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Approximate multi-Jensen-cubic mappings and a fixed point theorem

Abstract. In this paper, we introduce multi-Jensen-cubic mappings and unify the system of functional equations defining the multi-Jensen-cubic mapping to a single equation. Applying a fixed point theorem, we establish the generalized Hyers-Ulam stability of multi-Jensen-cubic mappings. As a known outcome, we show that every approximate multi-Jensen-cubic mapping can be multi-Jensen-cubic.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [38] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [22]. Later, the result of Hyers was significantly generalized by Aoki [1], Th. M. Rassias [36] (stability incorporated with sum of powers of norms), Găvruţa [21] (stability controlled by a general control function) and [35] (stability including mixed product-sum of powers of norms). Over the past few decades, many authors have published the generalized Hyers-Ulam stability theorems of various functional equations. One of them is the Jensen functional equation. Recall that the stability of the Jensen functional equation

$$J\Big(\frac{x+y}{2}\Big)=\frac{J(x)+J(y)}{2}$$

has been studied by a number of authors; see [29], [25] and [27] for more details.

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Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [20] that a mapping $f: V^n \to W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$) in each variable. Some facts on such mappings can be found in [28] and many other sources. In [20], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive mappings in Banach spaces. For the miscellaneous forms of multi-quadratic mappings, their characterizations and stability, we refer to [17], [37] and [40]. Prager and Schwaiger [33] introduced the notion of multi-Jensen mappings $f: V^n \to W$ (V and W being vector spaces over the set of all rational numbers) with the connection with generalized polynomials and obtained their general form. The aim of this note was to study the stability of the multi-Jensen equation. Next, the stability of multi-Jensen mappings in various normed spaces have been investigated by a number of mathematicians (see [18], [19], [34] and [39]).

The story of stability of cubic functional equation commences by introducing the cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) = 6f(y) \tag{1.1}$$

by J. M. Rassias in [35]. He found the solution of (1.1) and investigated the Hyers-Ulam stability problem for these cubic mappings. The following alternative cubic functional equations have been introduced by Jun and Kim in [23, 24].

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.2}$$

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x). \tag{1.3}$$

They studied the Hyers-Ulam stability problem for (1.2) and (1.3) in [24] and [23], respectively; for other forms of the (generalized) cubic functional equations and their stabilities on the various Banach spaces refer to [5], [6], [7] and [8].

It is worth mentioning that the fixed point theorems have been considered for various mappings and functional equations in [2], [3], [12], [16], [26] and [31]. Similar investigations have been carried out in the stability of linear recurrence; see [14], [15], and [32]. Moreover, the fixed point theorem were applied to obtain similar stability results in [11].

Recently, motivated by the cubic functional equations (1.2) and

$$8f\Big(\frac{x+2y}{2}\Big) + 8f\Big(\frac{x-2y}{2}\Big) = 4f(x+y) + 4f(x-y) - 6f(x) \tag{1.4}$$

some multi-cubic mappings are introduced in [10] and [30]. Furthermore, the stability of multi-cubic and multi-quartic mappings in Banach spaces via the fixed point method are investigated in [9] and [10], respectively.

In this paper, we define the multi-Jensen-cubic mappings which are Jensen in each of some k variables and are cubic in sense of satisfies equation (1.4) in each of the other variables and then we present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-Jensen-cubic mappings to obtain a single functional equation. We also prove the generalized Hyers-Ulam stability for the multi-Jensen-cubic mappings by using the fixed point method. Finally, we indicate some direct consequences of stability and hyperstability of multi-Jensen-cubic mappings in Banach spaces.

2. Characterization of multi-Jensen-cubic mappings

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$. Moreover, for the set X, we denote

$$X^n := \underbrace{X \times X \times \cdots \times X}_{n-\text{times}}.$$

For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $t = (t_1, \ldots, t_n) \in \{-2, 2\}^n$ and $x = (x_1, \ldots, x_n) \in V^n$ we write $lx := (lx_1, \ldots, lx_n)$ and $tx := (t_1x_1, \ldots, t_nx_n)$, where lx stands, as usual, for the scaler product of l on x in the linear space V.

Let V and W be linear spaces, $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. A mapping $f\colon V^n \to W$ is called k-Jensen and $n-k$-cubic (briefly, multi-Jensen-cubic) if f is Jensen in each of some k variables and is cubic in each of the other variables (see equation (1.4)). In this note, we suppose for simplicity that f is Jensen in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$ ($k = 0$), the above definition leads to the so-called multi-Jensen (multi-cubic) mappings; some basic facts on Jensen mappings can be found for instance in [33].

From now on, we assume that V and W are vector spaces over the set of all rational numbers. Moreover, we identify $x = (x_1, \ldots, x_n) \in V^n$ with $(x^k, x^{n-k}) \in V^k \times V^{n-k}$, where $x^k := (x_1, \ldots, x_k)$ and $x^{n-k} := (x_{k+1}, \cdots, x_n)$, and we adopt the convention that $(x^n, x^0) := x^n := (x^0, x^n)$. Put $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}, \ldots, x_{in}) \in V^{n-k}$ where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. In addition, we put

$$\mathcal{M} = \{\mathfrak{N}_n = (N_{k+1}, \cdots, N_n) : \ N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\},$$

where $j \in \{k+1, \ldots, n\}$. Consider

$$\mathcal{M}_T^{n-k} := \{\mathfrak{N}_n = (N_{k+1}, \ldots, N_n) \in \mathcal{M} : \ \operatorname{Card}\{N_j : \ N_j = x_{1j}\} = T\}.$$

We also use the following notations

$$f(\mathcal{M}_T^{n-k}) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^{n-k}} f(\mathfrak{N}_n),$$

$$f(x_i^k, \mathcal{M}_T^{n-k}) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^{n-k}} f(x_i^k, \mathfrak{N}_n) \qquad \text{for } i \in \{1, 2\}.$$

We say the mapping $f\colon V^n \to W$ satisfies *the r-power condition* in the jth variable if for all $(z_1, \ldots, z_n) \in V^n$,

$$f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^r f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).$$

Example 2.1
[30] Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in $\mathcal{A}$ (not necessarily unit). Define a mapping $h\colon \mathcal{A}^n \to \mathcal{A}$ by $h(a_1, \ldots, a_n) = \prod_{j=1}^n \|a_j\|^3 a_0$ for $(a_1, \ldots, a_n) \in \mathcal{A}^n$. It is easily verified that the mapping h satisfies 3-power condition in all variables but h is not multi-cubic even for $n = 1$, that is h does not satisfy in the equation (1.4).

In this paper, we use $\binom{n}{k}$ which is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $n!/(k!(n-k)!)$.

In this section, we wish to show that the mapping $f\colon V^n \to W$ is multi-Jensen-cubic if and only if it satisfies the following equation

$$\begin{aligned} 2^{3n-2k} \sum_{q\in\{-2,2\}^{n-k}} f\Big(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big) \\ = \sum_{l_1,\ldots,l_k\in\{1,2\}} \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m f(x_{l_1 1}, \cdots, x_{l_k k}, \mathcal{M}_m^{n-k}) \end{aligned} \tag{2.1}$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \ldots, x_{in}) \in V^{n-k}$, where $i \in \{1, 2\}$.

Here, we reduce the system of n equations defining the multi-Jensen-cubic mapping to obtain a single functional equation.

Theorem 2.2
Let $n \in \mathbb{N}$ and $k \in \{0, \cdots, n\}$. Then, the mapping $f\colon V^n \to W$ is multi-Jensen-cubic mapping if and only if f satisfies equation (2.1) *and the* 3*-power condition in the last $n-k$ variables.*

Proof. (Necessity) We firstly note that it is easily verified that f satisfies 3-power condition in the last $n-k$ variables. Suppose that f is a multi-Jensen-cubic mapping. Without loss of generality, we assume that $k \in \{0, \ldots, n-1\}$. For any $x^{n-k} \in V^{n-k}$, define the mapping $g_{x^{n-k}}\colon V^k \to W$ by $g_{x^{n-k}}(x^k) := f(x^k, x^{n-k})$ for $x^k \in V^k$. By assumption, $g_{x^{n-k}}$ is k-Jensen, and hence Lemma 1.1 from [34] implies that

$$2^k g_{x^{n-k}}\Big(\frac{x_1^k + x_2^k}{2}\Big) = \sum_{j_1,j_2,\ldots,j_k\in\{1,2\}} g_{x^{n-k}}(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k})$$

for all $x_1^k, x_2^k \in V^k$. It now follows from the above equality that

$$2^k f\Big(\frac{x_1^k + x_2^k}{2}, x^{n-k}\Big) = \sum_{j_1,j_2,\ldots,j_k\in\{1,2\}} f(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x^{n-k}) \tag{2.2}$$

for all $x_1^k, x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Similarly to the above, for any $x^k \in V^k$, consider the mapping $h_{x^k}\colon V^{n-k} \to W$ defined via $h_{x^k}(x^{n-k}) := f(x^k, x^{n-k})$ for $x^{n-k} \in V^{n-k}$ which is $n-k$-cubic. Now, [30, Theorem 3.2] implies that

$$8^{n-k} \sum_{q\in\{-2,2\}^{n-k}} h_{x^k}\Big(\frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big) = \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m h_{x^k}(\mathcal{M}_m^{n-k}) \tag{2.3}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. By the definition of h_{x^k}, relation (2.3) is equivalent to

$$8^{n-k} \sum_{q\in\{-2,2\}^{n-k}} f\Big(x^k, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big) = \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m f(x^k, \mathcal{M}_m^{n-k}) \tag{2.4}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. Inserting equality (2.2) into (2.4) we get

$$\begin{aligned}
2^{3n-2k} &\sum_{q\in\{-2,2\}^{n-k}} f\Big(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big)\\
&= \sum_{q\in\{-2,2\}^{n-k}} \sum_{j_1,j_2,\dots,j_k\in\{1,2\}} f\Big(x_{j_1 1}, x_{j_2 2}, \dots, x_{j_k k}, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big)\\
&= \sum_{j_1,\dots,j_k\in\{1,2\}} \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m f(x_{j_1 1}, \dots, x_{j_k k}, \mathcal{M}_m^{n-k})
\end{aligned}$$

for all $x_i^k = (x_{i1}, \dots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \dots, x_{in}) \in V^{n-k}$, which proves that f satisfies equation (2.1).

(Sufficiency) Assume that f satisfies (2.1). Putting $x_2^{n-k} = 0$ in (2.1) and using the assumption, we obtain

$$\begin{aligned}
2^{3n-2k}&\times 2^{n-k} f\Big(\frac{x_1^k + x_2^k}{2}, x_1^{n-k}\Big)\\
&= \sum_{j_1,j_2,\dots,j_k\in\{1,2\}} \sum_{m=0}^{n-k} 2^{n-k-m}(-6)^m 4^{n-k-m}\binom{n-k}{m}\\
&\qquad\qquad \times f(x_{j_1 1}, x_{j_2 2}, \dots, x_{j_k k}, 2x_1^{n-k})\\
&= \sum_{j_1,j_2,\dots,j_k\in\{1,2\}} (8-6)^{n-k} 2^{3(n-k)} f(x_{j_1 1}, x_{j_2 2}, \dots, x_{j_k k}, x_1^{n-k})\\
&= 2^{4n-4k} \sum_{j_1,j_2,\dots,j_k\in\{1,2\}} f(x_{j_1 1}, x_{j_2 2}, \dots, x_{j_k k}, x_1^{n-k}).
\end{aligned}$$

Thus

$$2^k f\Big(\frac{x_1^k + x_2^k}{2}, x_1^{n-k}\Big) = \sum_{j_1,j_2,\dots,j_k\in\{1,2\}} f(x_{j_1 1}, x_{j_2 2}, \dots, x_{j_k k}, x_1^{n-k}) \tag{2.5}$$

for all $x_1^k, x_2^k \in V^n$ and $x_1^{n-k} \in V^{n-k}$. In view of [34, Lemma 1.1], we see that f is Jensen in each of the k first variables. Furthermore, by putting $x_1^k = x_2^k$ in (2.1), we have

$$2^{3n-2k} \sum_{q\in\{-2,2\}^{n-k}} f\Big(x_1^k, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big) = 2^k \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m f(x_1^k, \mathcal{M}_m^{n-k})$$

and so

$$8^{n-k} \sum_{q\in\{-2,2\}^{n-k}} f\Big(x_1^k, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\Big) = \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m f(x_1^k, \mathcal{M}_m^{n-k})$$

for all $x_1^k \in V^k$ and $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. In view of [30, Theorem 3.2], we see that f is a multi-Jensen-cubic mapping.

3. Stability of multi-Jensen-cubic mappings

In this section, we prove the generalized Hyers-Ulam stability of equation (2.1) by a fixed point result (Theorem 3.1) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. Here, we introduce the following three hypotheses:

(A1) Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \ldots, g_j \colon \mathcal{S} \to \mathcal{S}$ and $L_1, \ldots, L_j \colon \mathcal{S} \to \mathbb{R}_+$;

(A2) $\mathcal{T} \colon Y^{\mathcal{S}} \to Y^{\mathcal{S}}$ is an operator satisfying the inequality

$$\|\mathcal{T}(\lambda)(x) - \mathcal{T}(\mu)(x)\| \leq \sum_{i=1}^{j} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|$$

for all $\lambda, \mu \in Y^{\mathcal{S}}$, $x \in \mathcal{S}$;

(A3) $\Lambda \colon \mathbb{R}_+^{\mathcal{S}} \to \mathbb{R}_+^{\mathcal{S}}$ is an operator defined through

$$\Lambda(\delta)(x) := \sum_{i=1}^{j} L_i(x) \delta(g_i(x)), \qquad \delta \in \mathbb{R}_+^{\mathcal{S}},\ x \in \mathcal{S}.$$

The following result presents a theorem in fixed point theory [12, Theorem 1] which plays a fundamental tool to reach our purpose in this paper.

Theorem 3.1
Let hypotheses (A1)–(A3) *hold and the function* $\theta \colon \mathcal{S} \to \mathbb{R}_+$ *and the mapping* $\phi \colon \mathcal{S} \to Y$ *fulfil the following two conditions:*

$$\|\mathcal{T}(\phi)(x) - \phi(x)\| \leq \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty$$

for all $x \in \mathcal{S}$. *Then, there exists a unique fixed point* ψ *of* $\mathcal{T}$ *such that*

$$\|\phi(x) - \psi(x)\| \leq \theta^*(x)$$

for all $x \in \mathcal{S}$. *Moreover,* $\psi(x) = \lim_{l \to \infty} \mathcal{T}^l(\phi)(x)$ *for all* $x \in \mathcal{S}$.

Here and subsequently, for the mapping $f \colon V^n \to W$, we consider the difference operator $\mathcal{D}f \colon V^n \times V^n \to W$ defined by

$$\begin{aligned}\mathcal{D}f(x_1, x_2) := 2^{3n-2k} &\sum_{q \in \{-2,2\}^{n-k}} f\Big(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + q x_2^{n-k}}{2}\Big) \\ &- \sum_{l_1,\ldots,l_k \in \{1,2\}} \sum_{m=0}^{n-k} 4^{n-k-m} (-6)^m f(x_{l_1 1}, \cdots, x_{l_k k}, \mathcal{M}_m^{n-k})\end{aligned}$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}, \ldots, x_{in}) \in V^{n-k}$.

The next lemma from [4] will be useful in the proof of our stability result. For simplicity, given an $m \in \mathbb{N}$, we write $S := \{0,1\}^m$, and S_i stands for the set of all elements of S having exactly i zeros, i.e.

$$S_i := \{(s_1,\dots,s_m) \in S :\ \mathrm{card}\{j :\ s_j = 0\} = i\}, \qquad i \in \{0,\dots,m\}.$$

Lemma 3.2
Let $m \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $\psi\colon S \to \mathbb{R}$. Then

$$\sum_{v=0}^{m}\sum_{w=0}^{m}\sum_{s\in S_w}\sum_{t\in S_v}(2^l-1)^w\psi(st) = \sum_{i=0}^{m}\sum_{p\in S_i}(2^{l+1}-1)^i\psi(p).$$

From now on S stands for $\{0,1\}^k$ and $S_i \subseteq S$ for $i \in \{0,\dots,k\}$. We have the following stability result for functional equation (2.1).

Theorem 3.3
Let V be a linear space and W be a Banach space. Suppose that $\phi\colon V^n \times V^n \to \mathbb{R}_+$ is a function satisfying the equality

$$\lim_{l\to\infty}\Big(\frac{1}{2^{3n-2k}}\Big)^l\sum_{i=0}^{n}\sum_{p\in S_i}(2^l-1)^i\phi((2^lpx_1^k,2x_1^{n-k}),(2^lpx_2^k,2x_2^{n-k})) = 0 \tag{3.1}$$

for all $x_1 = \left(x_1^k, x_1^{n-k}\right), x_2 = \left(x_2^k, x_2^{n-k}\right) \in V^n$ and

$$\begin{aligned}\Phi(x) := \frac{1}{2^{4n-3k}}&\sum_{l=0}^{\infty}\Big(\frac{1}{2^{3n-2k}}\Big)^l\\ &\times\sum_{i=0}^{n}\sum_{p\in S_i}(2^l-1)^i\phi((2^{l+1}px_1^k,2^2x_1^{n-k}),(0,0)) < \infty\end{aligned} \tag{3.2}$$

for all $x = \left(x_1^k, x_1^{n-k}\right) \in V^n$. Assume also that $f\colon V^n \to W$ is a mapping satisfying the inequality

$$\|\mathcal{D}f(x_1,x_2)\| \leqslant \phi(x_1,x_2) \tag{3.3}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{F}\colon V^n \to W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \le \Phi(x) \tag{3.4}$$

for all $x = (x_1^k, x_1^{n-k}) \in V^n$.

Proof. Replacing $x_1 = (x_1^k, x_1^{n-k})$, $x_2 = (x_2^k, x_2^{n-k})$ by $2x_1 = 2(x_1^k, x_1^{n-k})$, $(0,0)$ in (3.3), respectively, we have

$$\begin{aligned}\Bigg\|&2^{3n-2k}\times 2^{n-k}f(x)\\ &-\sum_{s\in S}\sum_{m=0}^{n-k}\tbinom{n-k}{m}2^{n-k-m}4^{n-k-m}(-6)^mf(2sx_1^k,2x_1^{n-k})\Bigg\| \le \phi(2x,0),\end{aligned} \tag{3.5}$$

where $x = x_1 = (x_1^k, x_1^{n-k}) \in V^n$. Since $\sum_{m=0}^{n-k} \binom{n-k}{m} 8^{n-k-m}(-6)^m = (8-6)^{n-k} = 2^{n-k}$, inequality (3.5) shows that

$$\left\| f(x) - \frac{1}{2^{3n-2k}} \sum_{s\in S} f(2sx_1^k, 2x_1^{n-k}) \right\| \le \frac{1}{2^{4n-3k}} \phi(2x, 0) \tag{3.6}$$

for all $x = x_1 = (x_1^k, x_1^{n-k}) \in V^n$. Set $\theta(x) := \frac{1}{2^{4n-3k}} \phi(2x, 0)$ and $\mathcal{T}(\theta)(x) := \frac{1}{2^{3n-2k}} \sum_{s\in S} \theta(2sx_1^k, 2x_1^{n-k})$, where $\theta \in W^{V^n}$, $x \in V^n$. Then, relation (3.6) can be modified as

$$\|f(x) - \mathcal{T}(f)(x)\| \le \theta(x), \qquad x \in V^n. \tag{3.7}$$

Define

$$\Lambda\eta(x) := \frac{1}{2^{3n-2k}} \sum_{s\in S} \eta(2sx_1^k, 2x_1^{n-k})$$

for all $\eta \in \mathbb{R}_+^{V^n}$, $x = x_1 = (x_1^k, x_1^{n-k}) \in V^n$. We now see that Λ has the form described in (A3) with $\mathcal{S} = V^n$, $g_i(x) = g_s(x) = (2sx_1^k, 2x_1^{n-k})$ and $L_i(x) = \frac{1}{2^{3n-2k}}$ for all i and $x \in V^n$. Furthermore, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we find

$$\begin{aligned} \|\mathcal{T}(\lambda)(x) - \mathcal{T}(\mu)(x)\| &= \left\| \frac{1}{2^{3n-2k}} \left[\sum_{s\in S} \left(\lambda(2sx_1^k, 2x_1^{n-k}) - \mu(2sx_1^k, 2x_1^{n-k}) \right) \right] \right\| \\ &\le \frac{1}{2^{3n-2k}} \sum_{s\in S} \|\lambda(2sx_1^k, 2x_1^{n-k}) - \mu(2sx_1^k, 2x_1^{n-k})\|. \end{aligned}$$

The above relation shows that the hypothesis (A2) holds. By induction on l, one can check for any $l \in \mathbb{N}_0$ and $x \in V^n$ that

$$\Lambda^l(\theta)(x) := \left(\frac{1}{2^{3n-2k}} \right)^l \sum_{i=0}^{n} (2^l - 1)^i \sum_{p\in S_i} \theta(2^l p x_1^k, 2x_1^{n-k}). \tag{3.8}$$

Fix an $x \in V^n$. Here, we adopt the convention that $0^0 = 1$. Hence, the relation (3.8) is trivially true for $l = 0$. Next, assume that (3.8) holds for a $l \in \mathbb{N}_0$. Using Lemma 3.2 for $m = n$ and $\psi(s) := \theta(2^{l+1} s x_1^k, 2x_1^{n-k})$, $s \in S$, we get

$$\begin{aligned} \Lambda^{l+1}(\theta)(x) = \Lambda(\Lambda^l(\theta))(x) &= \frac{1}{2^{3n-2k}} \sum_{v=0}^{n} \sum_{t\in S_v} \Lambda^l(\theta)(2tx_1^k, 2x_1^{n-k}) \\ &= \left(\frac{1}{2^{3n-2k}} \right)^{l+1} \sum_{v=0}^{n} \sum_{t\in S_v} \sum_{w=0}^{n} (2^l - 1)^w \sum_{s\in S_w} \theta(2^{l+1} stx_1^k, 2x_1^{n-k}) \\ &= \left(\frac{1}{2^{3n-2k}} \right)^{l+1} \sum_{v=0}^{n} \sum_{w=0}^{n} \sum_{s\in S_w} \sum_{t\in S_v} (2^l - 1)^w \theta(2^{l+1} stx_1^k, 2x_1^{n-k}) \\ &= \left(\frac{1}{2^{3n-2k}} \right)^{l+1} \sum_{i=0}^{n} \sum_{p\in S_i} (2^{l+1} - 1)^i \theta(2^{l+1} px_1^k, 2x_1^{n-k}). \end{aligned}$$

Therefore, (3.8) holds for any $l \in \mathbb{N}_0$ and $x \in V^n$. Now, relations (3.2) and (3.8) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a mapping $\mathcal{F}\colon V^n \to W$ such that

$$\mathcal{F}(x) = \lim_{l\to\infty} \mathcal{T}^l(f)(x) = \frac{1}{2^{3n-2k}} \sum_{s\in S} \mathcal{F}(2sx_1^k, 2x_1^{n-k})$$

for all $x = \left(x_1^k, x_1^{n-k}\right) \in V^n$, and also (3.4) holds. We argue by induction on l that

$$\begin{aligned} &\|\mathcal{D}\mathcal{T}^l(f)(x_1,x_2)\| \\ &\qquad \le \Big(\frac{1}{2^{3n-2k}}\Big)^l \sum_{i=0}^{n} \sum_{p\in S_i} (2^l-1)^i \phi((2^l px_1^k, 2x_1^{n-k}), (2^l px_2^k, 2x_2^{n-k})) \end{aligned} \tag{3.9}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. The inequality (3.9) is valid for $l = 0$ by (3.3). Assume that (3.9) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned} &\|\mathcal{D}\mathcal{T}^{l+1}(f)(x_1,x_2)\| \\ &\quad = \frac{1}{2^{3n-2k}} \Big\| \sum_{s\in S} \mathcal{D}\mathcal{T}^l(f)((2sx_1^k, 2x_1^{n-k}), (2sx_2^k, 2x_2^{n-k})) \Big\| \\ &\quad \le \Big(\frac{1}{2^{3n-2k}}\Big)^{l+1} \sum_{s\in S}\sum_{i=0}^{n} \sum_{t\in S_i} (2^l-1)^i \phi((2^{l+1} stx_1^k, 2x_1^{n-k}), (2^{l+1} stx_2^k, 2x_2^{n-k})) \\ &\quad = \Big(\frac{1}{2^{3n-2k}}\Big)^{l+1} \sum_{i=0}^{n} \sum_{p\in S_i} (2^{l+1}-1)^i \phi((2^{l+1} px_1^k, 2x_1^{n-k}), (2^{l+1} px_2^k, 2x_2^{n-k})) \end{aligned}$$

for all $x_1, x_2 \in V^n$. We note that the last equality follows from Lemma 3.2 with $m := n$ and $\psi(s) := \phi((2^{l+1}sx_1^k, x_1^{n-k}), (2^{l+1}sx_2^k, 2x_2^{n-k}))$, $s \in S$. Letting $l \to \infty$ in (3.9) and applying (3.1), we arrive at $\mathcal{D}\mathcal{F}(x_1,x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathcal{F}$ satisfies (2.1).

Finally, assume that $\mathfrak{F}\colon V^n \to W$ is another mapping satisfying equation (2.1) and inequality (3.4), and fix $x \in V^n$, $j \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|\mathcal{F}(x) - \mathfrak{F}(x)\| &= \Big\| \Big(\frac{1}{2^{3n-2k}}\Big)^j \mathcal{F}(2^j x) - \Big(\frac{1}{2^{3n-2k}}\Big)^j \mathfrak{F}(2^j x) \Big\| \\ &\le \Big(\frac{1}{2^{3n-2k}}\Big)^j (\|\mathcal{F}(2^j x) - f(2^j x)\| + \|\mathfrak{F}(2^j x) - f(2^j x)\|) \\ &\le 2\Big(\frac{1}{2^{3n-2k}}\Big)^j \Phi(2^j x) \\ &\le \frac{1}{2^{4n-3k-1}} \Big(\frac{1}{2^{3n-2k}}\Big)^j \sum_{l=0}^{n} \Big(\frac{1}{2^{3n-2k}}\Big)^l \\ &\qquad\qquad \times \sum_{i=0}^{n} \sum_{p\in S_i} (2^l-1)^i \phi((2^{l+1} px_1^k, 2^2 x_1^{n-k}), (0,0)). \end{aligned}$$

Consequently, letting $j \to \infty$ and using the fact that series (3.2) is convergent for all $x \in V^n$, we obtain $\mathcal{F}(x) = \mathfrak{F}(x)$ for all $x \in V^n$, and hence the proof is finished.

The next corollary which is a direct consequence of Theorem 3.3, shows that every multi-Jensen-cubic mapping can be stable when the norm of $\mathcal{D}f(x_1, x_2)$ is controlled by a positive number for all $x_1, x_2 \in V^n$, where V is a normed space.

Corollary 3.4
Given $\delta > 0$. Let also V be a normed space and W be a Banach space. If $f\colon V^n \to W$ is a mapping satisfying the inequality

$$\|\mathcal{D}f(x_1, x_2)\| \leq \delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{F}\colon V^n \to W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{\delta}{2^n(2^{3(n-k)} - 1)}$$

for all $x \in V^n$.

Proof. Setting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$, and applying Theorem 3.3, we have

$$\begin{aligned}
\Phi(x) &= \frac{1}{2^{4n-3k}} \sum_{l=0}^{\infty} \Big(\frac{1}{2^{3n-2k}}\Big)^l \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi((2^{l+1} p x_1^k, 2^2 x_1^{n-k}), (0,0)) \\
&= \frac{\delta}{2^{4n-3k}} \sum_{l=0}^{\infty} \Big(\frac{1}{2^{3n-2k}}\Big)^l \sum_{i=0}^{k} \tbinom{k}{i} (2^l - 1)^i \times 1^{n-i} \\
&= \frac{\delta}{2^{4n-3k}} \sum_{l=0}^{\infty} \Big(\frac{1}{2^{3n-2k}}\Big)^l 2^{kl} = \frac{\delta}{2^{4n-3k}} \sum_{l=0}^{\infty} \Big(\frac{1}{2^{3(n-k)}}\Big)^l \\
&= \frac{\delta}{2^n(2^{3(n-k)} - 1)}.
\end{aligned}$$

We note that Corollary 3.4 does not hold for $k = n$. In the upcoming result by putting $k = 0$ in Corollary 3.4, we show that every multi-cubic mappings is stable.

Corollary 3.5
Let $\delta > 0$. Suppose that V is a normed space and W is a Banach space. If $f\colon V^n \to W$ is a mapping satisfying the inequality

$$\Big\| 2^{3n} \sum_{q \in \{-2,2\}^n} f\Big(\frac{x_1^n + q x_2^n}{2}\Big) - \sum_{m=0}^{n} 4^{n-m} (-6)^m f(\mathcal{M}_m^n) \Big\| \leq \delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C}\colon V^n \to W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{\delta}{2^n(2^{3n} - 1)}$$

for all $x \in V^n$.

Let A be a nonempty set, (X,d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathcal{F}_1\varphi(a_1,\dots,a_n) = \mathcal{F}_2\varphi(a_1,\cdots,a_n) \tag{3.10}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathcal{F}_1\varphi_0(a_1,\dots,a_n),\mathcal{F}_2\varphi_0(a_1,\dots,a_n)) \le \psi(a_1,\dots,a_n), \qquad a_1,\dots,a_n \in A,$$

fulfils (3.10); this definition is introduced in [13]. In other words, a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. In the following corollary, we show that every multi-Jensen-cubic mapping is hyperstable.

Corollary 3.6
Let V be a normed space and W be a Banach space. Suppose that $\delta_{ij} > 0$ for $i \in \{1,2\}$ and $j \in \{1,\cdots,n\}$ fulfil $\sum_{i=1}^{2}\sum_{j=1}^{n}\delta_{ij} < 3n-2k$. If $f\colon V^n \to W$ is a mapping satisfying the inequality

$$\|\mathcal{D}f(x_1,x_2)\| \le \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\delta_{ij}}$$

for all $x_1,x_2 \in V^n$, then f satisfies equation (2.1).

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