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Counter examples for pseudo-amenability of some semigroup algebras

Abstract. In this short note, we give some counter examples which show that [11, Proposition 3.5] is not true. As a consequence, the arguments in [11, Proposition 4.10] are not valid.

A Banach algebra A is called *amenable* if there exists a bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ and $\pi_A(m_\alpha)a \rightarrow a$ for each $a \in A$, where $\pi_A: A \otimes_p A \rightarrow A$ is the product morphism given by $\pi_A(a \otimes b) = ab$, also $A \otimes_p A$ is denoted for the projective tensor product of A with A , for the history of amenability, see [12].

By removing the boundedness condition, Ghahramani and Zhang [8] gave a new notion of amenability, called pseudo-amenability. In fact A is called *pseudo-amenable* if there exists a not necessarily bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ and $\pi_A(m_\alpha)a \rightarrow a$ for each $a \in A$.

Pseudo-amenability of some Banach algebras like Segal algebras, semigroup algebras and matrix algebras were investigated in [6], [7], [13] and [5].

Recently [11] come to our attention. The paper is devoted to study the pseudo-amenability of some semigroup algebras related to inverse and restricted semigroups. We found out some result of the paper is not valid and we give some counter examples to show our claims.

We should recall some definitions from the semigroup theory here. A semigroup S is called *inverse semigroup*, if for every $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*s^*s^* = s^*$.

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Let S be an inverse semigroup. The restricted product of x and y of S is xy if $x^*x = yy^*$ and undefined otherwise. The set S with the restricted product forms a discrete groupoid. If we adjoin a zero element 0 to this groupoid and put $0^* = 0$, we get an inverse semigroup denoted by S_r . The multiplication of S_r defined as follows

$$x * y = \begin{cases} xy, & \text{if } x^*x = yy^* \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup S_r is called *restricted semigroup* see [2].

EXAMPLE 1

Let $S = \mathbb{N}$ be the set of all natural numbers. Equip this set with " \vee " operation, that is, $m \vee n = \max\{m, n\}$ for all $m, n \in S$. Also equip S with " \wedge " operation, that is, $m \wedge n = \min\{m, n\}$ for all $m, n \in S$. With these two productions S becomes a commutative infinite inverse semigroup.

EXAMPLE 2

Let $S = G$ be a discrete group. It is well-known that S is an inverse semigroup.

THEOREM 3 ([11, Proposition 3.5])

Let S be an inverse semigroup. Then $\ell^1(S)$ is pseudo-amenable if and only if S is finite.

Using the previous theorem, the following result is given.

THEOREM 4 ([11, Proposition 4.10])

Let S be an inverse semigroup. Then $\ell^1(S_r)$ is pseudo-amenable if and only if it is character amenable.

We recall that a Banach algebra A is *approximately amenable*, if for each Banach A -bimodule X and each bounded derivation $D: A \rightarrow X^*$ there exists a net (x_α) in X^* such that

$$D(a) = \lim_{\alpha} a \cdot x_\alpha - x_\alpha \cdot a, \quad a \in A,$$

for more details see [9] and [10].

THEOREM 5 ([14, Theorem 2.7(i)])

If A has a bounded approximate identity, then it is approximately amenable if and only if it is pseudo-amenable.

In the following examples we show that Theorem 3 and the proof of Theorem 4 are not valid.

EXAMPLE 6

Let $S = \mathbb{N}$. Equip this semigroup with \wedge and \vee operations. We denote S by \mathbb{N}_\wedge and \mathbb{N}_\vee , respectively. Using [5, Example 10.10], we see that $\ell^1(\mathbb{N}_\wedge)$ and $\ell^1(\mathbb{N}_\vee)$ are approximately amenable. Clearly, δ_1 is an identity for $\ell^1(\mathbb{N}_\wedge)$. Also $(\delta_n)_{n \in \mathbb{N}}$ is a bounded approximate identity for $\ell^1(\mathbb{N}_\wedge)$. Applying Theorem 5 we get that $\ell^1(\mathbb{N}_\wedge)$ and $\ell^1(\mathbb{N}_\vee)$ are pseudo-amenable semigroup algebras. Since \mathbb{N}_\wedge and \mathbb{N}_\vee are inverse semigroups, by Theorem 3, \mathbb{N}_\wedge and \mathbb{N}_\vee must be finite which is impossible.

EXAMPLE 7

Let $S = G$ be a discrete and infinite amenable group. By the Johnson Theorem [12, Theorem 2.1.8], $\ell^1(G)$ is amenable. So $\ell^1(G)$ is pseudo-amenable. Since G is an inverse semigroup by Theorem 3, G must be finite, which is impossible.

It is shown in the proof of [11, Proposition 3.5] that every amenable inverse semigroup is finite which is not true. For instance, every bicyclic inverse semigroup is amenable but not finite and hence one side of the mentioned proposition is not true. In fact, it should be changed as follows:

THEOREM 8

For every finite inverse semigroup S , $\ell^1(S)$ is pseudo-amenable.

In light of the above discussion, [11, Proposition 4.10] can not be valid. In other words, when $\ell^1(S)$ is pseudo-amenable, then by [11, Proposition 3.5], S is amenable (not finite) and in general case, amenability of S does not imply amenability or character amenability of $\ell^1(S)$. This happens when S is an abelian or left (right) cancellative unital semigroup. On the other hand, for module version of amenability, we have

THEOREM 9

For a discrete inverse semigroup S ,

- (i) $\ell^1(S)$ is module amenable if and only if S is amenable [1].
- (ii) $\ell^1(S)$ is module pseudo-amenable if and only if S is amenable [4].
- (iii) $\ell^1(S)$ is module character amenable if and only if S is amenable [3].

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