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## On the Chow ring of certain Fano fourfolds

**Abstract.** We prove that certain Fano fourfolds of K3 type constructed by Fatighenti–Mongardi have a multiplicative Chow–Künneth decomposition. We present some consequences for the Chow ring of these fourfolds.

### 1. Introduction

This note is a part of a program aimed at understanding the class of varieties admitting a *multiplicative Chow–Künneth decomposition*, in the sense of [18]. The concept of multiplicative Chow–Künneth decomposition was introduced in order to better understand the (conjectural) behaviour of the Chow ring of hyperkähler varieties, while also providing a systematic explanation of the peculiar behaviour of the Chow ring of K3 surfaces and abelian varieties. In [12], the following conjecture is raised.

#### CONJECTURE 1.1

*Let  $X$  be a smooth projective Fano variety of K3 type (i.e.  $\dim X = 2m$  and the Hodge numbers  $h^{p,q}(X)$  are 0 for all  $p \neq q$  except for  $h^{m-1,m+1}(X) = 1$  and  $h^{m+1,m-1}(X) = 1$ ). Then  $X$  has a multiplicative Chow–Künneth decomposition.*

This conjecture is verified in some special cases [11], [10], [12]. The aim of the present note is to provide some more evidence for Conjecture 1.1. We consider two families of Fano fourfolds of K3 type (these are the families labelled B1 and B2 in [5]).

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THEOREM 1 (=Theorem 4.1)

Let  $X$  be a smooth fourfold of one of the following types

- (i) a hypersurface of multidegree  $(2, 1, 1)$  in  $M = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ ;
- (ii) a hypersurface of multidegree  $(2, 1)$  in  $M = \text{Gr}(2, 4) \times \mathbb{P}^1$  (with respect to the Plücker embedding).

Then  $X$  has a multiplicative Chow–Künneth decomposition.

Theorem 4.1 has interesting consequences for the Chow ring  $A^*(X)_{\mathbb{Q}}$  of these fourfolds:

COROLLARY 1 (=Corollary 5.1)

Let  $X$  and  $M$  be as in Theorem 4.1. Let  $R^3(X) \subset A^3(X)_{\mathbb{Q}}$  be the subgroup generated by the Chern class  $c_3(T_X)$ , the image of the restriction map  $A^3(M)_{\mathbb{Q}} \rightarrow A^3(X)_{\mathbb{Q}}$ , and intersections  $A^1(X)_{\mathbb{Q}} \cdot A^2(X)_{\mathbb{Q}}$  of divisors with 2-cycles. The cycle class map induces an injection

$$R^3(X) \hookrightarrow H^6(X, \mathbb{Q}).$$

This is reminiscent of the famous result of Beauville–Voisin describing the Chow ring of a K3 surface [2]. More generally, there is a similar injectivity result for the Chow ring of certain self-products  $X^m$  (Corollary 5.1).

Another consequence is the existence of a multiplicative decomposition in the derived category for families of Fano fourfolds as in Theorem 4.1 (Corollary 5.4).

CONVENTIONS 1

In this note, the word *variety* will refer to a reduced irreducible scheme of finite type over  $\mathbb{C}$ . For a smooth variety  $X$ , we will denote by  $A^j(X)$  the Chow group of codimension  $j$  cycles on  $X$  with  $\mathbb{Q}$ -coefficients.

The notation  $A_{\text{hom}}^j(X)$  will be used to indicate the subgroups of homologically trivial cycles.

For a morphism between smooth varieties  $f: X \rightarrow Y$ , we will write  $\Gamma_f \in A^*(X \times Y)$  for the graph of  $f$ . The contravariant category of Chow motives (i.e. pure motives with respect to rational equivalence as in [17], [15]) will be denoted by  $\mathcal{M}_{\text{rat}}$ .

We will write  $H^*(X) := H^*(X, \mathbb{Q})$  for a singular cohomology with  $\mathbb{Q}$ -coefficients.

## 2. The Fano fourfolds

PROPOSITION 2.1

- (i) Let  $X \subset \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$  be a smooth hypersurface of multidegree  $(2, 1, 1)$  (following [5], we will say  $X$  is “of type B1”). Then  $X$  is Fano, and the

Hodge numbers of  $X$  are

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 0 & 0 \\
 & & & & & 0 & 3 & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 22 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 3 & 0 & & \\
 & & & & & 0 & 0 & & & \\
 & & & & & & & & & 1
 \end{array}$$

- (ii) Let  $X \subset \text{Gr}(2,4) \times \mathbb{P}^1$  be a smooth hypersurface of multidegree  $(2,1)$  with respect to the Plücker embedding (following [5], we will say  $X$  is “of type B2”). Then  $X$  is Fano, and the Hodge numbers of  $X$  are

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 0 & 0 \\
 & & & & & 0 & 2 & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 22 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 2 & 0 & & \\
 & & & & & 0 & 0 & & & \\
 & & & & & & & & & 1
 \end{array}$$

*Proof.* An easy way to determine the Hodge numbers is to use the following identification:

LEMMA 2.2

Let  $Z$  be a smooth projective variety of Picard number 1, and  $X \subset Z \times \mathbb{P}^1$  a general hypersurface of bidegree  $(d,1)$ . Then  $X$  is isomorphic to the blow-up of  $Z$  with centre  $S$ , where  $S \subset Z$  is a smooth dimensionally transversal intersection of 2 divisors of degree  $d$ .

Conversely, given a smooth dimensionally transversal intersection  $S \subset Z$  of 2 divisors of degree  $d$ , the blow-up of  $Z$  with centre  $S$  is isomorphic to a smooth hypersurface  $X \subset Z \times \mathbb{P}^1$  of bidegree  $(d,1)$ .

*Proof.* This is [5, Lemma 2.2]. The gist of the argument is that  $X$  determines a pencil of divisors in  $Z$ , of which  $S$  is the base locus. In terms of equations, if  $X$  is defined by  $y_0f + y_1g = 0$  (where  $[y_0 : y_1] \in \mathbb{P}^1$  and  $f, g \in H^0(Z, \mathcal{O}_Z(d))$ ) then  $S$  is defined by  $f = g = 0$ . It follows that for  $X$  general (in the usual sense of “being parametrized by a Zariski open in the parameter space”) the locus  $S$  is smooth.

In the case (i),  $Z = \mathbb{P}^3 \times \mathbb{P}^1$  and  $S$  is a genus 7 K3 surface. In the case (ii),  $Z = \text{Gr}(2,4)$  (which is a quadric in  $\mathbb{P}^5$ ), and  $S$  is a genus 5 K3 surface. This readily gives the Hodge numbers.

## REMARK 2.3

Fatighenti–Mongardi [5] give a long list of Fano varieties of K3 type. The Fano fourfolds of Proposition 2.1 (i) and (ii) are labelled B1 resp. B2 in their list.

### 3. Multiplicative Chow–Künneth decomposition

DEFINITION 3.1 (Murre [14])

Let  $X$  be a smooth projective variety of dimension  $n$ . We say that  $X$  has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \quad \text{in } A^n(X \times X),$$

such that the  $\pi_X^i$  are mutually orthogonal idempotents and the action of  $\pi_X^i$  on  $H^j(X)$  is the identity for  $i = j$  and zero for  $i \neq j$ . Given a CK decomposition for  $X$ , we set

$$A^i(X)_{(j)} := (\pi_X^{2i-j})_* A^i(X).$$

The CK decomposition is said to be *self-dual* if

$$\pi_X^i = {}^t \pi_X^{2n-i} \quad \text{in } A^n(X \times X) \text{ for all } i.$$

Here  ${}^t \pi$  denotes the transpose of a cycle  $\pi$ . (NB: “CK decomposition” is short-hand for “Chow–Künneth decomposition”.)

## REMARK 3.2

The existence of a Chow–Künneth decomposition for any smooth projective variety is part of Murre’s conjectures [14], [15]. It is expected that for any  $X$  with a CK decomposition, one has

$$A^i(X)_{(j)} \stackrel{??}{=} 0 \quad \text{for } j < 0, \quad A^i(X)_{(0)} \cap A_{num}^i(X) \stackrel{??}{=} 0.$$

These are Murre’s conjectures B and D, respectively.

DEFINITION 3.3 (Definition 8.1 in [18])

Let  $X$  be a smooth projective variety of dimension  $n$ . Let  $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$  be the class of the small diagonal

$$\Delta_X^{sm} := \{(x, x, x) : x \in X\} \subset X \times X \times X.$$

A CK decomposition  $\{\pi_X^i\}$  of  $X$  is *multiplicative* if it satisfies

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \otimes \pi_X^j) = 0 \quad \text{in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k.$$

In that case,

$$A^i(X)_{(j)} := (\pi_X^{2i-j})_* A^i(X)$$

defines a bigraded ring structure on the Chow ring; that is, the intersection product has the property that

$$\text{Im}(A^i(X)_{(j)} \otimes A^{i'}(X)_{(j')}) \xrightarrow{\cdot} A^{i+i'}(X) \subseteq A^{i+i'}(X)_{(j+j')}.$$

(For brevity, we will write *MCK decomposition* for “multiplicative Chow–Künneth decomposition”.)

## REMARK 3.4

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “(weak) splitting property” [1]. For more ample discussion, and examples of varieties admitting a MCK decomposition, we refer to [18, Chapter 8], as well as [20], [19], [7], [13].

## REMARK 3.5

It turns out that any MCK decomposition is self-dual, cf. [8, Footnote 24].

There are the following useful general results.

## PROPOSITION 3.6 (Shen–Vial [18])

*Let  $M, N$  be smooth projective varieties that have an MCK decomposition. Then the product  $M \times N$  has an MCK decomposition.*

*Proof.* This is [18, Theorem 8.6], which shows more precisely that the *product CK decomposition*

$$\pi_{M \times N}^i := \sum_{k+\ell=i} \pi_M^k \times \pi_N^\ell \in A^{\dim M + \dim N}((M \times N) \times (M \times N))$$

is multiplicative.

## PROPOSITION 3.7 (Shen–Vial [19])

*Let  $M$  be a smooth projective variety, and let  $f: \widetilde{M} \rightarrow M$  be the blow-up with centre a smooth closed subvariety  $N \subset M$ . Assume that*

- 1°  *$M$  and  $N$  have an MCK decomposition;*
- 2° *the Chern classes of the normal bundle  $\mathcal{N}_{N/M}$  are in  $A_{(0)}^*(N)$ ;*
- 3° *the graph of the inclusion morphism  $N \rightarrow M$  is in  $A_{(0)}^*(N \times M)$ ;*
- 4° *the Chern classes  $c_j(T_M)$  are in  $A_{(0)}^*(M)$ .*

*Then  $\widetilde{M}$  has an MCK decomposition, the Chern classes  $c_j(T_{\widetilde{M}})$  are in  $A_{(0)}^*(\widetilde{M})$ , and the graph  $\Gamma_f$  is in  $A_{(0)}^*(\widetilde{M} \times \widetilde{M})$ .*

*Proof.* This is [19, Proposition 2.4]. (NB: in loc. cit.,  $M$  and  $N$  are required to have a *self-dual* MCK decomposition; however, the self-duality is actually a redundant hypothesis, cf. Remark 3.5.)

In a nutshell, the construction of loc. cit. is as follows. Given MCK decompositions  $\pi_M^*$  and  $\pi_N^*$  (of  $M$  resp.  $N$ ), one defines

$$\pi_{\widetilde{M}}^j := \Psi \circ \left( \pi_M^j \oplus \bigoplus_{k=1}^r \pi_N^{j-2k} \right) \circ \Psi^{-1} \in A^{\dim \widetilde{M}}(\widetilde{M} \times \widetilde{M}), \quad (1)$$

where  $r + 1$  is the codimension of  $N$  in  $M$ , and  $\Psi, \Psi^{-1}$  are certain explicit correspondences (this is [19, Equation (13)]). Then one checks that the  $\pi_{\widetilde{M}}^*$  form an MCK decomposition.

## 4. Main result

THEOREM 4.1

Let  $X$  be a smooth fourfold of one of the following types:

- a hypersurface of multidegree  $(2, 1, 1)$  in  $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ ;
- a hypersurface of multidegree  $(2, 1)$  in  $\text{Gr}(2, 4) \times \mathbb{P}^1$  (with respect to the Plücker embedding).

Then  $X$  has an MCK decomposition. Moreover, the Chern classes  $c_j(T_X)$  are in  $A_{(0)}^*(X)$ .

*Proof.* The argument relies on the alternative description of the general  $X$  given by Lemma 2.2.

STEP 1: We restrict to  $X$  sufficiently general, in the sense that  $X$  is a blow-up as in Lemma 2.2 with *smooth* centre  $S$ .

To construct an MCK decomposition for  $X$ , we apply the general Proposition 3.7, with  $M$  being either  $\mathbb{P}^3 \times \mathbb{P}^1$  or  $\text{Gr}(2, 4)$ , and  $N$  being the K3 surface  $S \subset M$  determined by Lemma 2.2. All we need to do is to check that the assumptions of Proposition 3.7 are met with.

Assumption 1° is verified, since both varieties with trivial Chow groups  $M$  and K3 surfaces  $S$  have an MCK decomposition. For  $M$  there is no choice involved ( $M$  has a unique CK decomposition which is MCK). For  $S$ , we choose

$$\pi_S^0 := \mathfrak{o}_S \times S, \quad \pi_S^4 := S \times \mathfrak{o}_S, \quad \pi_S^2 := \Delta_S - \pi_S^0 - \pi_S^4 \in A^2(S \times S), \quad (2)$$

where  $\mathfrak{o}_S \in A^2(S)$  is the distinguished zero-cycle of [2]. This is an MCK decomposition for  $S$  [18, Example 8.17].

Assumption 4° is trivially satisfied: one has  $A_{hom}^*(M) = 0$  and so (because  $\pi_M^j$  acts as zero on  $H^{2i}(M)$  for  $j \neq 2i$ ) one has  $A^*(M) = A_{(0)}^*(M)$ .

To check assumptions 2° and 3°, we consider things family-wise. That is, we write

$$\bar{B} := \mathbb{P}H^0(M, \mathcal{L}^{\oplus 2}),$$

where the line bundle  $\mathcal{L}$  is either  $\mathcal{O}_M(2, 1)$  (in case  $M = \mathbb{P}^3 \times \mathbb{P}^1$ ) or  $\mathcal{O}_M(2)$  (in case  $M = \text{Gr}(2, 4)$ ), and we consider the universal complete intersection

$$\bar{\mathcal{S}} \rightarrow \bar{B}.$$

We write  $B_0 \subset \bar{B}$  for the Zariski open parametrizing smooth dimensionally transversal intersections, and  $\mathcal{S} \rightarrow B_0$  for the base change (so the fibres  $S_b$  of  $\mathcal{S} \rightarrow B_0$  are exactly the K3 surfaces that are the centres of the blow-up occurring in Lemma 2.2). We now make the following claim.

CLAIM 4.2

Let  $\Gamma \in A^i(\mathcal{S})$  be such that

$$\Gamma|_{S_b} = 0 \quad \text{in } H^{2i}(S_b) \text{ for all } b \in B_0.$$

Then also

$$\Gamma|_{S_b} = 0 \quad \text{in } A^i(S_b) \text{ for all } b \in B_0.$$

We argue that the claim implies that assumptions 2° and 3° of Proposition 3.7 are met with (and thus Proposition 3.7 can be applied to prove Theorem 4.1). Indeed, let  $p_j: \mathcal{S} \times_{B_0} \mathcal{S} \rightarrow \mathcal{S}$ ,  $j = 1, 2$ , denote the two projections. We observe that

$$\begin{aligned}\pi_{\mathcal{S}}^0 &:= \frac{1}{24} (p_1)^* c_2(T_{\mathcal{S}/B_0}) \in A^4(\mathcal{S} \times_{B_0} \mathcal{S}), \\ \pi_{\mathcal{S}}^4 &:= \frac{1}{24} (p_2)^* c_2(T_{\mathcal{S}/B_0}) \in A^4(\mathcal{S} \times_{B_0} \mathcal{S}), \\ \pi_{\mathcal{S}}^2 &:= \Delta_{\mathcal{S}} - \pi_{\mathcal{S}}^0 - \pi_{\mathcal{S}}^4 \in A^4(\mathcal{S} \times_{B_0} \mathcal{S})\end{aligned}$$

defines a “relative MCK decomposition”, in the sense that for any  $b \in B_0$ , the restriction  $\pi_{\mathcal{S}}^i|_{S_b \times S_b}$  defines an MCK decomposition for  $S_b$  which agrees with (2).

Let us now check that assumption 2° is satisfied. Since  $A^1(S_b) = A_{(0)}^1(S_b)$ , we only need to consider  $c_2$  of the normal bundle. That is, we need to check that for any  $b \in B_0$  there is vanishing

$$(\pi_{S_b}^2)_* c_2(\mathcal{N}_{S_b/M}) \stackrel{??}{=} 0 \quad \text{in } A^2(S_b). \quad (3)$$

But we can write

$$(\pi_{S_b}^2)_* c_2(\mathcal{N}_{S_b/M}) = ((\pi_{\mathcal{S}}^2)_* c_2(\mathcal{N}_{\mathcal{S}/(M \times B_0)}))|_{S_b} \quad \text{in } A^2(S_b)$$

(for the formalism of relative correspondences, cf. [15, Chapter 8]), and besides we know that  $(\pi_{S_b}^2)_* c_2(\mathcal{N}_{S_b/M})$  is homologically trivial ( $\pi_{S_b}^2$  acts as zero on  $H^4(S_b)$ ). Thus, Claim (4.2) implies the necessary vanishing (3).

Assumption 3° is checked similarly. Let  $\iota_b: S_b \rightarrow M$  and  $\iota: \mathcal{S} \rightarrow M \times B$  denote the inclusion morphisms. To check assumption 3°, we need to convince ourselves of the vanishing

$$(\pi_{S_b \times M}^\ell)_*(\Gamma_{\iota_b}) \stackrel{??}{=} 0 \quad \text{in } A^4(S_b \times M) \quad \text{for all } \ell \neq 8 \text{ and for all } b \in B_0. \quad (4)$$

Since  $\Gamma_{\iota_b} \in A^4(S_b \times M)$ , one knows that  $(\pi_{S_b \times M}^\ell)_*(\Gamma_{\iota_b})$  is homologically trivial for any  $\ell \neq 8$ . Furthermore, we can write the cycle we are interested in as the restriction of a universal cycle

$$(\pi_{S_b \times M}^\ell)_*(\Gamma_{\iota_b}) = \left( \left( \sum_{j+k=\ell} \pi_{\mathcal{S}}^j \times \pi_M^k \right)_*(\Gamma_{\iota}) \right)|_{S_b \times M} \quad \text{in } A^4(S_b \times M).$$

For any  $b \in B_0$ , there is a commutative diagram

$$\begin{array}{ccc} A^4(\mathcal{S} \times M) & \rightarrow & A^4(S_b \times M) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus A^*(\mathcal{S}) & \rightarrow & \bigoplus A^*(S_b) \end{array}$$

where horizontal arrows are restriction to a fibre, and where vertical arrows are isomorphisms because  $M$  has trivial Chow groups. Claim 4.2 applied to the lower horizontal arrow shows the vanishing (4), and so assumption 3° holds.

It is only left to prove the claim. Since  $A_{hom}^i(S_b) = 0$  for  $i \leq 1$ , the only non-trivial case is  $i = 2$ . Given  $\Gamma \in A^2(\mathcal{S})$  as in the claim, let  $\bar{\Gamma} \in A^2(\bar{\mathcal{S}})$  be a cycle restricting to  $\Gamma$ . We consider the two projections

$$\begin{array}{ccc} \bar{\mathcal{S}} & \xrightarrow{\pi} & M \\ \downarrow \phi & & \\ \bar{B} & & \end{array}$$

Since any point of  $M$  imposes exactly one condition on  $\bar{B}$ , the morphism  $\pi$  has the structure of a projective bundle. As such, any  $\bar{\Gamma} \in A^2(\bar{\mathcal{S}})$  can be written

$$\bar{\Gamma} = \sum_{\ell=0}^2 \pi^*(a_\ell) \cdot \xi^\ell \quad \text{in } A^2(\bar{\mathcal{S}}),$$

where  $a_\ell \in A^{2-\ell}(M)$  and  $\xi \in A^1(\bar{\mathcal{S}})$  is the relative hyperplane class.

Let  $h := c_1(\mathcal{O}_{\bar{B}}(1)) \in A^1(\bar{B})$ . There is a relation

$$\phi^*(h) = \alpha \xi + \pi^*(h_1) \quad \text{in } A^1(\bar{\mathcal{S}}),$$

where  $\alpha \in \mathbb{Q}$  and  $h_1 \in A^1(M)$ . As in [16, Proof of Lemma 1.1], one checks that  $\alpha \neq 0$  (if  $\alpha$  were 0, we would have  $\phi^*(h^{\dim \bar{B}}) = \pi^*(h_1^{\dim \bar{B}})$ , which is absurd since  $\dim \bar{B} > 4$  and so the right-hand side must be 0). Hence, there is a relation

$$\xi = \frac{1}{\alpha}(\phi^*(h) - \pi^*(h_1)) \quad \text{in } A^1(\bar{\mathcal{S}}).$$

For any  $b \in B_0$ , the restriction of  $\phi^*(h)$  to the fibre  $S_b$  vanishes, and so it follows that

$$\bar{\Gamma}|_{S_b} = a'_0|_{S_b} \quad \text{in } A^2(S_b)$$

for some  $a'_0 \in A^2(M)$ . But  $A^2(M)$  is generated by intersections of divisors in case  $M = \mathbb{P}^3 \times \mathbb{P}^1$ , and  $A^2(M)$  is generated by divisors and  $c_2$  of the tautological bundle in case  $M = \text{Gr}(2, 4)$ . In both cases, it follows that

$$\bar{\Gamma}|_{S_b} = a'_0|_{S_b} \in \mathbb{Q}[\mathfrak{o}_{S_b}] \subset A^2(S_b),$$

(in the second case, this is proven as in [16, Proposition 2.1]). Given  $\Gamma \in A^2(\mathcal{S})$  a cycle such that the fibrewise restriction has degree zero, this shows that the fibrewise restriction is zero in  $A^2(S_b)$ . Claim 4.2 is proven.

STEP 2: It remains to extend to *all* smooth hypersurfaces as in the theorem. That is, let  $B \subset \bar{B}$  be the open such that the Fano fourfold  $X_b$  (which is the blow-up of  $M$  with centre  $S_b$ ) is smooth. One has  $B \supset B_0$ . Let  $\mathcal{X} \rightarrow B$  and  $\mathcal{X}^0 \rightarrow B_0$  denote the universal families of Fano fourfolds over  $B$  resp.  $B_0$ .



From step 1, one knows that  $X_b$  has an MCK decomposition for any  $b \in B_0$ . A closer look at the proof reveals more: the family  $\mathcal{X}^0 \rightarrow B_0$  has a “universal MCK decomposition”, in the sense that there exist relative correspondences  $\pi_{\mathcal{X}^0}^* \in A^4(\mathcal{X}^0 \times_{B_0} \mathcal{X}^0)$  such that for each  $b \in B_0$  the restriction

$$\pi_{X_b}^* := \pi_{\mathcal{X}^0}^*|_b \in A^4(X_b \times X_b)$$

forms an MCK decomposition for  $X_b$ . (To see this, one observes that Proposition 2.2 is “universal”: given families  $\mathcal{M} \rightarrow B$ ,  $\mathcal{N} \rightarrow B$  and universal MCK decompositions  $\pi_{\mathcal{M}}^*$ ,  $\pi_{\mathcal{N}}^*$ , the result of (1) is a universal MCK decomposition for  $\widetilde{\mathcal{M}} \rightarrow B$ .)

A standard argument now allows to spread out the MCK property from  $B_0$  to the larger base  $B$ . That is, we define

$$\pi_{\mathcal{X}}^j := \bar{\pi}_{\mathcal{X}^0}^j \in A^4(\mathcal{X} \times_B \mathcal{X}),$$

where  $\bar{\pi}$  refers to the closure of a representative of  $\pi$ . The “spread lemma” [22, Lemma 3.2] (applied to  $\mathcal{X} \times_B \mathcal{X}$ ) gives that the  $\pi_{\mathcal{X}}^*$  are a fibrewise CK decomposition, and the same spread lemma (applied to  $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ ) gives that the  $\pi_{\mathcal{X}}^*$  are a fibrewise MCK decomposition. This ends step 2.

REMARK 4.3

Claim 4.2 states that the families  $\mathcal{S} \rightarrow B_0$  verify the “Franchetta property” as studied in [6]. It is worth mentioning that the Franchetta property for the universal K3 surface of genus  $g \leq 10$  (and for some other values of  $g$ ) was already proven in [16]; the families considered in Claim 4.2 are different, however, so Claim 4.2 is not covered by [16] (e.g. in case  $M = \mathbb{P}^3 \times \mathbb{P}^1$  the K3 surfaces of Claim 4.2 have Picard number at least 2, so they correspond to a Noether–Lefschetz divisor in  $\mathcal{F}_7$ ).

As a corollary of Claim 4.2, the universal families  $\mathcal{X} \rightarrow B$  of Fano fourfolds of type B1 or B2 also verify the Franchetta property. (Indeed, in view of [22, Lemma 3.2] it suffices to prove this for  $\mathcal{X}^0 \rightarrow B_0$ . In view of Lemma 2.2,  $\mathcal{X}^0$  can be constructed as the blow-up of  $M \times B_0$  with centre  $\mathcal{S}$ . This blow-up yields a relative correspondence from  $\mathcal{X}^0$  to  $\mathcal{S}$ , inducing a commutative diagram

$$\begin{array}{ccc} A^j(\mathcal{X}^0) & \rightarrow & A^{j-1}(\mathcal{S}) \oplus \bigoplus \mathbb{Q} \\ \downarrow & & \downarrow \\ A^j(X_b) & \rightarrow & A^{j-1}(S_b) \oplus \bigoplus \mathbb{Q} \end{array}$$

where horizontal arrows are injective (by the blow-up formula). The Franchetta property for  $\mathcal{S} \rightarrow B_0$  thus implies the Franchetta property for  $\mathcal{X}^0 \rightarrow B_0$ .)

REMARK 4.4

One would expect that for Fano varieties of K3 type, there is a *unique* MCK decomposition. I cannot prove this unicity for the Fano fourfolds of Theorem 4.1. (At least, one may observe that for  $X$  as in Theorem 4.1 the induced splitting of the Chow ring is canonical, since  $A^j(X) = A_{(0)}^j(X)$  for  $j \neq 3$ , whereas  $A^3(X) = A_{(0)}^3(X) \oplus A_{(2)}^3(X)$  with  $A_{(0)}^3 = A^2(X) \cdot A^1(X)$  and  $A_{(2)}^3(X) = A_{hom}^3(X)$ ).

## 5. Some consequences

### 5.1. An injectivity result

COROLLARY 5.1

Let  $X$  and  $M$  be as in Theorem 4.1, and let  $m \in \mathbb{N}$ . Let  $R^*(X^m) \subset A^*(X^m)$  be the  $\mathbb{Q}$ -subalgebra

$$R^*(X^m) := \langle (p_i)^* A^1(X), (p_i)^* A^2(X), (p_{ij})^*(\Delta_X), (p_i)^* c_j(T_X), \\ (p_i)^* \text{Im}(A^i(M) \rightarrow A^i(X)) \rangle \subset A^*(X^m).$$

(Here  $p_i: X^m \rightarrow X$  and  $p_{ij}: X^m \rightarrow X^2$  denote a projection to the  $i$ th factor, resp. to the  $i$ th and  $j$ th factor.)

The cycle class map induces injections  $R^j(X^m) \hookrightarrow H^{2j}(X^m)$  in the following cases:

1.  $m = 1$  and  $j$  arbitrary;
2.  $m = 2$  and  $j \geq 5$ ;
3.  $m = 3$  and  $j \geq 9$ .

*Proof.* Theorem 4.1, in combination with Proposition 3.6, ensures that  $X^m$  has an MCK decomposition, and so  $A^*(X^m)$  has the structure of a bigraded ring under the intersection product. The corollary is now implied by the conjunction of the two following claims.

CLAIM 5.2

There is inclusion

$$R^*(X^m) \subset A_{(0)}^*(X^m).$$

CLAIM 5.3

The cycle class map induces injections

$$A_{(0)}^j(X^m) \hookrightarrow H^{2j}(X^m)$$

provided  $m = 1$ , or  $m = 2$  and  $j \geq 5$ , or  $m = 3$  and  $j \geq 9$ .

To prove Claim 5.2, we note that  $A_{hom}^k(X) = 0$  for  $k \neq 3$ , which readily implies the equality  $A^k(X) = A_{(0)}^k(X)$  for  $k \neq 3$ . The fact that  $c_3(T_X)$  is in  $A_{(0)}^3(X)$  is part of Theorem 4.1. The fact that  $\Delta_X \in A_{(0)}^4(X \times X)$  is a general fact for any  $X$  with a (necessarily self-dual) MCK decomposition [19, Lemma 1.4]. It remains to prove that codimension three cycles coming from the ambient space  $M$  are in  $A_{(0)}^3(X)$ . To this end, we observe that such cycles are universally defined, i.e.

$$\text{Im}(A^3(M) \rightarrow A^3(X)) \subset \text{Im}(A^3(\mathcal{X}) \rightarrow A^3(X)),$$

where  $\mathcal{X} \rightarrow B$  is the universal family as before, and  $X = X_{b_0}$  for some  $b_0 \in B$ . Given  $a \in A^3(\mathcal{X})$ , applying the Franchetta property (Remark 4.3) to

$$\Gamma := (\pi_{\mathcal{X}}^j)_*(a) \in A^3(\mathcal{X}), \quad j \neq 6,$$

one finds that the restriction  $a|_X \in A^3(X)$  lives in  $A^3_{(0)}(X)$ . In particular, it follows that

$$\mathrm{Im}(A^3(M) \rightarrow A^3(X)) \subset A^3_{(0)}(X),$$

as desired. Since the projections  $p_i$  and  $p_{ij}$  are pure of grade 0 [19, Corollary 1.6], and  $A^*_{(0)}(X^m)$  is a ring under the intersection product, this proves Claim 5.2.

To prove Claim 5.3, we observe that Manin's blow-up formula [17, Theorem 2.8] gives an isomorphism of motives

$$h(X) \cong h(S)(1) \oplus \bigoplus \mathbf{1}(\ast) \quad \text{in } \mathcal{M}_{\mathrm{rat}}.$$

Moreover, in view of Proposition 3.7 (cf. also [19, Proposition 2.4]), the correspondence inducing this isomorphism is of pure grade 0. In particular, for any  $m \in \mathbb{N}$  we have isomorphisms of Chow groups

$$A^j(X^m) \cong A^{j-m}(S^m) \oplus \bigoplus_{k=0}^4 A^{j-m+1-k}(S^{m-1})^{b_k} \oplus \bigoplus A^*(S^{m-2}) \oplus \bigoplus_{\ell \geq 3} A^*(S^{m-\ell}),$$

and this isomorphism respects the  $A^*_{(0)}()$  parts. Claim 5.3 now follows from the fact that for any surface  $S$  with an MCK decomposition, and any  $m \in \mathbb{N}$ , the cycle class map induces injections

$$A^i_{(0)}(S^m) \hookrightarrow H^{2i}(S^m) \quad \text{for all } i \geq 2m - 1$$

(this is noted in [20, Introduction], cf. also [9, Proof of Lemma 2.20]).

## 5.2. Decomposition in the derived category

Given a smooth projective morphism  $\pi: \mathcal{X} \rightarrow B$ , Deligne [3] has proven a decomposition in the derived category of sheaves of  $\mathbb{Q}$ -vector spaces on  $B$ :

$$R\pi_*\mathbb{Q} \cong \bigoplus_i R^i\pi_*\mathbb{Q}[-i]. \quad (5)$$

As explained in [21], for both sides of this isomorphism there is a cup-product: on the right-hand side, this is the direct sum of the usual cup-products of local systems, while on the left-hand side, this is the derived cup-product (inducing the usual cup-product in cohomology). In general, the isomorphism (5) is *not* compatible with these cup-products, even after shrinking the base  $B$  (cf. [21]). In some rare cases, however, there is such a compatibility (after shrinking): this is the case for families of abelian varieties [4], and for families of K3 surfaces [21], [22, Section 5.3] (cf. also [20, Theorem 4.3] and [7, Corollary 8.4] for some further cases).

Given the close link to K3 surfaces, it is not surprising that the Fano fourfolds of Theorem 4.1 also have such a multiplicative decomposition.

### COROLLARY 5.4

*Let  $\mathcal{X} \rightarrow B$  be a family of Fano fourfolds of type B1 or B2. There is a non-empty Zariski open  $B' \subset B$ , such that the isomorphism (5) becomes multiplicative after shrinking to  $B'$ .*

*Proof.* This is a formal consequence of the existence of a relative MCK decomposition, cf. [20, Proof of Theorem 4.2] and [7, Section 8].

Given a family  $\mathcal{X} \rightarrow B$  and  $m \in \mathbb{N}$ , let us write  $\mathcal{X}^{m/B}$  for the  $m$ -fold fibre product

$$\mathcal{X}^{m/B} := \mathcal{X} \times_B \mathcal{X} \times_B \cdots \times_B \mathcal{X}.$$

Corollary 5.4 has the following concrete consequence, which is similar to a result for families of K3 surfaces obtained by Voisin [21, Proposition 0.9]:

**COROLLARY 5.5**

*Let  $\mathcal{X} \rightarrow B$  be a family of Fano fourfolds of type B1 or B2. Let  $z \in A^r(\mathcal{X}^{m/B})$  be a polynomial in (pullbacks of) divisors and codimension 2 cycles on  $\mathcal{X}$ . Assume the fibrewise restriction  $z|_b$  is homologically trivial, for some  $b \in B$ . Then there exists a non-empty Zariski open  $B' \subset B$  such that*

$$z = 0 \quad \text{in } H^{2r}((\mathcal{X}')^{m/B'}, \mathbb{Q}).$$

*Proof.* The argument is the same as [21, Proposition 0.9]. First, one observes that divisors  $d_i$  and codimension 2 cycles  $e_j$  on  $\mathcal{X}$  admit a cohomological decomposition (with respect to the Leray spectral sequence)

$$\begin{aligned} d_i &= d_{i0} + \pi^*(d_{i2}) \quad \text{in } H^0(B, R^2\pi_*\mathbb{Q}) \oplus \pi^*H^2(B, \mathbb{Q}) \cong H^2(\mathcal{X}, \mathbb{Q}), \\ e_j &= e_{j0} + \pi^*(e_{j2}) + \pi^*(e_{j4}) \quad \text{in } H^0(B, R^4\pi_*\mathbb{Q}) \oplus \pi^*H^2(B)^{\oplus 2} \\ &\quad \oplus \pi^*H^4(B) \cong H^4(\mathcal{X}, \mathbb{Q}). \end{aligned}$$

We claim that the cohomology classes  $d_{ik}$  and  $e_{jk}$  are *algebraic*. This claim implies the corollary: indeed, given a polynomial  $z = p(d_i, e_j)$ , one may take  $B'$  to be the complement of the support of the cycles  $d_{i2}$ ,  $e_{j2}$  and  $e_{j4}$ . Then over the restricted base one has equality

$$z := p(d_i, e_j) = p(d_{i0}, e_{j0}) \quad \text{in } H^{2r}((\mathcal{X}')^{m/B'}, \mathbb{Q}).$$

Multiplicativity of the decomposition ensures that (after shrinking the base some more)

$$p(d_{i0}, e_{j0}) \in H^0(B', R^{2r}(\pi^m)_*\mathbb{Q}) \subset H^{2r}((\mathcal{X}')^{m/B'}, \mathbb{Q}),$$

and so the conclusion follows.

The claim is proven for divisor classes  $d_i$  in [21, Lemma 1.4]. For codimension 2 classes  $e_j$ , the argument is similar to loc. cit.: let  $h \in H^2(\mathcal{X})$  be an ample divisor class, and let  $h_0$  be the part that lives in  $H^0(B, R^2\pi_*\mathbb{Q})$ . One has

$$e_j(h_0)^4 = e_{j0}(h_0)^4 + \pi^*(e_{j2})(h_0)^4 + \pi^*(e_{j4})(h_0)^4 \quad \text{in } H^{12}(\mathcal{X}, \mathbb{Q}).$$

By multiplicativity, after some shrinking of the base the first two summands are contained in  $H^0(B', R^{12}\pi_*\mathbb{Q})$ , resp. in  $H^2(B', R^{10}\pi_*\mathbb{Q})$ , hence they are zero as  $\pi$  has 4-dimensional fibres. The above equality thus simplifies to

$$e_j(h_0)^4 = \pi^*(e_{j4})(h_0)^4 \quad \text{in } H^{12}(\mathcal{X}, \mathbb{Q}).$$

Pushing forward to  $B'$ , one obtains

$$\pi_*(e_j(h_0)^4) = \pi_*((h_0)^4)e_{j4} = \lambda e_{j4} \quad \text{in } H^4(B'),$$

for some  $\lambda \in \mathbb{Q}^*$ . As the left-hand side is algebraic, so is  $e_{j4}$ .

Next, one considers

$$e_j(h_0)^3 = e_{j0}(h_0)^3 + \pi^*(e_{j2})(h_0)^3 + \pi^*(e_{j4})(h_0)^3 \quad \text{in } H^{10}(\mathcal{X}, \mathbb{Q}).$$

The first summand is again zero for dimension reasons, and so

$$\pi^*(e_{j2})(h_0)^3 = e_j(h_0)^3 - \pi^*(e_{j4})(h_0)^3 \in H^{10}(\mathcal{X}, \mathbb{Q})$$

is algebraic. A fortiori,  $\pi^*(e_{j2})(h_0)^4$  is algebraic, and so

$$\pi_*(\pi^*(e_{j2})(h_0)^4) = \pi_*((h_0)^4)e_{j2} = \mu e_{j2} \quad \text{in } H^2(B'), \quad \mu \in \mathbb{Q}^*$$

is algebraic.

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## References

- [1] Beauville, Arnaud. "On the splitting of the Bloch-Beilinson filtration. In *Algebraic cycles and motives. Vol. 2.*", Vol. 344 of *London Math. Soc. Lecture Note Ser.* Cambridge: Cambridge Univ. Press, 2007. Cited on 43.
- [2] Beauville, Arnaud, and Claire Voisin. "On the Chow ring of a K3 surface." *J. Algebraic Geom.* 13, no. 3 (2004): 417-426. Cited on 40 and 44.
- [3] Deligne, Pierre. "Théorème de Lefschetz et critères de dégénérescence de suites spectrales." *Inst. Hautes Études Sci. Publ. Math.* 35 (1968): 259-278. Cited on 49.
- [4] Deninger, Christopher, and Jacob Murre. "Motivic decomposition of abelian schemes and the Fourier transform." *J. Reine Angew. Math.* 422 (1991): 201-219. Cited on 49.
- [5] Fatighenti, Enrico, and Giovanni Mongardi. "Fano varieties of K3 type and IHS manifolds." *arXiv* (2019): 1904.05679. Cited on 39, 40, 41 and 42.
- [6] Fu, Lie, Robert Laterveer, and Charles Vial. "The generalized Franchetta conjecture for some hyper-Kähler varieties." *J. Math. Pures Appl. (9)* 130 (2019): 1-35. Cited on 47.
- [7] Fu, Lie, Zhiyu Tian, and Charles Vial. "Motivic hyper-Kähler resolution conjecture, I: generalized Kummer varieties." *Geom. Topol.* 23, no. 1 (2019): 427-492. Cited on 43, 49 and 50.
- [8] Fu, Lie, and Charles Vial. "Distinguished cycles on varieties with motive of abelian type and the Section Property", *arXiv* (2017): 1709.05644v2. Cited on 43.
- [9] Laterveer, Robert. "Algebraic cycles on some special hyperkähler varieties." *Rend. Mat. Appl. (7)* 38, no. 2 (2017): 243-276. Cited on 49.

- [10] Laterveer, Robert. "A remark on the Chow ring of K uchle fourfolds of type  $d3$ ." *Bull. Aust. Math. Soc.* 100, no. 3 (2019): 410-418. Cited on 39.
- [11] Laterveer, Robert. "Algebraic cycles and Verra fourfolds." *Tohoku Math. J.* (to appear) Cited on 39.
- [12] Laterveer, Robert. "On the Chow ring of Fano varieties of type  $S2$ ." preprint. Cited on 39.
- [13] Laterveer, Robert and Charles Vial. "On the Chow ring of Cynk–Hulek Calabi–Yau varieties and Schreieder varieties." *Canadian Journal of Math.* DOI: 10.4153/S0008414X19000191. Cited on 43.
- [14] Murre, Jacob P. "On a conjectural filtration on the Chow groups of an algebraic variety I and II." *Indag. Math. (N.S.)* 4, no. 2 (1993): 177-201. Cited on 42.
- [15] Murre, Jacob P., Jan Nagel, and Chris A.M. Peters. *Lectures on the theory of pure motives*. Providence, RI: American Mathematical Society, 2013. Cited on 40, 42 and 45.
- [16] Pavic, Nebojsa, Junliang Shen, and Qizheng Yin. "On O’Grady’s generalized Franchetta conjecture." *Int. Math. Res. Not. IMRN* 2017, no. 16 (2017): 4971-4983. Cited on 46 and 47.
- [17] Scholl, Anthony J. "Classical motives." In Vol. 55 of *Proc. Sympos. Pure Math.*, 163–187. Providence, RI: Amer. Math. Soc., 1994. Cited on 40 and 49.
- [18] Shen, Mingmin, and Charles Vial. "The Fourier transform for certain hyperk ahler fourfolds." *Mem. Amer. Math. Soc.* 240 (2016): no. 1139. Cited on 39, 42, 43 and 44.
- [19] Shen, Mingmin, and Charles Vial. "The motive of the Hilbert cube  $X^{[3]}$ ." *Forum Math. Sigma* 4 (2016): e30. Cited on 43, 48 and 49.
- [20] Vial, Charles. "On the motive of some hyperK ahler varieties." *J. Reine Angew. Math.* 725 (2017): 235-247. Cited on 43, 49 and 50.
- [21] Voisin, Claire. "Chow rings and decomposition theorems for families of  $K3$  surfaces and Calabi-Yau hypersurfaces." *Geom. Topol.* 16, no. 1 (2012): 433-473. Cited on 49 and 50.
- [22] Voisin, Claire. *Chow rings, decomposition of the diagonal, and the topology of families*. Vol. 187 of *Annals of Mathematics Studies*. Princeton, NJ: Princeton University Press, 2014. Cited on 47 and 49.

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