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Second Hukuhara derivative and cosine family of linear set-valued functions

Abstract. Let K be a closed convex cone with the nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F_t : t \geq 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t: K \longrightarrow cc(K)$, $x \in F_t(x)$ for $t \geq 0$, $x \in K$ and $F_t \circ F_s = F_s \circ F_t$ for $s, t \geq 0$, then

$$D^2 F_t(x) = F_t(H(x))$$

for $x \in K$ and $t \geq 0$, where $D^2 F_t(x)$ denotes the second Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

Let X be a vector space. Through this paper all vector spaces are supposed to be real. We introduce the notations

$$A + B := \{a + b : a \in A, b \in B\},$$
$$\lambda A := \{\lambda a : a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$.

A subset K of X is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

Let X and Y be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F: K \longrightarrow n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *additive* if

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in K$. If additionally F satisfies

$$F(\lambda x) = \lambda F(x)$$

for all $x \in K$ and $\lambda \geq 0$, then F is called *linear*.

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A set-valued function $F\colon [0,+\infty) \longrightarrow n(Y)$ is said to be *concave* if

$$F(\lambda t + (1-\lambda)s) \subset \lambda F(t) + (1-\lambda)F(s)$$

for all $s,t \in [0,+\infty)$ and $\lambda \in (0,1)$.

From now on we assume that X is a normed vector space, $c(X)$ denotes the family of all compact members of $n(X)$ and $cc(X)$ stands for the family of all convex sets of $c(X)$.

Let A, B, C be sets of $cc(X)$. We say that the set C is the *Hukuhara difference* of A and B when $C = A - B$ if $B + C = A$. By Rådström Cancellation Lemma [7] it follows that if this difference exists, then it is unique.

Let A, A_1, $A_2, \dots$ be elements of the space $cc(X)$. We say that the *sequence* $(A_n)_{n\in\mathbb{N}}$ *is convergent to* A and we write $A_n \to A$ if $d(A, A_n) \to 0$, where d denotes the Hausdorff metric induced by the norm in X.

Lemma 1
Let X be a Banach space, $A, A_1, A_2, \dots, B, B_1, B_2, \dots \in cc(X)$. If $A_n \to A$, $B_n \to B$ and there exist the Hukuhara differences $A_n - B_n$ in $cc(X)$ for $n \in \mathbb{N}$, then there exists the Hukuhara difference $A - B$ and $A_n - B_n \to A - B$.

Proof. Let $C_n = A_n - B_n$ for $n \in \mathbb{N}$. By the definition of the Hukuhara difference $A_n = B_n + C_n$ for $n \in \mathbb{N}$. By properties of the Hausdorff metric for $m, n \in \mathbb{N}$ we have

$$\begin{aligned} d(C_m, C_n) &= d(B_n + B_m + C_m, B_m + B_n + C_n) \\ &= d(B_n + A_m, B_m + A_n) \\ &\le d(B_n, B_m) + d(A_m, A_n). \end{aligned}$$

Sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are Cauchy sequences thus by the last inequality $(C_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, too. By the completness of $cc(X)$ (see Theorem II.3 in [1]) there exists $C \in cc(X)$ such that $C_n \to C$. Moreover, $B_n + C_n \to B + C$ since

$$\begin{aligned} d(B_n + C_n, B + C) &\le d(B_n + C_n, B_n + C) + d(B_n + C, B + C) \\ &= d(C_n, C) + d(B_n, B). \end{aligned}$$

On the other hand $A_n \to A$ and $A_n = B_n + C_n$ so $A = B + C$, i.e., there exists the Hukuhara difference $A - B = C$.

Let $F, G\colon K \longrightarrow cc(K)$. We can define the multifunctions $F + G$ and $F - G$ on K as follows

$$(F+G)(x) := F(x) + G(x) \qquad \text{for } x \in K$$

and

$$(F-G)(x) := F(x) - G(x)$$

if the Hukuhara differences $F(x) - G(x)$ exist for all $x \in K$.

Lemma 2
For each set $A \subset K$ *the inclusion*

$$(F+G)(A) \subset F(A) + G(A) \tag{1}$$

holds. Moreover, if there exist the Hukuhara difference $F(A) - G(A)$ *and the multifunction* $F - G$*, then*

$$F(A) - G(A) \subset (F-G)(A). \tag{2}$$

Proof. Inclusion (1) is obvious. To prove (2) we observe that $(F-G)+G = F$. Hence by (1) we obtain

$$F(A) \subset (F-G)(A) + G(A). \tag{3}$$

Since $F(A) = G(A) + (F(A) - G(A))$, (3) and Rådström Cancellation Lemma yield inclusion (2).

Lemma 3 (Lemma 3 in [8])
Let X *and* Y *be two normed vector spaces and let* K *be a closed convex cone in* X*. Assume that* $F\colon K \longrightarrow cc(K)$ *is continuous additive set-valued function and* $A, B \in cc(K)$*. If there exists the difference* $A-B$*, then there exists* $F(A)-F(B)$ *and* $F(A) - F(B) = F(A-B)$*.*

Lemma 4 (Lemma 3 in [5])
If $(A_n)_{n\in\mathbb{N}}$ *is a sequence of elements of the set* $c(X)$ *such that* $A_{n+1} \subset A_n$ *for* $n \in \mathbb{N}$*, then*

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n\,.$$

Lemma 5 (Lemma 3 in [9])
Let K *be a closed convex cone such that* $\operatorname{int} K \neq \emptyset$ *in a Banach space* X *and let* Y *be a normed space. If* $(F_n)_{n\in\mathbb{N}}$ *is a sequence of continuous additive set-valued functions* $F_n\colon K \longrightarrow cc(Y)$ *such that* $F_{n+1}(x) \subset F_n(x)$ *for all* $x \in K$ *and* $n \in \mathbb{N}$*, then the formula*

$$F_0(x) := \bigcap_{n=1}^{\infty} F_n(x), \qquad x \in K,$$

defines a continuous additive set-valued function $F_0\colon K \longrightarrow cc(Y)$*. Moreover,*

$$\lim_{n\to\infty} F_n(x) = F_0(x), \qquad x \in K, \tag{4}$$

and the convergence in (4) *is uniform on every nonempty compact subset of* K*.*

Lemma 6 (Lemma 4 in [5])
If $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of $c(X)$ satisfying $A_n \subset A_{n+1} \subset B$ for $n \in \mathbb{N}$ and a compact set B, then

$$\lim_{n\to\infty} A_n = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Lemma 7
Let K be a closed convex cone such that $\operatorname{int} K \neq \emptyset$ in a Banach space X and let Y be a normed space. If $(F_n)_{n\in\mathbb{N}}$ is a sequence of continuous additive set-valued functions $F_n\colon K \longrightarrow cc(Y)$ such that

1) *$F_n(x) \subset F_{n+1}(x)$ for all $x \in K$ and $n \in \mathbb{N}$,*
2) *$F_n(x) \subset G(x)$ for all $x \in K$, $n \in \mathbb{N}$ and a set-valued function $G\colon K \longrightarrow c(Y)$,*

then the formula

$$F_0(x) := \operatorname{cl}\left(\bigcup_{n=1}^{\infty} F_n(x)\right), \qquad x \in K, \tag{5}$$

defines a continuous additive set-valued function $F_0\colon K \longrightarrow cc(Y)$. Moreover,

$$\lim_{n\to\infty} F_n(x) = F_0(x), \qquad x \in K, \tag{6}$$

and the convergence in (6) is uniform on every nonempty compact subset of K.

Proof. The sets $F_0(x)$ defined by the formula (5) are obviously closed and convex. Since $F_0(x) \subset G(x)$ and $G(x)$ are compact, they belong to $cc(Y)$ for every $x \in K$. Equality (6) holds according to Lemma 6. By Lemma 5.6 in [4] we have

$$F_0(x+y) = \lim_{n\to\infty} F_n(x+y) = \lim_{n\to\infty} \left(F_n(x) + F_n(y)\right) = F_0(x) + F_0(y)$$

for all $x, y \in K$. Thus the set-valued function F_0 is additive. Since $F_1(x) \subset F_0(x)$ for $x \in K$ and F_1 is continuous, the set-valued function F_0 is continuous on $\operatorname{int} K$ (see Theorem 5.2 in [4]). Fix $y \in \operatorname{int} K$ and $x_0 \in K$, then $\frac{x_0+y}{2} \in \operatorname{int} K$ (see Chapter V, §1, Lemma 8 in [3]). Let (x_n) be a sequence of elements of K convergent to x_0. Then

$$\begin{aligned} d\left(F_0(x_n), F_0(x_0)\right) &= d\left(F_0(x_n) + F_0(y), F_0(x_0) + F_0(y)\right) \\ &= 2d\left(F_0\left(\frac{x_n+y}{2}\right), F_0\left(\frac{x_0+y}{2}\right)\right). \end{aligned}$$

The continuity of F_0 at $\frac{x_0+y}{2}$ implies that

$$\lim_{n\to\infty} F_0(x_n) = F_0(x_0).$$

This means that F_0 is continuous on K. The sequence $(d(F_n(x), F_0(x)))$ $n \in \mathbb{N}$ is a decreasing sequence of continuous functions convergent to the zero function and according to Dini Theorem this function is the uniform limit of this sequence on every nonempty compact subset of K.

Let $F\colon [0, +\infty) \longrightarrow cc(X)$ be a set-valued function such that there exist the Hukuhara differences $F(t) - F(s)$ for $0 \le s \le t$. The *Hukuhara derivative* of F at $t > 0$ is defined by the formula

$$DF(t) = \lim_{h\to 0^+} \frac{F(t+h) - F(t)}{h} = \lim_{h\to 0^+} \frac{F(t) - F(t-h)}{h},$$

whenever both these limits exist (see [2]). Moreover,

$$DF(0) = \lim_{h\to 0^+} \frac{F(h) - F(0)}{h}.$$

Let $(K, +)$ be a semigroup. A one-parameter family $\{F_t : \ t \ge 0\}$ of set-valued functions $F_t\colon K \longrightarrow n(K)$ is said to be a *cosine family* if

$$F_0(x) = \{x\} \qquad \text{for } x \in K$$

and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) := 2\bigcup\{F_t(y) : \ y \in F_s(x)\} \tag{7}$$

for $x \in K$ and $0 \le s \le t$.

Let X be a normed space. A cosine family $\{F_t : \ t \ge 0\}$ is said to be *regular* if

$$\lim_{t\to 0^+} d(F_t(x), \{x\}) = 0.$$

Lemma 8

Let X be a Banach space and let K be a closed convex cone in X such that $\operatorname{int} K \ne \emptyset$. *Assume that $\{F_t : \ t \ge 0\}$ is a regular cosine family of continuous additive set-valued functions $F_t\colon K \longrightarrow cc(K)$ and $x \in F_t(x)$ for all $x \in K$ and $t \ge 0$. Then there exist the Hukuhara differences $F_t(x) - F_s(x)$ for all $0 \le s \le t$ and $x \in K$.*

Proof. We first prove, by induction on n, that there exist the Hukuhara differences

$$F_{ns}(x) - F_{(n-1)s}(x) \tag{8}$$

for all $s \ge 0$, $x \in K$, $n \in \mathbb{N}$.

For $n = 1$ it suffices to show that

$$F_s(x) - x \subset K$$

for $x \in K$ and $s \geq 0$. Let $x \in K$ and $s \geq 0$. Putting $t = s$ in (7) we have

$$F_{2s}(x) + x = 2F_s(F_s(x)). \tag{9}$$

Hence and by the assumption $x \in F_t(x)$ we get

$$F_s(x) \subset \frac{1}{2}F_{2s}(x) + \frac{1}{2}x.$$

Replacing s by $2s$ in the last inclusion we obtain

$$F_{2s}(x) \subset \frac{1}{2}F_{4s}(x) + \frac{1}{2}x,$$

whence

$$F_s(x) \subset \frac{1}{4}F_{4s}(x) + \frac{1}{4}x + \frac{1}{2}x.$$

By induction we can prove that

$$F_s(x) \subset \frac{1}{2^p}F_{2^p s}(x) + \frac{1}{2^p}x + \cdots + \frac{1}{2}x$$

for all $p \in \mathbb{N}$. Therefore

$$F_s(x) \subset K + \left(1 - 2^{-p}\right) x.$$

Let $y \in F_s(x)$. Then $y - (1 - 2^{-p})\, x \in K$ and letting $p \to \infty$ we have $y - x \in K$. Thus $F_s(x) - x \subset K$.

By (9) and by the additivity of F_s we obtain

$$F_{2s}(x) + x = 2F_s(F_s(x) - x) + 2F_s(x)$$

and

$$F_{2s}(x) - F_s(x) = 2F_s(F_s(x) - x) + F_s(x) - x.$$

Let $k \in \mathbb{N}$. Assuming (8) to hold for $n = k$, we will prove it for $n = k + 1$. Putting $t = ks$ in (7) we get

$$F_{(k+1)s}(x) + F_{(k-1)s}(x) = 2F_{ks}(F_s(x)),$$

whence and by the additivity of F_s

$$F_{(k+1)s}(x) + F_{(k-1)s}(x) = 2F_{ks}(F_s(x) - x) + 2F_{ks}(x).$$

By the induction assumption we obtain

$$F_{(k+1)s}(x) = 2F_{ks}(F_s(x) - x) + \big(F_{ks}(x) - F_{(k-1)s}(x)\big) + F_{ks}(x).$$

Thus

$$F_{(k+1)s}(x) - F_{ks}(x) = 2F_{ks}(F_s(x) - x) + \big(F_{ks}(x) - F_{(k-1)s}(x)\big)\,.$$

From this we see that there exist the Hukuhara differences

$$F_{ns}(x) - F_{ms}(x) \tag{10}$$

for all $m, n \in \mathbb{N}$, $m \le n$, $s \ge 0$. Suppose that $0 \le s \le t$. Replacing s by $\frac{t}{n}$ in (10) we can assert that there exist the Hukuhara differences

$$F_t(x) - F_{\frac{m}{n}t}(x).$$

There exists a sequence $a_n \in \mathbb{Q} \cap [0,1]$ such that $a_n t$ is convergent to s. By the continuity of $t \mapsto F_t(x)$ (Theorem 2 in [10]), $F_{a_n t}(x) \to F_s(x)$ and by Lemma 1, there exists the difference

$$F_t(x) - F_s(x) = \lim_{n\to\infty} \left(F_t(x) - F_{a_n t}(x)\right).$$

A cosine family $\{F_t :\ t \ge 0\}$ of set-valued functions $F_t\colon K \longrightarrow cc(K)$ is said to be *differentiable* if all set-valued functions $t \mapsto F_t(x)$, $x \in K$, have Hukuhara derivative on $[0, +\infty)$.

Lemma 9

Let X be a Banach space and let K be a closed convex cone in X such that $\operatorname{int} K \neq \emptyset$. Suppose that $\{F_t :\ t \ge 0\}$ is a regular cosine family of continuous additive set-valued functions $F_t\colon K \longrightarrow cc(K)$ and $x \in F_t(x)$ for all $x \in K$ and $t \ge 0$. Then multifunctions $t \mapsto F_t(x)$ $(x \in K)$ are concave, there exist set-valued functions $G_t^+\colon K \longrightarrow cc(K)$ and $G_t^-\colon K \longrightarrow cc(K)$ such that

$$G_t^+(x) = \lim_{h\to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h}, \qquad G_t^-(x) = \lim_{h\to 0^+} \frac{F_t(x) - F_{t-h}(x)}{h}$$

for all $t > 0$, $x \in K$ and the convergence is uniform on every nonempty compact subset of K. Moreover, G_t^+ and G_t^- are additive, continuous,

$$G_t^+(x) = \bigcap_{h>0} \frac{F_{t+h}(x) - F_t(x)}{h}, \qquad G_t^-(x) = \operatorname{cl}\left(\bigcup_{t\ge h>0} \frac{F_t(x) - F_{t-h}(x)}{h}\right)$$

and $G_t^-(x) \subset G_t^+(x)$ for $x \in K$.

Proof. Let us fix $x \in K$. We consider the mutlifunction $t \mapsto F_t(x)$ for $t \geq 0$. Setting $t = \frac{v+u}{2}$, $s = \frac{v-u}{2}$, $0 \leq u \leq v$ in (7) we get

$$F_v(x) + F_u(x) = 2F_{\frac{v+u}{2}}(F_{\frac{v-u}{2}}(x).$$

Since $x \in F_t(x)$ for all $t \geq 0$, we have

$$F_{\frac{v+u}{2}}(x) \subset \frac{F_v(x) + F_u(x)}{2}.$$

Hence, by the continuity (Theorem 2 in [10]) and by Theorem 4.1 in [4] the multifunction $t \mapsto F_t(x)$ is concave. Moreover, by Lemma 8 there exist the Hukuhara differences

$$F_{t+h}(x) - F_t(x), \qquad F_t(x) - F_{t-h}(x)$$

for all $0 \leq h \leq t$. Thus (Theorem 3.2 in [6]) there exist limits

$$G_t^+(x) = \lim_{h \to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h}, \quad G_t^-(x) = \lim_{h \to 0^+} \frac{F_t(x) - F_{t-h}(x)}{h} \tag{11}$$

for all $t > 0$. As $t \mapsto F_t(x)$ is concave we see that $h \mapsto \frac{F_{t+h}(x) - F_t(x)}{h}$ is increasing, $h \mapsto \frac{F_t(x) - F_{t-h}(x)}{h}$ is decreasing in $(0, t)$ and $\frac{F_t(x) - F_{t-h}(x)}{h} \subset G_t^+(x)$.

Lemmas 5 and 7 respectively imply that the convergence in (11) is uniform on every nonempty compact subset of K and G_t^+, G_t^- are additive and continuous.

Theorem

Let X be a Banach space and let K be a closed convex cone with the nonempty interior. Suppose that $\{F_t : t \geq 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t \colon K \longrightarrow cc(K)$, $x \in F_t(x)$ for all $x \in K$ and $t > 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t > 0$. Then this cosine family is twice differentiable and

$$D^2 F_t(x) = F_t(H(x))$$

for $x \in K$, $t \geq 0$, where $D^2 F_t(x)$ denotes the second Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

Proof. Let us fix $x \in K$. Consider the multifunction $t \mapsto F_t(x)$ for $t \geq 0$. By Lemma 8 there exist the Hukuhara differences $F_t(x) - F_s(x)$ for $0 \leq s \leq t$. By Lemma 9 the multifunction $t \mapsto F_t(x)$ is concave and there exist

$$G_t^+(x) = \lim_{h \to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h} \qquad \text{and} \qquad G_t^-(x) = \lim_{h \to 0^+} \frac{F_t(x) - F_{t-h}(x)}{h}$$

for $t > 0$ and $G_t^-(x) \subset G_t^+(x)$. The same argument may be used to prove that there exists

$$\lim_{t \to 0^+} \frac{F_t(x) - x}{t}.$$

It follows from (7) that

$$\frac{F_{2t}(x) - x}{2t} = F_t\left(\frac{F_t(x) - x}{t}\right) + \frac{F_t(x) - x}{t}.$$

Letting $t \to 0^+$ we get

$$\lim_{t \to 0^+} F_t\left(\frac{F_t(x) - x}{t}\right) = \{0\}$$

and since

$$0 \in \frac{F_t(x) - x}{t} \subset F_t\left(\frac{F_t(x) - x}{t}\right)$$

we have

$$DF_0(x) = \lim_{t \to 0^+} \frac{F_t(x) - x}{t} = \{0\}. \tag{12}$$

Let $0 < h \le t$. By (7) and the additivity of F_t we obtain

$$F_{t+h}(x) - F_t(x) = 2F_t(F_h(x) - x) + F_t(x) - F_{t-h}(x).$$

Dividing the last equality by h we get

$$\frac{F_{t+h}(x) - F_t(x)}{h} = 2F_t\left(\frac{F_h(x) - x}{h}\right) + \frac{F_t(x) - F_{t-h}(x)}{h}.$$

Letting $h \to 0^+$, by Lemma 9 and (12) we have

$$G_t^+(x) = G_t^-(x) =: G_t(x) \qquad \text{for } t > 0.$$

This and (12) imply that the family $\{F_t :\ t \ge 0\}$ is differentiable.

Next we will show that there exist the Hukuhara differences $G_t(x) - G_s(x)$ for $0 \le s \le t$. It is enough to consider the case $0 < s < t$. Let $h > 0$ be such that $t - s - h \ge 0$. By Lemma 8 there exist the differences

$$F_{\frac{1}{2}t-\frac{1}{2}s+\frac{1}{2}h}(x) - F_{\frac{1}{2}t-\frac{1}{2}s-\frac{1}{2}h}(x), \quad F_{t+h}(x) - F_t(x) \quad \text{and} \quad F_{s+h}(x) - F_s(x)$$

in $cc(K)$. Since $F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}$ is linear and continuous with respect to Lemma 3 there exists the difference

$$F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s+\frac{1}{2}h}(x)) - F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s-\frac{1}{2}h}(x)).$$

By (7) we have

$$\begin{aligned} &2F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s+\frac{1}{2}h}(x)) - 2F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s-\frac{1}{2}h}(x)) \\ &\qquad = F_{t+h}(x) + F_s(x) - (F_t(x) + F_{s+h}(x)) \\ &\qquad = (F_{t+h}(x) - F_t(x)) - (F_{s+h}(x) - F_s(x)) \,. \end{aligned}$$

Because of Lemma 1 there exists

$$G_t(x) - G_s(x) = \lim_{h\to 0^+} \left(\frac{F_{t+h}(x) - F_t(x)}{h} - \frac{F_{s+h}(x) - F_s(x)}{h} \right).$$

Our next claim is that the multifunction $t \mapsto G_t(x)$ is concave and differentiable. Replacing in (7) t by $t+h$, $h > 0$ and substracting $F_{t+s}(x) + F_{t-s}(x)$ from both the sides of this equality we get

$$F_{t+s+h}(x) - F_{t+s}(x) + F_{t-s+h}(x) - F_{t-s}(x) = 2F_{t+h}(F_s(x)) - 2F_t(F_s(x)).$$

The equality $F_t \circ F_s = F_s \circ F_t$, $s,t \geq 0$ leads to

$$\frac{F_{t+s+h}(x) - F_{t+s}(x)}{h} + \frac{F_{t-s+h}(x) - F_{t-s}(x)}{h} = 2F_s\left(\frac{F_{t+h}(x) - F_t(x)}{h} \right),$$

whence, as $h \to 0^+$,

$$G_{t+s}(x) + G_{t-s}(x) = 2F_s(G_t(x)). \tag{13}$$

Setting $t = \frac{v+u}{2}$, $s = \frac{v-u}{2}$, where $0 \leq u \leq v$ in (13) yields

$$G_v(x) + G_u(x) = 2F_{\frac{v-u}{2}}(G_{\frac{v+u}{2}}(x)).$$

By the assumption $x \in F_t(x)$ we get

$$G_{\frac{v+u}{2}}(x) \subset \frac{G_v(x) + G_u(x)}{2}.$$

Fix an interval $[a,b] \subset [0,\infty)$ and let $t \in [a,b]$. Since the multifunctions $t \mapsto F_t(x)$, $x \in K$, are concave and differences $F_t(x) - F_s(x)$ exist for $x \in K$ and $0 \leq s \leq t$, the multifunctions $t \mapsto G_t(x)$ are increasing (Theorem 3.2 in [6]) and we have $G_t(x) \subset G_b(x)$. Therefore the multifunctions $t \mapsto G_t(x)$ are bounded on $[a,b]$. By Theorem 4.4 in [4] the multifunction $t \mapsto G_t(x)$ is continuous in $(0,\infty)$ and by Theorem 4.1 in [4] it is concave. In virtue of Theorem 3.2 in [6] there exist

$$H_t^+(x) = \lim_{h\to 0^+} \frac{G_{t+h}(x) - G_t(x)}{h} \qquad \text{and} \qquad H_t^-(x) = \lim_{h\to 0^+} \frac{G_t(x) - G_{t-h}(x)}{h}$$

for $t > 0$ and $H_t^-(x) \subset H_t^+(x)$. Since $\frac{G_{\lambda t}(x)}{\lambda t} \subset \frac{G_t(x)}{t}$ for $t > 0$ and $\lambda \in (0,1)$, there also exists

$$\lim_{t\to 0^+} \frac{G_t(x)}{t} =: H(x)$$

and $H(x) \in cc(K)$.

Let $0 < s \le t$. The relation $F_t \circ F_s = F_s \circ F_t$ and Lemmas 2, 3 and 9 yield

$$\begin{aligned}
&F_s(G_t(x))\\
&\quad = F_s\left(\lim_{h\to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h}\right) = \lim_{h\to 0^+} \frac{F_s(F_{t+h}(x)) - F_s(F_t(x))}{h}\\
&\quad = \lim_{h\to 0^+} \frac{F_{t+h}(F_s(x)) - F_t(F_s(x))}{h}\\
&\quad \subset \lim_{h\to 0^+} \frac{(F_{t+h} - F_t)(F_s(x))}{h}\\
&\quad = G_t(F_s(x))
\end{aligned}$$

which together with (13) lead to

$$G_{t+s}(x) + G_{t-s}(x) \subset 2G_t(F_s(x)).$$

By the additivity of G_t we get

$$G_{t+s}(x) + G_{t-s}(x) \subset 2G_t(F_s(x) - x) + 2G_t(x),$$

whence

$$G_{t+s}(x) - G_t(x) \subset 2G_t(F_s(x) - x) + G_t(x) - G_{t-s}(x).$$

Dividing the last inclusion by s and letting $s \to 0^+$ we obtain

$$H_t^+(x) \subset H_t^-(x).$$

Therefore

$$H_t^+(x) = H_t^-(x) =: H_t(x)$$

for $t > 0$ and the family $\{F_t :\ t \ge 0\}$ is twice differentiable.

It remains to prove the equality in the assertion. Let $0 < s < t$. Lemmas 1, 3 and (7) lead to

$$\begin{aligned}
2F_t(G_s(x)) &= 2F_t\left(\lim_{h\to 0^+} \frac{F_{s+h}(x) - F_s(x)}{h}\right)\\
&= \lim_{h\to 0^+} \frac{2F_t(F_{s+h}(x)) - 2F_t(F_s(x))}{h}\\
&= \lim_{h\to 0^+} \frac{F_{t+s+h}(x) + F_{t-s-h}(x) - (F_{t+s}(x) + F_{t-s}(x))}{h}
\end{aligned}$$

$$= \lim_{h\to 0^+}\left[\frac{F_{t+s+h}(x) - F_{t+s}(x)}{h} - \frac{F_{t-s}(x) - F_{t-s-h}(x)}{h}\right]$$
$$= G_{t+s}(x) - G_{t-s}(x)$$
$$= G_{t+s}(x) - G_t(x) + G_t(x) - G_{t-s}(x).$$

Dividing the last equality by s we get

$$2F_t\left(\frac{G_s(x)}{s}\right) = \frac{G_{t+s}(x) - G_t(x)}{s} + \frac{G_t(x) - G_{t-s}(x)}{s},$$

letting $s \to 0^+$ and dividing by 2 we have

$$F_t(H(x)) = H_t(x).$$

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