. The spectra of F_{km}

In this chapter we are interested in the case $k \geqslant 3$. Then $F_{km}\colon D_{km} \longrightarrow D_{km}$ is a Banach space automorphism. Let us repeat that

$$\rho(F_{km}) = \{\lambda \in \mathbb{R};\ (\lambda I - F_{km})^{-1} \in L(D_{km}, D_{km})\}.$$

$$\sigma_c(F_{km}) = \{\lambda \in \mathbb{R};\ \lambda I - F_{km} \text{ injective, not surjective, } \operatorname{cl}(\lambda I - F_{km})[D_{km}] = D_{km}\},$$

$$\sigma_r(F_{km}) = \{\lambda \in \mathbb{R};\ \lambda I - F_{km} \text{ injective, not surjective, } \operatorname{cl}(\lambda I - F_{km})[D_{km}] \neq D_{km}\}.$$

So, to compute the continuous and the residual spectra of F_{km} we first have to determine, for which $\lambda \in \mathbb{R}$ the operator $(\lambda I - F_{km})$ is injective and not surjective. The injectivity information is provided by the point spectrum $\sigma_p(F_{km})$, which is known from [1] in all cases $3 \le k \le 4$, $1 \le m \le 4$. The surjectivity information has to be discussed for the individual F_{km}, but some general remarks are possible. The following first statement is easily verified.

Proposition 3
The operator $\lambda I - F_{km}$ is surjective if and only if

$$\forall f \in D_{km}\, \exists \varphi \in D_{km}\, \forall x \in \mathbb{R}:\ (\lambda - 1)\varphi(x) - \frac{1}{2}\lambda\varphi(2x) = f(x) - \frac{1}{2}f(2x). \tag{4}$$

Now we show that equation (4) has, for any given $\lambda \in \mathbb{R}\backslash\{\frac{2}{3}, 2\}$ and bounded $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, exactly one bounded solution φ, which we shall denote by $\Phi_{\lambda,f}$.

For $\lambda = 1$ we get $-\frac{1}{2}\varphi(2x) = f(x) - \frac{1}{2}f(2x)$, hence $\Phi_{1,f}(x) = f(x) - 2f(\frac{x}{2})$.

For $\lambda \neq 1$, equation (4) can be written in the equivalent form

$$\varphi(x) - \gamma\varphi(2x) = (\lambda - 1)^{-1}\tilde{f}(x), \tag{5}$$

where γ is as in (3), $\tilde{f}(x) := f(x) - \frac{1}{2}f(2x)$. Note that $\tilde{f}(x)$ is the left hand side of the de Rham equation (2) and that the associated homogeneous equation of (5) is just the Schröder equation (3).

For $|\gamma| < 1$, i.e., $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, iteration of (5) gives

$$(\lambda - 1)\Phi_{\lambda,f}(x) = \sum_{\nu=0}^{\infty} \gamma^{\nu}\tilde{f}(2^{\nu}x). \tag{6}$$

For $|\gamma| > 1$, i.e., $\lambda \in (\frac{2}{3}, 2) \setminus \{1\}$, iteration of

$$\varphi(x) = \frac{1}{\gamma}\varphi\left(\frac{x}{2}\right) - \frac{1}{\gamma(\lambda - 1)}\tilde{f}\left(\frac{x}{2}\right)$$

gives

$$(1 - \lambda)\Phi_{\lambda,f}(x) = \sum_{\nu=1}^{\infty} \gamma^{-\nu}\tilde{f}(2^{-\nu}x). \tag{7}$$

It is straightforward to check that in each case $\Phi_{\lambda,f}$ is in fact a bounded solution of (4).

After these preparations we are ready to supply the announced information on $\rho(F_{km})$, $\sigma_c(F_{km})$ and $\sigma_r(F_{km})$ for $k = 3$ in the following

THEOREM
We have

a) $\rho(F_{31}) = \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, $\sigma_c(F_{31}) = \emptyset$, $\sigma_r(F_{31}) = \emptyset$,

b) $\rho(F_{32}) = (-\infty, \frac{2}{3}) \cup (2, \infty)$, $\sigma_c(F_{32}) = \emptyset$, $\sigma_r(F_{32}) = (\frac{2}{3}, 2)$,

c) $\rho(F_{33}) = \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, $\sigma_c(F_{33}) = \emptyset$, $\sigma_r(F_{33}) = \emptyset$,

d) $\rho(F_{34}) = (-\infty, \frac{2}{3}) \cup (2, \infty)$, $\sigma_c(F_{34}) = \emptyset$, $\sigma_r(F_{34}) = (\frac{2}{3}, 2)$.

Proof. In [1] it is shown that $\sigma_p(F_{3m}) = \{\frac{2}{3}, 2\}$, $1 \le m \le 4$. So the operator $\lambda I - F_{3m}$ is injective for every $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$.

a) This was already proved in [2].

b) Let $\lambda \in \mathbb{R}\setminus\{\frac{2}{3}, 2\}$. Then the operator $\lambda I - F_{32}\colon D_{32} \longrightarrow D_{32}$ is surjective iff (4) holds with $k = 3$, $m = 2$.

For $f \in D_{32}$ (i.e. f bounded and 1-periodic) and $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$ we get $\Phi_{\lambda,f} \in D_{32}$, because then also $\tilde{f} \in D_{32}$ and therefore also $\Phi_{\lambda,f}$, given by (6), is bounded and of period 1 (note that $|\gamma| < 1$).

On the other hand, in case $\lambda \in (\frac{2}{3}, 2)$ there are $f \in D_{32}$ such that neither of the functions $\Phi_{1,f}$ nor $\Phi_{\lambda,f}$, given by (7), are 1-periodic. Explicit examples are given below.

Therefore, $\lambda I - F_{32}\colon D_{32} \longrightarrow D_{32}$ is bijective exactly for $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, and by the inverse operator theorem, this set coincides with the resolvent set.

So far we have shown that $\sigma_c(F_{32}) \cup \sigma_r(F_{32}) = (\frac{2}{3}, 2)$.

We shall prove now that $\sigma_c(F_{32}) = \emptyset$. To this end, take first the case $\lambda = 1$. Consider the particular element $d \in D_{32}$ given by

$$d(x) := \operatorname{dist}(x, \mathbb{Z}).$$

Then $\Phi_{1,d}(x) = d(x) - 2d(\frac{x}{2})$ is not of period 1, hence $\Phi_{1,d} \notin D_{32}$. Therefore there is no $\varphi \in D_{32}$ such that $(I - F_{32})[\varphi] = d$.

Now let $h \in D_{32}$ such that $\|h - d\| < \frac{1}{10}$. We get

$$\Phi_{1,h}(0) = h(0) - 2h(0) \in [-\tfrac{1}{10},\ \tfrac{1}{10}]\ ,$$
$$\Phi_{1,h}(1) = h(1) - 2h\left(\frac{1}{2}\right) \in [-\tfrac{13}{10}, -\tfrac{7}{10}],$$

which means that also $\Phi_{1,h} \notin D_{32}$. So $(I - F_{32})[D_{32}] \cap U_{\frac{1}{10}}(d) = \emptyset$, i.e., $\mathrm{cl}(I - F_{32})[D_{32}] \neq D_{32}$ and $1 \in \sigma_r(F_{32})$.

Finally consider the case $\lambda \in (\frac{2}{3}, 2) \setminus \{1\}$ (i.e., $|\gamma| > 1$). The surjectivity of $\lambda I - F_{32}$ would imply in particular that $\Phi_{\lambda,d}$, given by (7), is 1-periodic. Here we have

$$(1-\lambda)\Phi_{\lambda,d}(x) = \sum_{\nu=1}^{\infty} \gamma^{-\nu} \tilde{d}(2^{-\nu}x)$$

with $\tilde{d}(x) = d(x) - \frac{1}{2}d(2x)$. The function $\tilde{d}$ is piecewise affine with vertices through $(0,0)$, $(\frac{1}{4}, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{4}, 0)$, $(1,0)$ and is of period one. However $\Phi_{\lambda,d}$ is not 1-periodic, because

$$\begin{aligned}(1-\lambda)\Phi_{\lambda,d}(0) &= 0,\\ (1-\lambda)\Phi_{\lambda,d}(1) &= \frac{1}{\gamma}\cdot\frac{1}{2} + 0 \neq 0.\end{aligned}$$

Now let $h \in D_{32}$ such that $\|h - d\| < \varepsilon$. Then a short calculation shows that $\|\tilde{h} - \tilde{d}\| < 2\varepsilon$. We want to prove that $\Phi_{\lambda,h}$, given by

$$(1-\lambda)\Phi_{\lambda,h}(x) = \sum_{\nu=1}^{\infty} \gamma^{-\nu} \tilde{h}(2^{-\nu}x),$$

is not 1-periodic, if

$$\varepsilon < \frac{|\gamma| - 1}{4|\gamma|}.$$

Consider first the case $\gamma > 1$ (i.e. $1 < \lambda < 2$). We have $\tilde{d}(t) - 2\varepsilon \leq \tilde{h}(t) \leq \tilde{d}(t) + 2\varepsilon$ for every $t \in \mathbb{R}$. Therefore

$$\begin{aligned}\sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{-\nu}x) - 2\varepsilon] &= (1-\lambda)\Phi_{\lambda,d}(x) - \frac{2\varepsilon}{\gamma - 1}\\ &\leq (1-\lambda)\Phi_{\lambda,h}(x) \leq \sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{-\nu}x) + 2\varepsilon]\\ &= (1-\lambda)\Phi_{\lambda,d}(x) + \frac{2\varepsilon}{\gamma - 1}.\end{aligned}$$

In particular, we get

$$-\frac{2\varepsilon}{\gamma - 1} \leq (1-\lambda)\Phi_{\lambda,h}(0) \leq \frac{2\varepsilon}{\gamma - 1}$$

and

$$-\frac{2\varepsilon}{\gamma - 1} + \frac{1}{2\gamma} \leq (1-\lambda)\Phi_{\lambda,h}(1) \leq \frac{2\varepsilon}{\gamma - 1} + \frac{1}{2\gamma}.$$

So we obtain $\Phi_{\lambda,h}(0) \neq \Phi_{\lambda,h}(1)$ as $4\varepsilon\gamma < \gamma - 1$.

Now we consider the case $\gamma < -1$ (i.e. $\frac{2}{3} < \lambda < 1$). As γ is negative, we get

$$\begin{aligned}\sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{-\nu}x) + (-1)^{\nu-1}2\varepsilon] &= (1-\lambda)\Phi_{\lambda,d}(x) + \frac{2\varepsilon}{\gamma+1} \\ &\le (1-\lambda)\Phi_{\lambda,h}(x) \\ &\le \sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{\nu}x) + (-1)^{\nu}2\varepsilon] \\ &= (1-\lambda)\Phi_{\lambda,d}(x) - \frac{2\varepsilon}{\gamma+1},\end{aligned}$$

hence

$$(1-\lambda)\Phi_{\lambda,d}(x) - \frac{2\varepsilon}{|\gamma|-1} \le (1-\lambda)\Phi_{\lambda,h}(x) \le (1-\lambda)\Phi_{\lambda,d}(x) + \frac{2\varepsilon}{|\gamma|-1}.$$

So we obtain as above $\Phi_{\lambda,h}(0) \neq \Phi_{\lambda,h}(1)$ as $4\varepsilon \mid \gamma \mid < \mid \gamma \mid -1$.

Consequently $(\lambda I - F_{32})[D_{32}] \cap U_\varepsilon(d) = \emptyset$, i.e., $\mathrm{cl}(\lambda I - F_{32})[D_{32}] \neq D_{32}$ and $\lambda \in \sigma_r(F_{32})$.

c) Let $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$. Then the operator $\lambda I - F_{33} \colon D_{33} \longrightarrow D_{33}$ is surjective iff (4) holds with $k = m = 3$.

If f is bounded and even, then $\tilde{f}$ and the function $x \mapsto f(x) - 2f(\frac{x}{2})$ are bounded and even, hence the unique bounded solutions $\Phi_{\lambda,f}$ of (4) are bounded and even as well.

Therefore $\lambda I - F_{33}$ is bijective for every $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, and again by the inverse operator theorem, $\rho(F_{33}) = \mathbb{R} \setminus \{\frac{2}{3}, 2\}$.

d) Let $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$. Then the operator $\lambda I - F_{34} \colon D_{34} \longrightarrow D_{34}$ is surjective iff (4) holds with $k = 3$, $m = 4$.

If $f \in D_{34}$ and $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, the representation (6) shows that also $\Phi_{\lambda,f} \in D_{34}$.

On the other hand, for $\lambda \in (\frac{2}{3}, 2)$, the counterexample $d \in D_{32}$ from b) can be used here as well, because we have also $d \in D_{34}$ (d is 1-periodic and even). With the same argument as in b) we see that $(\lambda I - F_{34})[D_{34}] \cap U_\varepsilon(d) = \emptyset$, provided that $\varepsilon < \frac{1}{10}$ in case $\lambda = 1$, $\varepsilon < \frac{|\gamma|-1}{4|\gamma|}$ in case $\lambda \in (\frac{2}{3}, 2) \setminus \{1\}$. So $\mathrm{cl}(\lambda I - F_{34})[D_{34}] \neq D_{34}$ and $\lambda \in \sigma_r(F_{34})$.

Determining the corresponding statements about $\rho(F_{4k})$, $\sigma_c(F_{4k})$ and $\sigma_r(F_{4k})$ remains the subject of future research.

References

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