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Maximal functions for Weinstein operator

Abstract. In the present paper, we study in the harmonic analysis associated to the Weinstein operator, the boundedness on L^p of the uncentered maximal function. First, we establish estimates for the Weinstein translation of characteristic function of a closed ball with radius ε centered at 0 on the upper half space $\mathbb{R}^{d-1}\times\,]0,+\infty[$. Second, we prove weak-type L^1-estimates for the uncentered maximal function associated with the Weinstein operator and we obtain the L^p-boundedness of this operator for $1 < p \leq +\infty$. As application, we define a large class of operators such that each operator of this class satisfies these L^p-inequalities. In particular, the maximal operator associated respectively with the Weinstein heat semigroup and the Weinstein-Poisson semigroup belong to this class.

1. Introduction

For a real parameter $\alpha > -\frac{1}{2}$ and $d \geq 2$, the Weinstein operator (also called Laplace-Bessel operator) is an elliptic partial differential operator $\Delta_{d,\alpha}$ defined on the upper half space $\mathbb{R}^d_+ = \mathbb{R}^{d-1}\times\,]0,+\infty[$ by

$$\Delta_{d,\alpha} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}.$$

The operator $\Delta_{d,\alpha}$ can be written as

$$\Delta_{d,\alpha} = \Delta_{d-1} + \mathcal{L}_\alpha,$$

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where Δ_{d-1} is the Laplacian operator on $\mathbb{R}^{d-1}$ and $\mathcal{L}_\alpha$ is the Bessel operator on $]0,+\infty[$ with respect to the variable x_d given by

$$\mathcal{L}_\alpha = \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}.$$

For $d > 2$, the operator $\Delta_{d,\alpha}$ arises as the Laplace-Beltrami operator on the Riemannian space $\mathbb{R}^d_+$ equipped with the metric

$$ds^2 = x_d^{\frac{4\alpha+2}{d-2}} \sum_{i=1}^{d} dx_i^2.$$

The Weinstein operator $\Delta_{d,\alpha}$ has important applications in both pure and applied mathematics, especially in the fluid mechanics (see [19]). Many authors were interested in the study of the Weinstein equation $\Delta_{d,\alpha} u = 0$, one can cite for instance M. Brelot [5] and H. Leutwiler [12]. The harmonic analysis associated with the Weinstein operator was studied in [2, 3]. In particular, the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator also called the Weinstein transform.

The Hardy-Littlewood maximal function was first introduced by Hardy and Littlewood in 1930 for functions defined on the circle (see [10]). Later it was extended to various Lie groups, symmetric spaces, some weighted measure spaces (see [6, 9, 13, 15, 16, 17]) and different hypergroups (see [7, 8, 14]).

In this paper, we denote by $B^+(0,\varepsilon) = B(0,\varepsilon)\cap\mathbb{R}^d_+$, the closed ball on $\mathbb{R}^d_+$ with radius ε centered at 0. For $x \in \mathbb{R}^d_+$, we establish in a first step, estimates of the Weinstein translation (see next section) of the characteristic function of $B^+(0,\varepsilon)$, $\tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d)$, based on the inversion formula, where we put $y = (y', y_d)$ in $\mathbb{R}^d_+$ with $y' = (y_1, \dots, y_{d-1})$. Using these estimates, we prove in the second step, the weak-type $(1,1)$ of the uncentered maximal function $M_\alpha f$, defined for each integrable function f on $(\mathbb{R}^d_+, \nu_\alpha)$ and $x \in \mathbb{R}^d_+$ by

$$M_\alpha f(x) = \sup_{\varepsilon>0,\, z\in B^+(x,\varepsilon)} \frac{1}{\nu_\alpha(B^+(0,\varepsilon))}\left| \int_{\mathbb{R}^d_+} f(y)\tau_z(\chi_{B^+(0,\varepsilon)})(-y', y_d)d\nu_\alpha(y)\right|,$$

where ν_α is a weighted Lebesgue measure associated with the Weinstein operator (see next section) and $B^+(x,\varepsilon) = B(x,\varepsilon)\cap\mathbb{R}^d_+$ is the closed ball on $\mathbb{R}^d_+$ with radius ε centered at x. We can write also

$$M_\alpha f(x) = \sup_{\varepsilon>0,\, z\in B^+(x,\varepsilon)} \frac{1}{\nu_\alpha(B^+(0,\varepsilon))}\left|f *_\alpha \chi_{B^+(0,\varepsilon)}(z)\right|,$$

where $*_\alpha$ is the Weinstein-convolution operator (see next section). As a consequence, we obtain that M_α is of strong type (p,p) when $1 < p \le +\infty$. We recall that the operator M_α is said to be of weak-type $(1,1)$ if there exists a positive constant c such that for all $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ and $\lambda > 0$, we have

$$\nu_\alpha(\{x \in \mathbb{R}^d_+ :\ M_\alpha f(x) > \lambda\}) \le c\frac{\|f\|_{1,\alpha}}{\lambda}.$$

Also, we recall that the operator M_α is said to be of strong-type (p,p) for $1 < p \leq +\infty$, if it is bounded from $L^p(\mathbb{R}^d_+, \nu_\alpha)$ into itself.

Bloom and Xu in [4] have obtained analogous results for the Chébli-Trimèche hypergroups. Later, similar results have been established in [1] for the harmonic analysis involving the Dunkl operators on $\mathbb{R}^d$. Finally, since the strong-type (p,p) of an uncentered maximal function is an important tool in harmonic analysis, we define as application in this paper, a large class of operators such that each operator of this class satisfies these inequalities, and such that, in particular, the maximal operator associated respectively with the Weinstein heat semigroup and the Weinstein-Poisson semigroup belong to this class.

The contents of this paper are as follows. In section 2, we collect some basic definitions and results about harmonic analysis associated with Weinstein operator. In section 3, we establish estimates of $\tau_x(\chi_{B^+(0,\varepsilon)})$, $x \in \mathbb{R}^d_+$. Using these estimates, we prove that the uncentered maximal function $M_\alpha f$ is of weak type $(1,1)$ and we obtain the L^p-boundedness of this operator for $1 < p \leq +\infty$. In section 4, as application, we define a large class of operators such that each operator of this class satisfies these L^p-inequalities and we give two important examples.

Along this paper, we denote $\langle .,. \rangle$ the usual Euclidean inner product in $\mathbb{R}^d$ as well as its extension to $\mathbb{C}^d \times \mathbb{C}^d$, we write for $x \in \mathbb{R}^d$, $\|x\| = \sqrt{\langle x, x\rangle}$. In the sequel c represents a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $D_*(\mathbb{R}^d)$ the space of C^∞-functions which are of compact support, even with respect to the last variable;
- $S_*(\mathbb{R}^d)$ the space of C^∞-functions which are rapidly decreasing together with their derivatives, even with respect to the last variable.

2. Preliminaries

In this section, we recall some notations and results about harmonic analysis associated with the Weinstein operator. For every $1 \leq p \leq +\infty$, we denote by $L^p(\mathbb{R}^d_+, \nu_\alpha)$ the space of measurable functions f on $\mathbb{R}^d_+$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}^d_+} |f(x)|^p d\nu_\alpha(x) \right)^{1/p} < +\infty, \qquad \text{if } p < +\infty$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d_+} |f(x)| < +\infty,$$

where ν_α is a measure defined by

$$d\nu_\alpha(x) = \frac{x_d^{2\alpha+1}}{(2\pi)^{\frac{d-1}{2}} 2^\alpha \Gamma(\alpha+1)} dx = \frac{x_d^{2\alpha+1}}{(2\pi)^{\frac{d-1}{2}} 2^\alpha \Gamma(\alpha+1)} dx_1 \dots dx_d.$$

For a radial function $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$, the function F defined on $]0, +\infty[$ such that $f(x) = F(\|x\|)$ for all $x \in \mathbb{R}^d_+$, is integrable with respect to the measure $r^{2\alpha+d} dr$.

More precisely, we have

$$\int_{\mathbb{R}^d_+} f(x)d\nu_\alpha(x) = \frac{1}{2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+\frac{d+1}{2})}\int_0^{+\infty} F(r)r^{2\alpha+d}dr. \tag{2.1}$$

For all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$ the system

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), \quad j = 1, \ldots, d-1, \\ \mathcal{L}_\alpha u(x) = -\lambda_d^2 u(x), \\ \quad u(0) = 1, \quad \dfrac{\partial u}{\partial x_d}(0) = 0, \quad \dfrac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad j = 1, \ldots, d-1, \end{cases}$$

has a unique solution on $\mathbb{R}^d$, denoted by Ψ_λ called the Weinstein kernel and given by

$$\Psi_\lambda(x) = e^{-i\langle x', \lambda'\rangle} j_\alpha(x_d\lambda_d).$$

Here $x = (x', x_d) \in \mathbb{R}^d_+$, $x' = (x_1, \ldots, x_{d-1})$, $\lambda' = (\lambda_1, \ldots, \lambda_{d-1})$ and j_α is the normalized Bessel function of the first kind and order α, defined by

$$j_\alpha(\lambda x) = \begin{cases} 2^\alpha \Gamma(\alpha+1)\dfrac{J_\alpha(\lambda x)}{(\lambda x)^\alpha}, & \text{if } \lambda x \neq 0, \\ 1, & \text{if } \lambda x = 0, \end{cases} \tag{2.2}$$

where J_α is the Bessel function of the first kind and order α (see [18]). For all $x \in \mathbb{R}$, we have that the function $\lambda \to j_\alpha(\lambda x)$ is even on $\mathbb{R}$.

The Weinstein kernel $\Psi_\lambda(x)$ has a unique extension to $\mathbb{C}^d \times \mathbb{C}^d$. It has the following properties:

(i) $\forall \lambda, z \in \mathbb{C}^d$ $\Psi_\lambda(z) = \Psi_z(\lambda)$.

(ii) $\forall \lambda, z \in \mathbb{C}^d$ $\Psi_\lambda(-z) = \Psi_{-\lambda}(z)$.

(iii) $\forall \lambda, x \in \mathbb{R}^d$

$$|\Psi_\lambda(x)| \leq 1. \tag{2.3}$$

There exists an analogue of the classical Fourier transform with respect to the Weinstein kernel called the Weinstein transform and denoted by $\mathcal{F}_W$. The Weinstein transform enjoys properties similar to those of the classical Fourier transform and is defined for $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ by

$$\mathcal{F}_W(f)(\lambda) = \int_{\mathbb{R}^d_+} f(y)\Psi_\lambda(y)d\nu_\alpha(y), \qquad \lambda \in \mathbb{R}^d_+.$$

We list some known properties of this transform.

(i) For all $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$, we have

$$\|\mathcal{F}_W(f)\|_\infty \leq \|f\|_{1,\alpha}. \tag{2.4}$$

(ii) Let $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$. If $\mathcal{F}_W(f) \in L^1(\mathbb{R}^d_+, \nu_\alpha)$, then we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d_+} \Psi_\lambda(-x', x_d)\mathcal{F}_W(f)(\lambda) d\nu_\alpha(\lambda), \qquad x \in \mathbb{R}^d_+. \tag{2.5}$$

(iii) The Weinstein transform $\mathcal{F}_W$ on $S_*(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d_+, \nu_\alpha)$.

(iv) Plancherel formula: For all $f \in L^2(\mathbb{R}^d_+, \nu_\alpha)$, we have

$$\int_{\mathbb{R}^d_+} |f(x)|^2 d\nu_\alpha(x) = \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(x)|^2 d\nu_\alpha(x).$$

(v) Let $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ be a radial function, then the function F such that $f(x) = F(\|x\|)$ is integrable on $]0, +\infty[$ with respect to the measure $r^{2\alpha+d}dr$ and its Weinstein transform is given, for $y \in \mathbb{R}^d_+$, by

$$\mathcal{F}_W(f)(y) = \mathcal{F}_B^{\alpha + \frac{d-1}{2}}(F)(\|y\|), \tag{2.6}$$

where $\mathcal{F}_B^\gamma$ is the Fourier-Bessel transform of order $\gamma > -\frac{1}{2}$, given by

$$\mathcal{F}_B^\gamma(F)(\lambda) = \frac{1}{2^\gamma \Gamma(\gamma+1)} \int_0^{+\infty} F(r) j_\gamma(\lambda r) r^{2\gamma+1} dr, \qquad \lambda \in (0, +\infty).$$

For $x, y \in \mathbb{R}^d_+$ and f a continuous function on $\mathbb{R}^d$ which is even with respect to the last variable, the Weinstein translation operator τ_x is given by

$$\tau_x(f)(y) = \int_0^{+\infty} f(x' + y', \rho) W_\alpha(x_d, y_d, \rho) \rho^{2\alpha+1} d\rho, \tag{2.7}$$

where the kernel W_α is given by

$$\begin{aligned} &W_\alpha(x_d, y_d, \rho) \\ &= \frac{\Gamma(\alpha+1)((x_d+y_d)^2 - \rho^2)^{\alpha - \frac{1}{2}} (\rho^2 - (x_d - y_d)^2)^{\alpha - \frac{1}{2}}}{2^{2\alpha-1}\sqrt{\pi}\, \Gamma(\alpha + \frac{1}{2})(x_d y_d \rho)^{2\alpha}} \chi_{]|x_d - y_d|, x_d + y_d[}(\rho). \end{aligned} \tag{2.8}$$

For all $x_d, y_d > 0$, we have

$$\int_0^{+\infty} W_\alpha(x_d, y_d, \rho) \rho^{2\alpha+1} d\rho = 1. \tag{2.9}$$

The Weinstein translation operator satisfies the following properties.

(i) For every continuous function f on $\mathbb{R}^d$ which is even with respect to the last variable and $x, y \in \mathbb{R}^d_+$, we have

$$\tau_x(f)(y) = \tau_y(f)(x) \quad \text{and} \quad \tau_0 f = f.$$

(ii) For every $f \in S_*(\mathbb{R}^d)$ and all $y \in \mathbb{R}^d_+$, the function $x \to \tau_x f(y)$ belongs to $S_*(\mathbb{R}^d)$.

(iii) For every $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, $1 \le p \le +\infty$ and all $x \in \mathbb{R}^d_+$, we have

$$\|\tau_x f\|_{p,\alpha} \le \|f\|_{p,\alpha}.$$

For a function $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, with $p = 1$ or $p = 2$ and $x \in \mathbb{R}^d_+$, the Weinstein translation τ_x is also defined by the following relation

$$\mathcal{F}_W(\tau_x f)(\lambda) = \Psi_\lambda(-x', x_d)\mathcal{F}_W(f)(\lambda), \qquad \lambda \in \mathbb{R}^d_+. \tag{2.10}$$

By using the Weinstein translation, we define the convolution product $f *_\alpha g$ of functions $f, g \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ as follows:

$$(f *_\alpha g)(x) = \int_{\mathbb{R}^d_+} \tau_x(f)(-y', y_d)g(y)d\nu_\alpha(y), \qquad x \in \mathbb{R}^d_+.$$

This convolution is commutative and associative and satisfies the following properties.

(i) Let $1 \le p, q, r \le +\infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ (the Young condition). If $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$ and $g \in L^q(\mathbb{R}^d_+, \nu_\alpha)$, then $f *_\alpha g \in L^r(\mathbb{R}^d_+, \nu_\alpha)$ and we have

$$\|f *_\alpha g\|_{r,\alpha} \le \|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(ii) For every $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ and every $g \in L^p(\mathbb{R}^d_+, \nu_\alpha)$ with $p = 1$ or $p = 2$, we have

$$\mathcal{F}_W(f *_\alpha g) = \mathcal{F}_W(f)\mathcal{F}_W(g). \tag{2.11}$$

3. Weak-type (1,1) of the uncentered maximal function

In this section, we establish the estimates of $\tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d)$, $x, y \in \mathbb{R}^d_+$ and we prove the weak-type $(1,1)$ of the uncentered maximal function $M_\alpha f$ and we obtain that it is bounded on L^p for $1 < p \le +\infty$. The following remark plays a key role.

Remark 3.1
For any $x, y \in \mathbb{R}^d_+$ and $\varepsilon > 0$, we get

$$\tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d) = \int_0^{+\infty} \chi_{B^+(0,\varepsilon)}(x' - y', \rho)W_\alpha(x_d, y_d, \rho)\rho^{2\alpha+1}d\rho. \tag{3.1}$$

Put $u = (x' - y', \rho)$, we have $\|u\|^2 = \sum_{i=1}^{d-1}(x_i - y_i)^2 + \rho^2$. Then by (2.7) and (2.8), we obtain

$$\sqrt{\sum_{i=1}^{d-1}(x_i - y_i)^2 + (x_d - y_d)^2} < \|u\| < \sqrt{\sum_{i=1}^{d-1}(x_i - y_i)^2 + (x_d + y_d)^2}. \tag{3.2}$$

From (3.1), we have $u \in B^+(0,\varepsilon)$, which according to (3.2) gives

$$\tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d) = 0,$$

when $\|x-y\| = \sqrt{\sum_{i=1}^{d-1}(x_i-y_i)^2 + (x_d-y_d)^2} \geq \varepsilon$. Then we can assume that $y \in \mathbb{R}^d_+$ satisfies $\|x-y\| < \varepsilon$. Note that $\|x-y\| < \varepsilon$ implies $|x_d - y_d| < \varepsilon$.

Lemma 3.1
Let $\lambda \in \mathbb{R}^d_+$ and $\varepsilon \in]0,+\infty[$, then we have

$$|\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)| \leq c\,\varepsilon^{2\alpha+d+1}, \tag{3.3}$$

and

$$|\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)| \leq c\,\varepsilon^{\alpha+\frac{d}{2}}\|\lambda\|^{-(\alpha+\frac{d}{2}+1)}. \tag{3.4}$$

Here c is a constant which depends only on α and d.

Proof. By (2.6) we can write for $\lambda \in \mathbb{R}^d_+$ and $\varepsilon \in]0,+\infty[$,

$$\begin{aligned} &\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda) \\ &\quad = \frac{1}{2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+\frac{d+1}{2})}\int_0^{+\infty} \chi_{B^+(0,\varepsilon)}(r) j_{\alpha+\frac{d}{2}-\frac{1}{2}}(\|\lambda\| r) r^{2\alpha+d}dr \qquad (3.5) \\ &\quad = \frac{\varepsilon^{2\alpha+d+1}}{2^{\alpha+\frac{d+1}{2}}\Gamma(\alpha+\frac{d+3}{2})}\, j_{\alpha+\frac{d}{2}+\frac{1}{2}}(\|\lambda\|\varepsilon). \end{aligned}$$

Since $|j_{\alpha+\frac{d}{2}+\frac{1}{2}}(\|\lambda\|\varepsilon)| \leq 1$, we get

$$|\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)| \leq c\,\varepsilon^{2\alpha+d+1}.$$

Now, from (2.2), (3.5) and the fact that the function $z \mapsto \sqrt{z}J_\alpha(z)$ is bounded on $]0,+\infty[$, we can see that

$$\begin{aligned} |\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)| &= \frac{\varepsilon^{2\alpha+d+1}}{2^{\alpha+\frac{d+1}{2}}\Gamma(\alpha+\frac{d+3}{2})}|j_{\alpha+\frac{d}{2}+\frac{1}{2}}(\|\lambda\|\varepsilon)| \\ &= \frac{1}{2^{\alpha+\frac{d+1}{2}}\Gamma(\alpha+\frac{d+3}{2})}\frac{\varepsilon^{\alpha+\frac{d}{2}}}{\|\lambda\|^{\alpha+\frac{d}{2}+1}}\sqrt{\|\lambda\|\varepsilon}\,|J_{\alpha+\frac{d}{2}+\frac{1}{2}}(\|\lambda\|\varepsilon)| \\ &\leq c\,\varepsilon^{\alpha+\frac{d}{2}}\|\lambda\|^{-(\alpha+\frac{d}{2}+1)}. \end{aligned}$$

Hence the lemma is proved.

Lemma 3.2
For $\alpha > \frac{d}{2}-1$, there exists a $c > 0$ such that for any $x \in \mathbb{R}^d_+$ with $x_d > 0$ and $\varepsilon > 0$, we have

$$0 \leq \tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d) \leq c\Big(\frac{\varepsilon}{x_d}\Big)^{2\alpha+1}, \qquad a.e.\ \ y \in \mathbb{R}^d_+. \tag{3.6}$$

Here c is a constant which depends only on α and d.

Proof. Let $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$. Using (2.7) and (2.9) we have

$$0 \le \tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d) \le 1, \qquad \text{a.e. } y \in \mathbb{R}^d_+. \tag{3.7}$$

If $0 < x_d \le 2\varepsilon$, we obtain that

$$1 \le \Big(\frac{2\varepsilon}{x_d}\Big)^{2\alpha+1},$$

hence, by (3.7) we deduce (3.6). Therefore we can assume in the following argument that $x_d > 2\varepsilon$, and in view of Remark 3.1 that $y \in \mathbb{R}^d_+$ satisfies $|x_d - y_d| < \varepsilon$. Take $\psi \in D_*(\mathbb{R}^d)$ satisfying $0 \le \psi(x) \le 1$, $\operatorname{supp}\psi \subset B^+(0,1)$ and $\|\psi\|_{1,\alpha} = 1$. Put

$$\psi_t(x) = \frac{1}{t^{2\alpha+d+1}}\,\psi\Big(\frac{x}{t}\Big), \qquad t > 0,\ x \in \mathbb{R}^d_+,$$

the dilation of ψ. We have $\psi_t \in D_*(\mathbb{R}^d)$ which gives $\mathcal{F}_W(\psi_t) \in S_*(\mathbb{R}^d)$, then we can assert that both of $\tau_x(\chi_{B^+(0,\varepsilon)} *_\alpha \psi_t)$ and $\mathcal{F}_W(\tau_x(\chi_{B^+(0,\varepsilon)} *_\alpha \psi_t))$ are in $L^1(\mathbb{R}^d_+, \nu_\alpha)$. Using (2.5), (2.10) and (2.11), we obtain for $y \in \mathbb{R}^d_+$,

$$\begin{aligned} &\tau_x(\chi_{B^+(0,\varepsilon)} *_\alpha \psi_t)(y)\\ &= \int_{\mathbb{R}^d_+} \Psi_\lambda(-x', x_d)\Psi_\lambda(-y', y_d)\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)\mathcal{F}_W(\psi_t)(\lambda)d\nu_\alpha(\lambda) \qquad (3.8)\\ &= \int_{\mathbb{R}^d_+} e^{i[\langle x',\lambda'\rangle + \langle y',\lambda'\rangle]} j_\alpha(\lambda_d x_d) j_\alpha(\lambda_d y_d)\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)\mathcal{F}_W(\psi_t)(\lambda)d\nu_\alpha(\lambda). \end{aligned}$$

Clearly we have $\|\psi_t\|_{1,\alpha} = 1$. According to (2.4), we have

$$|\mathcal{F}_W(\psi_t)(\lambda)| \le \|\mathcal{F}_W(\psi_t)\|_\infty \le \|\psi_t\|_{1,\alpha} = 1, \qquad \text{a.e. } \lambda \in \mathbb{R}^d_+. \tag{3.9}$$

Put

$$A(\lambda) = e^{i[\langle x',\lambda'\rangle + \langle y',\lambda'\rangle]} j_\alpha(\lambda_d x_d) j_\alpha(\lambda_d y_d)\mathcal{F}_W(\chi_{B^+(0,\varepsilon)})(\lambda)\mathcal{F}_W(\psi_t)(\lambda).$$

Let us decompose (3.8) as a sum of three terms

$$\begin{aligned} \tau_x(\chi_{B^+(0,\varepsilon)} *_\alpha \psi_t)(y) &= \int_{\|\lambda\| \le x_d^{-1}} A(\lambda)d\nu_\alpha(\lambda) + \int_{x_d^{-1} \le \|\lambda\| \le \varepsilon^{-1}} A(\lambda)d\nu_\alpha(\lambda)\\ &\quad + \int_{\varepsilon^{-1} \le \|\lambda\|} A(\lambda)d\nu_\alpha(\lambda) \qquad (3.10)\\ &= I_1 + I_2 + I_3. \end{aligned}$$

From (2.1), (2.3), (3.3) and (3.9), we obtain

$$\begin{aligned} |I_1| &\le c\,\varepsilon^{2\alpha+d+1} \int_{\|\lambda\| \le x_d^{-1}} d\nu_\alpha(\lambda) \le c\Big(\frac{\varepsilon}{x_d}\Big)^{2\alpha+d+1}\\ &\le c\Big(\frac{\varepsilon}{x_d}\Big)^{2\alpha+1} \qquad \text{for } x_d > 2\varepsilon. \end{aligned} \tag{3.11}$$

To estimate I_2, we observe that for $x_d > 2\varepsilon$ and $|x_d - y_d| < \varepsilon$, we have

$$\frac{1}{2}x_d < x_d - \varepsilon < y_d < x_d + \varepsilon < \frac{3}{2}x_d,$$

so we deduce

$$0 < y_d^{-(\alpha+\frac{1}{2})} < c\, x_d^{-(\alpha+\frac{1}{2})}. \tag{3.12}$$

By (2.2) and the fact that the function $z \mapsto \sqrt{z}J_\alpha(z)$ is bounded on $]0, +\infty[$, we can write

$$|j_\alpha(z)| \le c\, z^{-(\alpha+\frac{1}{2})}, \tag{3.13}$$

then from (2.1), (3.3), (3.9), (3.12) and (3.13), we get

$$\begin{aligned}
|I_2| &\le c\,\varepsilon^{2\alpha+d+1} x_d^{-(\alpha+\frac{1}{2})} y_d^{-(\alpha+\frac{1}{2})} \int_{x_d^{-1} \le \|\lambda\| \le \varepsilon^{-1}} \lambda_d^{-(2\alpha+1)} d\nu_\alpha(\lambda) \\
&\le c\,\varepsilon^{2\alpha+d+1} x_d^{-2\alpha-1} \Big(\frac{1}{\varepsilon^d} - \frac{1}{x_d^d}\Big) \le c\,\varepsilon^{2\alpha+1} x_d^{-2\alpha-1} \qquad (3.14) \\
&\le c\left(\frac{\varepsilon}{x_d}\right)^{2\alpha+1} \qquad \text{for } x_d > 2\varepsilon.
\end{aligned}$$

For I_3, we use (2.1), (3.4), (3.9), (3.12) and (3.13) and we find that

$$\begin{aligned}
|I_3| &\le c\,\varepsilon^{\alpha+\frac{d}{2}} x_d^{-(\alpha+\frac{1}{2})} y_d^{-(\alpha+\frac{1}{2})} \int_{\varepsilon^{-1} \le \|\lambda\|} \|\lambda\|^{-(\alpha+\frac{d}{2}+1)} \lambda_d^{-(2\alpha+1)} d\nu_\alpha(\lambda) \\
&\le c\,\varepsilon^{\alpha+\frac{d}{2}} x_d^{-2\alpha-1} \int_{\varepsilon^{-1}}^{+\infty} r^{-\alpha+\frac{d}{2}-2} dr.
\end{aligned}$$

Since $\alpha > \frac{d}{2} - 1$, we obtain

$$|I_3| \le c\,\varepsilon^{\alpha+\frac{d}{2}} x_d^{-(2\alpha+1)} \varepsilon^{\alpha-\frac{d}{2}+1} \le c\left(\frac{\varepsilon}{x_d}\right)^{2\alpha+1} \qquad \text{for } x_d > 2\varepsilon. \tag{3.15}$$

Thus we get by (3.10), (3.11), (3.14) and (3.15),

$$0 \le \tau_x(\chi_{B^+(0,\varepsilon)} *_\alpha \psi_t)(-y', y_d) \le c\left(\frac{\varepsilon}{x_d}\right)^{2\alpha+1} \qquad \text{for } x_d > 2\varepsilon.$$

Now using (2.7) and Fatou's Lemma, we can assert that

$$\lim_{t \to 0^+} \tau_x(\chi_{B^+(0,\varepsilon)} *_\alpha \psi_t)(-y', y_d) = \tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d), \qquad \text{a.e. } y \in \mathbb{R}^d_+.$$

Hence, we deduce that

$$0 \le \tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d) \le c\left(\frac{\varepsilon}{x_d}\right)^{2\alpha+1}$$

for $x_d > 2\varepsilon$, a.e. $y \in \mathbb{R}^d_+$ with $|x_d - y_d| < \varepsilon$, therefore (3.6) is established.

Before the next result we introduce some notation. For $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$, we put

$$C^+(x,\varepsilon) = B_{d-1}(x',\varepsilon)\times]\max\{0, x_d-\varepsilon\}, x_d+\varepsilon[,$$

with $x = (x', x_d)$ and $B_{d-1}(x',\varepsilon)$ is the closed ball on $\mathbb{R}^{d-1}$ with radius ε centered at x'.

Lemma 3.3
For $\alpha > \frac{d}{2}-1$, there exists a $c > 0$ such that for any $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$, we have

$$0 \le \tau_x(\chi_{B^+(0,\varepsilon)})(-y', y_d) \le c\,\frac{\nu_\alpha(B^+(0,\varepsilon))}{\nu_\alpha(C^+(x,\varepsilon))} \qquad a.e.\ \ y \in \mathbb{R}^d_+. \tag{3.16}$$

Here c is a constant which depends only on α and d.

Proof. Let $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$. Using (2.1), we have

$$\nu_\alpha(B^+(0,\varepsilon)) = \frac{\varepsilon^{2\alpha+d+1}}{2^{\alpha+\frac{d-1}{2}}(2\alpha+d+1)\Gamma(\alpha+\frac{d+1}{2})}. \tag{3.17}$$

On the one hand, we get for $x_d \le \varepsilon$,

$$C^+(x,\varepsilon) = B_{d-1}(x',\varepsilon)\times [0, x_d+\varepsilon[,$$

then, we obtain

$$\begin{aligned}
\nu_\alpha(C^+(x,\varepsilon)) &= \frac{1}{(2\pi)^{\frac{d-1}{2}}2^\alpha\Gamma(\alpha+1)}\int_{B_{d-1}(x',\varepsilon)} dy_1\ldots dy_{d-1}\int_0^{x_d+\varepsilon} y_d^{2\alpha+1}dy_d\\
&\le c\,\varepsilon^{2(\alpha+1)}\int_{B_{d-1}(x',\varepsilon)} dy_1\ldots dy_{d-1}\\
&\le c\,\varepsilon^{2\alpha+d+1}.
\end{aligned}$$

Using (3.17), we deduce

$$\nu_\alpha(C^+(x,\varepsilon)) \le c\,\nu_\alpha(B^+(0,\varepsilon)),$$

then by (3.7), we obtain (3.16) for $x_d \le \varepsilon$.

On the other hand, we have for $x_d > \varepsilon$,

$$C^+(x,\varepsilon) = B_{d-1}(x',\varepsilon)\times [x_d-\varepsilon, x_d+\varepsilon[,$$

then

$$\begin{aligned}
\nu_\alpha(C^+(x,\varepsilon)) &= \frac{1}{(2\pi)^{\frac{d-1}{2}}2^\alpha\Gamma(\alpha+1)}\int_{B_{d-1}(x',\varepsilon)} dy_1\ldots dy_{d-1}\int_{x_d-\varepsilon}^{x_d+\varepsilon} y_d^{2\alpha+1}dy_d\\
&\le c\,\varepsilon^{d-1}(x_d+\varepsilon)^{2\alpha+1}\times\varepsilon\\
&\le c\,\varepsilon^d x_d^{2\alpha+1}.
\end{aligned}$$

Using (3.17), we get

$$\nu_\alpha(C^+(x,\varepsilon)) \le c\,\nu_\alpha(B^+(0,\varepsilon))\Big(\frac{x_d}{\varepsilon}\Big)^{2\alpha+1},$$

thus by (3.6), we obtain (3.16) for $x_d > \varepsilon$, which proves the result.

According to ([13], Lemma 1.6), we have the following Vitali covering lemma.

Lemma 3.4
Let E be a measurable subset of $\mathbb{R}^d_+$ (with respect to ν_α) which is covered by the union of a family $\{B_j^+\}$ where $B_j^+ = B^+(x_j, r_j)$. Then from this family we can select a subfamily, $B_1^+, B_2^+, \ldots$ (which may be finite) such that $B_i^+ \cap B_j^+ = \emptyset$ for $i \neq j$ and

$$\sum_h \nu_\alpha(B_h^+) \geq c\, \nu_\alpha(E).$$

We recall that for $x \in \mathbb{R}^d_+$,

$$M_\alpha f(x) = \sup_{\varepsilon>0,\, z\in B^+(x,\varepsilon)} \frac{1}{\nu_\alpha(B^+(0,\varepsilon))} \Big| \int_{\mathbb{R}^d_+} f(y)\tau_z(\chi_{B^+(0,\varepsilon)})(-y', y_d)d\nu_\alpha(y)\Big|,$$

so, we can write also

$$M_\alpha f(x) = \sup_{\varepsilon>0,\, z\in B^+(x,\varepsilon)} \frac{1}{\nu_\alpha(B^+(0,\varepsilon))} \big| f *_\alpha \chi_{B^+(0,\varepsilon)}(z)\big|.$$

Theorem 3.1
The uncentered maximal function $M_\alpha f$ is of weak type $(1,1)$.

Proof. Let $\varepsilon > 0$, $x \in \mathbb{R}^d_+$, $z \in B^+(x,\varepsilon)$ and $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$. Using Remark 3.1, we have

$$\big| f *_\alpha \chi_{B^+(0,\varepsilon)}(z)\big| \leq \int_{B^+(z,\varepsilon)} |f(y)|\tau_z(\chi_{B^+(0,\varepsilon)})(-y', y_d)d\nu_\alpha(y).$$

By (3.16), we obtain

$$\big| f *_\alpha \chi_{B^+(0,\varepsilon)}(z)\big| \leq c\, \frac{\nu_\alpha(B^+(0,\varepsilon))}{\nu_\alpha(C^+(z,\varepsilon))} \int_{B^+(z,\varepsilon)} |f(y)|d\nu_\alpha(y).$$

Since $B^+(z,\varepsilon) \subset C^+(z,\varepsilon)$ we get

$$\big| f *_\alpha \chi_{B^+(0,\varepsilon)}(z)\big| \leq c\, \frac{\nu_\alpha(B^+(0,\varepsilon))}{\nu_\alpha(B^+(z,\varepsilon))} \int_{B^+(z,\varepsilon)} |f(y)|d\nu_\alpha(y).$$

Hence we deduce that

$$M_\alpha f(x) \leq c\, \tilde{M}_\alpha f(x), \tag{3.18}$$

where $\tilde{M}_\alpha f$ is defined by

$$\tilde{M}_\alpha f(x) = \sup_{\varepsilon>0,\, z\in B^+(x,\varepsilon)} \frac{1}{\nu_\alpha(B^+(z,\varepsilon))} \int_{B^+(z,\varepsilon)} |f(y)|d\nu_\alpha(y).$$

For $\lambda > 0$, put

$$\tilde{E}_\lambda = \{x \in \mathbb{R}^d_+ :\ \tilde{M}_\alpha f(x) > \lambda\}.$$

Then, for each $x \in \tilde{E}_\lambda$ there exist $\varepsilon > 0$ and $z \in B^+(x, \varepsilon)$ such that

$$\int_{B^+(z,\varepsilon)} |f(y)| d\nu_\alpha(y) > \lambda\, \nu_\alpha(B^+(z,\varepsilon)). \tag{3.19}$$

Furthermore, note that $x \in B^+(z,\varepsilon)$, then if x runs through the set $\tilde{E}_\lambda$, the union of the corresponding $B^+(z,\varepsilon)$ covers $\tilde{E}_\lambda$. Thus using Lemma 3.4, we can select a disjoint subfamily $B^+(z_1,\varepsilon_1), B^+(z_2,\varepsilon_2), \dots$ (which may be finite) such that

$$\sum_h \nu_\alpha(B^+(z_h,\varepsilon_h)) \geq c\, \nu_\alpha(\tilde{E}_\lambda). \tag{3.20}$$

We have

$$\int_{y \in \bigcup B^+(z_h,\varepsilon_h)} |f(y)| d\nu_\alpha(y) = \sum_h \Big(\int_{y \in B^+(z_h,\varepsilon_h)} |f(y)| d\nu_\alpha(y) \Big),$$

applying (3.19) and (3.20) to each of the mutually disjoint subfamily, we get

$$\int_{y \in \bigcup B^+(z_h,\varepsilon_h)} |f(y)| d\nu_\alpha(y) > \lambda \sum_h \nu_\alpha(B^+(z_h,\varepsilon_h) \geq \lambda c\, \nu_\alpha(\tilde{E}_\lambda).$$

But since the first member of this inequality is majorized by $\|f\|_{1,\alpha}$, we obtain

$$\nu_\alpha(\tilde{E}_\lambda) \leq c\, \frac{\|f\|_{1,\alpha}}{\lambda},$$

which gives that $\tilde{M}_\alpha f$ is of weak type $(1,1)$ and hence from (3.18) the same is true for $M_\alpha f$.

As consequence of Theorem 3.1, we obtain the following corollary.

Corollary 3.1
If $1 < p \leq +\infty$ and $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, then we have

$$M_\alpha f \in L^p(\mathbb{R}^d_+, \nu_\alpha) \quad \textit{and} \quad \|M_\alpha f\|_{p,\alpha} \leq c\, \|f\|_{p,\alpha}.$$

Proof. Using Theorem 3.1, ([11], Corollary 21.72) and proceeding in the same manner as in the proof on Euclidean spaces (see for example Theorem 1 in [13], section 1.3) completes the proof.

4. Application

Since the weak type $(1,1)$ and the strong type (p,p) for $p > 1$ of the uncentered maximal function are an important tool in harmonic analysis, we define as application in this paper, a large class of operators such that each operator of this class satisfies these L^p-inequalities, and such that, in particular, the maximal operator associated respectively with the Weinstein heat semigroup and the Weinstein-Poisson semigroup belong to this class.

Let $\varphi \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ be a radial function, that is $\varphi(x) = \varphi_0(\|x\|)$, for every $x \in \mathbb{R}^d_+$. Then we denote by $\mathcal{H}_\alpha$ the following operator, defined for each function $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ and $x \in \mathbb{R}^d_+$ by

$$\mathcal{H}_\alpha f(x) = \sup_{t>0} |f *_\alpha \varphi_t(x)|,$$

where φ_t is the dilation of φ given by $\varphi_t(x) = \frac{1}{t^{2\alpha+d+1}}\varphi\big(\frac{x}{t}\big)$.

Proposition 4.1
Let $\varphi \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ be a radial function, that is $\varphi(x) = \varphi_0(\|x\|)$ for every $x \in \mathbb{R}^d_+$, such that φ_0 is differentiable and satisfies

$$\lim_{r\to+\infty} \varphi_0(r) = 0, \qquad \int_0^{+\infty} r^{2\alpha+d+1}\Big|\frac{d}{dr}\varphi_0(r)\Big|dr < +\infty,$$

then we have for $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$,

$$\mathcal{H}_\alpha f(x) \le c\, M_\alpha f(x).$$

Proof. For $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ and g a radial function in $L^1(\mathbb{R}^d_+, \nu_\alpha)$, that is $g(x) = g_0(\|x\|)$. We can write for every $x \in \mathbb{R}^d_+$

$$\begin{aligned} f *_\alpha g(x) &= \int_{\mathbb{R}^d_+} \tau_x(f)(-y', y_d) g(y) d\nu_\alpha(y) \\ &= \int_{\mathbb{R}^d_+} \tau_y(f)(-x', x_d) g(y) d\nu_\alpha(y) \qquad (4.1)\\ &= \int_0^{+\infty} \Big(g_0(r) \int_{S^{d-1}_+} \tau_{rz}(f)(-x', x_d) d\nu_\alpha(z)\Big) r^{2\alpha+d} dr, \end{aligned}$$

where S^{d-1}_+ is the unit sphere on $\mathbb{R}^d_+$. Using (4.1) and according to the proof of Theorem 7.5 in [16], we have for such a function φ and for $x \in \mathbb{R}^d_+$,

$$\begin{aligned} |f *_\alpha \varphi(x)| &\le c \sup_{\varepsilon>0} \frac{1}{\nu_\alpha(B^+(0,\varepsilon))}\Big|f *_\alpha \chi_{B^+(0,\varepsilon)}(x)\Big| \int_0^{+\infty} r^{2\alpha+d+1}\big|\frac{d}{dr}\varphi_0(r)\big|dr \\ &\le c\, M_\alpha f(x) \int_0^{+\infty} r^{2\alpha+d+1}\big|\frac{d}{dr}\varphi_0(r)\big|dr. \end{aligned}$$

Then we can write

$$|f *_\alpha \varphi_t(x)| \le c\, M_\alpha f(x) \int_0^{+\infty} \frac{r^{2\alpha+d+1}}{t^{2\alpha+d+2}}\big|\frac{d}{dr}\varphi_0(\frac{r}{t})\big|dr.$$

By a change of variables, we obtain

$$|f *_\alpha \varphi_t(x)| \le c\, M_\alpha f(x) \int_0^{+\infty} r^{2\alpha+d+1}\big|\frac{d}{dr}\varphi_0(r)\big|dr,$$

from which we deduce that

$$\mathcal{H}_\alpha f(x) \le c\, M_\alpha f(x).$$

We give below two important examples that satisfy the conditions of Proposition 4.1.

Example 4.1
Put $\varphi(x) = e^{-\frac{\|x\|^2}{2}}$, $x \in \mathbb{R}^d_+$, then we get for $t > 0$,

$$\varphi_{\sqrt{2t}}(x) = \frac{1}{(2t)^{\alpha+\frac{d+1}{2}}} e^{-\frac{\|x\|^2}{4t}}.$$

In this case, $\varphi_{\sqrt{2t}} = q_t$ is the Weinstein heat kernel and $\mathcal{H}_\alpha$ is the maximal operator associated with the Weinstein heat semigroup.

Example 4.2
For $x \in \mathbb{R}^d_+$, take $\varphi(x) = \frac{c_{\alpha,d}}{(1+\|x\|^2)^{\alpha+\frac{d}{2}+1}}$ with $c_{\alpha,d} = \frac{2^{\alpha+\frac{d+1}{2}}}{\sqrt{\pi}}\Gamma(\alpha + \frac{d}{2} + 1)$, then we get for $t > 0$,

$$\varphi_t(x) = c_{\alpha,d} \frac{t}{(t^2 + \|x\|^2)^{\alpha+\frac{d}{2}+1}}.$$

In this case, $\varphi_t = P_t$ is the Weinstein-Poisson kernel and $\mathcal{H}_\alpha$ is the maximal operator associated with the Weinstein-Poisson semigroup.

References

[1] Abdelkefi, Chokri. "Dunkl operators on $\mathbb{R}^d$ and uncentered maximal function." *J. Lie Theory* 20, no. 1 (2010): 113-125. Cited on 107.

[2] Ben Nahia, Zouhir, and Néjib Ben Salem. "Spherical harmonics and applications associated with the Weinstein operator." *Potential Theory ICPT 94 (Kouty, 1994)*, 233-241. Berlin: de Gruyter, 1996. Cited on 106.

[3] Ben Nahia, Zouhir, and Néjib Ben Salem. "On a mean value property associated with the Weinstein operator." *Potential Theory ICPT 94 (Kouty, 1994)*, 243-253. Berlin: de Gruyter, 1996. Cited on 106.

[4] Bloom, Walter R., and Zeng Fu Xu. "The Hardy-Littlewood maximal function for Chébli-Trimèche hypergroups." *Applications of hypergroups and related measure algebras (Seattle, WA, 1993)*, 45–70. Vol. 183 of *Contemp. Math.* Providence, RI: Amer. Math. Soc., 1995. Cited on 107.

[5] Brelot, Marcel. "Équation de Weinstein et potentiels de Marcel Riesz" *Séminaire de Théorie du Potentiel, no. 3 (Paris, 1976/1977)*, 18–38. Vol. 681 of *Lecture Notes in Math.* Berlin: Springer, 1978. Cited on 106.

[6] Clerc, Jean-Louis, and Elias Menachem Stein. "L^p-multipliers for noncompact symmetric spaces." *Proc. Nat. Acad. Sci. U.S.A.* 71 (1974): 3911-3912. Cited on 106.

[7] Connett, William C, and Alan L. Schwartz. "The Littlewood-Paley theory for Jacobi expansions." *Trans. Amer. Math. Soc.* 251 (1979): 219-234. Cited on 106.

[8] Connett, William C., and Alan L. Schwartz. "A Hardy-Littlewood maximal inequality for Jacobi type hypergroups." *Proc. Amer. Math. Soc.* 107, no. 1 (1989): 137-143. Cited on 106.

[9] Gaudry, Garth Ian, et all. "Hardy-Littlewood maximal functions on some solvable Lie groups." *J. Austral. Math. Soc. Ser. A* 45, no. 1 (1988): 78-82. Cited on 106.

[10] Hardy, Godfrey Harold, and John Edensor Littlewood. "A maximal theorem with function-theoretic applications." *Acta Math.* 54, no. 1 (1930): 81-116. Cited on 106.

[11] Hewitt, Edwin, and Karl Stromberg. *Real and abstract analysis. A modern treatment of the theory of functions of a real variable.* New-York: Springer-Verlag, 1965. Cited on 116.

[12] Leutwiler, Heinz. "Best constants in the Harnack inequality for the Weinstein equation." *Aequationes Math.* 34, no. 2-3 (1987): 304-315. Cited on 106.

[13] Stein, Elias Menachem. *Singular integrals and differentiability properties of functions.* Vol. 30 of *Princeton Mathematical Series.* Princeton, New Jersey: Princeton University Press, 1970. Cited on 106, 115 and 116.

[14] Stempak, Krzysztof. "La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel." *C. R. Acad. Sci. Paris Sér. I Math.* 303, no. 1 (1986): 15-18. Cited on 106.

[15] Strömberg, Jan-Olov. "Weak type L^1 estimates for maximal functions on noncompact symmetric spaces." *Ann. of Math. (2)* 114, no. 1 (1981): 115-126. Cited on 106.

[16] Thangavelu, Sundaram, and Yuan Xu. "Convolution operator and maximal function for the Dunkl transform." *J. Anal. Math.* 97 (2005): 25-55. Cited on 106 and 117.

[17] Torchinsky, Alberto. *Real-variable nethods in harmonic analysis.* Vol. 123 of *Pure and applied mathematics.* Orlando: Academic Press, 1986. Cited on 106.

[18] Watson, George Neville. *A treatise on the theory of Bessel functions.* Cambridge-New York-Oakleigh: Cambridge University Press, 1966. Cited on 108.

[19] Weinstein, Alexander. "Singular partial differential equations and their applications." *Fluid dynamics and applied mathematics*, 29-49. New York: Gordon and Breach, 1962. Cited on 106.

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