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New results for almost increasing sequences

Abstract. In the present paper, two theorems of absolute summability have been proved by using the definition of almost increasing sequence.

1. Introduction

Let $\sum a_n$ be a given infinite series with its partial sums (s_n). Let (φ_n) be a sequence of positive real numbers. The series $\sum a_n$ is said to be *summable* $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [14])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |\gamma_n - \gamma_{n-1}|^k < \infty,$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1),$$

and the sequence-to-sequence transformation

$$\gamma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (γ_n) of the Riesz mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]).

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For $\varphi_n = \frac{P_n}{p_n}$, $\varphi - |\bar{N}, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n; \delta|_k$ summability (see [3]). Also, for $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, $\varphi - |\bar{N}, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability (see [2]). Some different applications of absolute summability can be find in [4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

A positive sequence (d_n) is said to be *almost increasing* if there is a positive increasing sequence (c_n) and two positive constants M and N such that

$$Mc_n \leq d_n \leq Nc_n$$

(see [1]). In [10], the following theorems of absolute summability have been proved by means of this sequence.

Theorem 1 ([10])
Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{3}$$

$$|\lambda_n|X_n = O(1) \quad as \quad n \to \infty, \tag{4}$$

where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. If

$$\sum_{v=1}^{n} \frac{|s_v|^k}{v} = O(X_n) \quad as \quad n \to \infty,$$

$$\sum_{v=1}^{n} \frac{p_v}{P_v}|s_v|^k = O(X_n) \quad as \quad n \to \infty, \tag{5}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem 2 ([10])
Let (X_n) be an almost increasing sequence. If conditions (1)–(4), (5) of Theorem 1 and conditions

$$\sum_{n=1}^{\infty} P_n X_n |\Delta\beta_n| < \infty, \tag{6}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{P_n} = O(X_m) \quad as \quad m \to \infty,$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

2. Main Results

In this section, two general theorems will be proved.

Theorem 3
Let (X_n) be an almost increasing sequence and $\varphi_n p_n = O(P_n)$. If conditions (1)–(4) of Theorem 1 and

$$\sum_{v=1}^{n} \varphi_v^{\delta k} \frac{1}{v} |s_v|^k = O(X_n) \quad as \quad n \to \infty, \tag{7}$$

$$\sum_{v=1}^{n} \varphi_v^{\delta k-1} |s_v|^k = O(X_n) \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k} \frac{1}{P_v}\right) \quad as \quad m \to \infty, \tag{9}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Theorem 4
Let (X_n) be an almost increasing sequence and $\varphi_n p_n = O(P_n)$. If conditions (1)–(4), (6), (8)–(9) and

$$\sum_{n=1}^{m} \varphi_n^{\delta k} \frac{|s_n|^k}{P_n} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

For $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, Theorem 3 and Theorem 4 reduce to Theorem 1 and Theorem 2, respectively.

Lemma 1 ([10])
If (X_n) is an almost increasing sequence, then under conditions (2)–(3), we have

$$nX_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{12}$$

Lemma 2 ([10])
If (X_n) is an almost increasing sequence, then under conditions (2) and (6), we have

$$P_n X_n \beta_n = O(1) \quad as \quad n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} p_n X_n \beta_n < \infty. \tag{14}$$

Proof of Theorem 3. Let (I_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, we have

$$I_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v.$$

For $n \geq 1$, we get

$$I_n - I_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v.$$

From Abel's transformation, we obtain

$$\begin{aligned} I_n - I_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\left(P_{v-1}\lambda_v\right)s_v + \frac{p_ns_n\lambda_n}{P_n} \\ &= \frac{p_ns_n\lambda_n}{P_n} - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vs_v\lambda_v + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vs_v\Delta\lambda_v \\ &= I_{n,1} + I_{n,2} + I_{n,3}. \end{aligned}$$

In order to prove that $\sum a_n\lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, we will show

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|I_{n,r}|^k < \infty \qquad \text{for } r = 1, 2, 3.$$

First, using condition (4) and the fact that (X_n) is an almost increasing sequence, we obtain $|\lambda_n|^{k-1} = O(1)$. Moreover, using the fact that $\varphi_np_n = O(P_n)$, we have

$$\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}|I_{n,1}|^k = O(1)\sum_{n=1}^{m}\varphi_n^{\delta k-1}|\lambda_n||s_n|^k.$$

By Abel's transformation,

$$\begin{aligned} \sum_{n=1}^{m}\varphi_n^{\delta k+k-1}|I_{n,1}|^k &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\varphi_r^{\delta k-1}|s_r|^k + O(1)|\lambda_m|\sum_{n=1}^{m}\varphi_n^{\delta k-1}|s_n|^k \\ &= O(1)\sum_{n=1}^{m-1}\beta_nX_n + O(1)|\lambda_m|X_m = O(1) \quad \text{as} \quad m \to \infty, \end{aligned}$$

by virtue of (1), (8), (12) and (4).

Now, by means of Hölder's inequality, using the fact that $\varphi_np_n = O(P_n)$ and conditions (4) and (9), we get

$$\begin{aligned} \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v|s_v||\lambda_v|\right)^k \\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v|\lambda_v|^k|s_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \end{aligned}$$

$$= O(1)\sum_{v=1}^{m} p_v|\lambda_v||s_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} \varphi_v^{\delta k-1}|\lambda_v||s_v|^k = O(1) \quad \text{as} \quad m\to\infty,$$

as in $I_{n,1}$.
Finally, again using the fact that $\varphi_n p_n = O(P_n)$, Hölder's inequality and condition (1), we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1}|I_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v|s_v|^k\beta_v\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\beta_v\right)^{k-1}.$$

Here, (12) yields

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1}|I_{n,3}|^k = O(1)\sum_{v=1}^{m} P_v|s_v|^k\beta_v \sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}.$$

Now, from (9), we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1}|I_{n,3}|^k = O(1)\sum_{v=1}^{m} \varphi_v^{\delta k} v\beta_v\frac{|s_v|^k}{v}.$$

Then,

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1}|I_{n,3}|^k &= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{1}{r}|s_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{1}{v}|s_v|^k\\ &= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m\\ &= O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by using Abel's transformation, (7), (3), (12) and (11). Therefore, the proof of Theorem 3 is completed.

Proof of Theorem 4. For $r=1$ and $r=2$, the proof of Theorem 4 as in the proof of Theorem 3. Thus, they can be omitted. Now, we will show

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1}|I_{n,r}|^k < \infty$$

only for $r=3$, by using the hypotheses of Theorem 4, Lemma 1 and Lemma 2.

For $r = 3$, we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |s_v|^k \beta_v \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v \right)^{k-1}$$
$$= O(1) \sum_{v=1}^{m} P_v |s_v|^k \beta_v \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}}.$$

Here, from (9), we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} P_v \beta_v \frac{|s_v|^k}{P_v}.$$

Thus, we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k = O(1) \sum_{v=1}^{m-1} \Delta(P_v \beta_v) \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{|s_r|^k}{P_r} + O(1) P_m \beta_m \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{|s_v|^k}{P_v}$$
$$= O(1) \sum_{v=1}^{m-1} P_v |\Delta \beta_v| X_v + O(1) \sum_{v=1}^{m-1} p_v \beta_v X_v + O(1) P_m \beta_m X_m$$
$$= O(1) \quad \text{as} \quad m \to \infty,$$

by using Abel's transformation, (10), (6), (14) and (13). Hence, the proof of Theorem 4 is completed.

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