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Report of Meeting

9th International Conference on Functional Equations and Inequalities, Złockie, September 7-13, 2003

The Ninth International Conference on Functional Equations and Inequalities, in the series of those organized by the Institute of Mathematics of the Pedagogical University of Cracow since 1984 (biannual since 1991), was held from September 7 to September 13, 2003, for the fifth time in the hotel "Geovita" at Złockie.

A support of the State Committee for Scientific Research (KBN) as well as the Bank Przemysłowo-Handlowy PBK SA is acknowledged with gratitude.

The Conference was opened by the address of Prof. Dr. Eugeniusz Wachnicki, Deputy Rector of the Pedagogical University of Cracow, who spoke on behalf of Prof. Dr. Michał Śliwa, Rector Magnificus. He greeted the participants, thanked the organizers and wished a fruitful and nice stay in this beatiful region of Poland.

After honorary doctorates conferred to Professor János Aczél by the Universities in Karlsruhe, Graz, Katowice and Miskolc, him was promoted to the Degree of Doctor Honoris Causa by the University of Debrecen in June 2003. The participants signed an address on this occasion with congratulations saying, among others: *Thanks to Your valuable and commemorable presence here in September 1999 at the 7th Conference we feel authorized to call ourselves students of the created by You World School on Functional Equations.*

Best greetings were also sent to Mrs Irena Gołąb, wife of Professor Stanisław Gołąb, on the occassion of her 100th birthday, and to Professor Stanisław Midura on his 70th birthday.

There were 57 participants who came from 8 countries: Austria (1, Innsbruck), France (1, Nantes), Germany (4, Clausthal-Zellerfeld, Karlsruhe, Munich), Greece (1, Athens), Hungary (12, Debrecen, Gödöllő, Gyöngyös, Miskolc), Italy (2, Torino), Japan (1, Kobe) and from Poland (35, Bielsko-Biała, Gdańsk, Katowice, Kraków, Rzeszów, Zielona Góra).

During 17 sessions 51 talks were delivered. The papers presented may be divided into four groups: A) functional equations in several variables,B) functional equations in a single variable and iteration theory, C) functional inequalities, D) related topics.

Here is a more detailed list of topics:

A) Properties of additive functions and derivations; the generalized Gołąb-Schinzel, Hosszú and Wilson equations; conditional and alternative equations (d'Alembert's, Cauchy's, Wigner's, of isometries); regularity type theorems with applications; equations stemming from the Cauchy–Riemann ones and from Mean Value Theorems; analytic solutions of H. Haruki's type equations; stability problems for various equations; the translation and cocycle equations; iteration semigroups, also of multifunctions; determining translative and quasi-commutative operations; comparing of utility representations.

B) Schröder equation, also in iteration theory and in Banach spaces; flows on the plane; iterative roots; regular iteration groups on a circle and in Banach spaces; sum type operators in Banach spaces; dilation equation; linear equation with iterates of the unknown function; k-difference equation and application to image processing.

C) Convexity: approximate, Jensen, Wright, t- and λ-convexity; Prékopa–Leindler inequality; simultaneous difference inequalities; monotonicity of sequences in the sense of Leja; characterization of quasi-monotonicity.

D) Reciprocal polynomials and number theory; Hermite–Hadamard inequality; recursive sequence; decision functions; inequalities between Gini and Stolarsky means; characterization of continuous functions; Rădulescu problem; metric space of multimeasures.

There were presented several remarks and open problems in 4 sessions.

Prof. Dr. Roman Ger exhibited a draft of his article entitled *Functional Equations and Inequalities*, to appear (in Polish) in a jubilee volume *50 Years of Matematics in Upper Silesia* asking for any comments. There has been stressed the significant role and merits of Professor Marek Kuczma and his School of Functional Equations in the development of research during the last 40 years.

The participants enjoyed a picnic on Tuesday and a banquet on Thursday. On Wednesday afternoon (which was free of sessions), due to a very nice weather, several groups of participants hiked along trails of Beskid Sądecki, some others visited Krynica Zdrój, and also Stará L'ubovňa and L'ubotín in Slovakia.

The organizing Committee was chaired by Prof. Dobiesław Brydak in co-operation with Prof. Bogdan Choczewski from the AGH University of Science and Technology in Kraków. Dr. Jacek Chmieliński acted as scientific secretary. Miss Janina Wiercioch and Mr Władysław Wilk (technical assistant) worked in the course of preparation of the meeting and in the Conference Office at Złockie, with a help of other academics from the Institute of Mathematics of the Pedagogical University of Cracow, in particular of Mr Paweł Solarz and Dr. Joanna Szczawińska.

The Conference was closed by Professor Dobiesław Brydak. Cordial thanks were addressed to the participants who all presented valuable contributions and created the unique friendly, both scientific and social, atmosphere. Thanks were extended to the members of the whole office staff at Złockie for their effective and dedicated work and helpful assistance, and to the managers of the hotel “Geovita” for their hospitality and quality of services.

The 10th ICFEI is planned to be held in September, 2005.

The abstracts of talks are printed in the alphabetical order, whereas the problems and remarks are presented chronologically. The careful and efficient work of Dr. J. Chmieliński on completing the material and preparing (together with Mr W. Wilk) the present report for printing is acknowledged with thanks.

Bogdan Choczewski

Abstracts of Talks

Mirosław Adamek *A characterization of λ-convex functions*

Let $I \subseteq \mathbb{R}$ be an interval and $\lambda\colon I^2 \longrightarrow (0,1)$ be a fixed function. A real-valued function $f\colon I \longrightarrow \mathbb{R}$ is called *λ-convex* if

$$f(\lambda(x,y)x + (1-\lambda(x,y))y) \le \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) \quad \text{for } x, y \in I.$$

The main result shows that λ-convex functions can be characterized in terms of a lower second-order generalized derivative.

Anna Bahyrycz *On the conditional equation of the exponential function*

We consider the conditional equation of the exponential function:

$$\forall\, x, y \in \mathbb{R}(n) \quad f(x)\cdot f(y) \ne 0_m \implies f(x+y) = f(x)\cdot f(y), \tag{1}$$

where

$$\begin{aligned} &f\colon \mathbb{R}(n) := [0,+\infty)^n \setminus \{0_n\} \longrightarrow \mathbb{R}(m), \quad n, m \in \mathbb{N},\\ &0_m := (0,\dots,0) \in \mathbb{R}^m,\\ &x+y := (x_1+y_1,\dots,x_k+y_k) \quad \text{and} \quad x\cdot y := (x_1\cdot y_1,\dots,x_k\cdot y_k)\\ &\qquad\qquad\qquad \text{for } x=(x_1,\dots,x_k),\ y=(y_1,\dots,y_k) \in \mathbb{R}(k). \end{aligned}$$

We study the general solution of the equation (1); exactly, we find a description and properties of the system of the cones over $\mathbb{Q}$ giving this solution.

Karol Baron *Dense sets of additive functions*

Joint work with Peter Volkmann.

We consider the topological vector space $\mathcal{A}$ of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ with the Tychonoff topology induced by $\mathbb{R}^{\mathbb{R}}$ and we prove that the following subsets of $\mathcal{A}$ and their complements (with respect to $\mathcal{A}$) are dense: the set of all additive injections, surjections, bijections, involutions, additive functions with countable image, additive functions with big graph. We are using a lemma which characterizes the density of subsets of $\mathcal{A}$.

Lech Bartłomiejczyk *Derivations with big graph*

A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a *derivation* (cf. [1; Ch. XIV]) iff it is additive and satisfies the equation

$$f(xy) = xf(y) + yf(x)$$

for all $x, y \in \mathbb{R}$.

We say that a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ has a *big graph* iff $B \cap \mathrm{Graph}\,(f) \neq \emptyset$ for every Borel subset B of $\mathbb{R}^2$ such that

$$\mathrm{card}\,\{x \in \mathbb{R} : \ (x, y) \in B\} = \mathfrak{c}.$$

The well known theorem of F.B. Jones [1; Theorem 3, p. 287] says that there exist additive functions with big graph; we prove that there exist derivations with big graph. This answers the question of Professor Ludwig Reich.

[1] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach 489, Państwowe Wydawnictwo Naukowe & Uniwersytet Śląski, Warszawa – Kraków – Katowice, 1985.

Mihály Bessenyei *On generalized Hermite–Hadamard inequality*

Joint work with Zsolt Páles.

The classical Hermite–Hadamard inequality provides the following lower and upper estimates for the integral average of a convex function $f\colon [a, b] \longrightarrow \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

The aim of our talk is to present Hermite–Hadamard inequalities under more general assumption than ordinary convexity. These kind of inequalities offer a lower and upper estimate for the integral average of a function involving certain base points of the domain.

Dobiesław Brydak *On the nonlinear iterative functional inequality*

We shall present a comparison theorem concerning the inequality

$$\psi[f(x)] \leq F[x, \psi(x)]$$

where ψ is an unknown function.

Janusz Brzdęk *On measurable solutions of some functional equations connected with multiplicative symmetry*

Let $(X,\cdot)$ be a group endowed with a topology, $F:\mathbb{C}\longrightarrow X$ be continuous at 0 and -1, and $F(-1)=F(1)^m$, $F(0)=F(1)^k$ with some $k,m\in\mathbb{Z}$. Under suitable assumptions on X, we describe the solutions $f:X\longrightarrow\mathbb{C}$ of the functional equation

$$f(F(f(y))\cdot x)=f(y)f(x)$$

that are continuous at a point or (universally, Baire, Christensen or Haar) measurable.

Jacek Chmieliński *Stability of angle-preserving mappings on the plane*

We say that a mapping $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is *angle-preserving* iff it satisfies:

$$|\cos(f(x),f(y))|=|\cos(x,y)|,\qquad x,y\in\mathbb{R}^2\setminus\{0\}$$

and $f(x)=0 \Leftrightarrow x=0$.

We prove that this property is stable and we apply this result to prove some kind of stability of the Wigner equation on the plane.

Bogdan Choczewski *k-difference equations and image processing*

The equations of the form

$$\varphi(kx)=g(\varphi(x))$$

are useful in a procedure of image processing proposed by S. Mann (*Intelligent Image Processing*, Willey and Sons, 2002). An example of such equation will be discussed.

Krzysztof Ciepliński *On rational iteration groups on the circle*

Denote by $\mathbb{S}^1$ the unit circle and let V be a linear space over $\mathbb{Q}$ with $\dim V\geq 1$. We recall that family $\{F^v:\mathbb{S}^1\longrightarrow\mathbb{S}^1,\ v\in V\}$ of homeomorphisms such that

$$F^{v_1}\circ F^{v_2}=F^{v_1+v_2},\qquad v_1,v_2\in V$$

is called an *iteration group*. An iteration group is said to be *rational*, if $V=\mathbb{Q}$.

In this talk we deal with rational iteration groups. We also give a general construction of all iteration groups $\{F^v:\mathbb{S}^1\longrightarrow\mathbb{S}^1,\ v\in V\}$ such that $\emptyset\neq\{z\in\mathbb{S}^1:\ F^v(z)=z,\ v\in V\}\neq\mathbb{S}^1$.

Péter Czinder *Comparison inequalities for Gini and Stolarsky means*

We investigate some inequalities concerning the two variable Gini and Stolarsky means, defined (in the most general case) by the formulae

$$G_{a,b}(x,y)=\left(\frac{x^a+y^a}{x^b+y^b}\right)^{\frac{1}{a-b}};\qquad S_{a,b}(x,y)=\left(\frac{x^a-y^a}{a}\,\frac{b}{x^b-y^b}\right)^{\frac{1}{a-b}}.$$

After giving the summary of preliminary results (comparison theorems for means of the same kind), we present our new results regarding the comparison of Gini and Stolarsky means.

Zoltán Daróczy *On translative and quasi-commutative operations*

We determine all the continuous operations $\circ\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ that are translative $((x+z)\circ(y+z) = x\circ y + z)$ and quasi-commutative $(x\circ(y\circ z) = y\circ(x\circ z))$.

Joachim Domsta *On a linear equation in two variables*

For a bijective $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\mathbb{R}_+ := (0,\infty)$, and two constants $a, b > 0$ let $g\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined as follows,

$$g(x,y) = \varphi^{-1}(a\cdot\varphi(x) + b\cdot\varphi(y)), \qquad \text{for } x, y \in \mathbb{R}_+. \tag{$*$}$$

This function appears in problems considered by J. Matkowski et al., see e.g. [1] and the references therein, cf. also the proceedings of the 40^{th} ISFE. Here we are proving the following fact: For every pair of a and b, function g determines φ uniquely, up to a multiplicative constant. Also a construction of the solution is presented, which goes through a special family of simultaneous Schröder equations in one variable derived from the equation $(*)$.

[1] J. Matkowski, *On a characterization of L^p-norm and a converse of Minkowski's inequality*, Hiroshima Math. J. **26**;2 (1996), 277-287.

Borbála Fazekas *Decision functions and their properties*

Our main aim is to characterize the relation between the properties of the so called decision functions and the properties of the decision generating functions. A function $D\colon \bigcup_{i=1}^{\infty} I^i \longrightarrow I$ is called a *decision function*, if it is symmetric, reflexive, regular and internal. We can generate a decision function D_d with a generalization of the least squares method using a decision generating function $d\colon I\times I \longrightarrow \mathbb{R}$. The reverse statement is also true, for every decision function D there exists a decision generating function d, that generates it. The main result, that characterizes the monotonicity property, is the following

Theorem
A decision function $D_d\colon \bigcup_{i=1}^{\infty} I^i \longrightarrow I$, generated by the decision generating function $d\colon I\times I \longrightarrow \mathbb{R}$, is monotonic if and only if

$$d(x_1,y_1) + d(x_2,y_2) \le d(x_1,y_2) + d(x_2,y_1)$$

holds for every $x_1, x_2, y_1, y_2 \in I$, $x_1 \le x_2$, $y_1 \le y_2$.

Margherita Fochi *On a conditional-alternative functional equation*

Let X be a real inner product space with $\dim X \geq 3$ and let $f\colon X \longrightarrow \mathbb{R}$.

Taking into account known results about functional equations on orthogonal vectors, we investigate the relations between the class of the solutions of the three following equations: the alternative equation on the whole space

$$f(x+y)^2 = (f(x)+f(y))^2 \qquad \text{for all } x,y \in X, \tag{1}$$

the corresponding equation restricted to the pairs of orthogonal vectors

$$f(x+y)^2 = (f(x)+f(y))^2 \qquad \text{for all } x,y \in X \text{ with } x \perp y \tag{2}$$

and the conditional Cauchy equation

$$f(x+y) = f(x)+f(y) \qquad \text{for all } x,y \in X \text{ with } x \perp y. \tag{3}$$

First of all we prove that (1), (2) and (3) are not equivalent, afterwards we shall characterize the common solutions introducing suitable auxiliary conditions.

Roman Ger *Logarithmic concavity and the Prékopa–Leindler inequality*

The Prékopa–Leindler inequality states that:

Given a $\lambda \in (0,1)$ and functions $f,g,h\colon \mathbb{R}^n \longrightarrow [0,\infty)$ that are Lebesgue integrable and satisfying the functional inequality

$$h(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$$

for all $x,y \in \mathbb{R}^n$, one has

$$\int_{\mathbb{R}^n} h\,d\ell_n \geq \left(\int_{\mathbb{R}^n} f\,d\ell_n\right)^\lambda \left(\int_{\mathbb{R}^n} g\,d\ell_n\right)^{1-\lambda}.$$

We discuss the pexiderized functional inequality of logarithmic concavity occurring here as the assumption. Among the corollaries issued from that studies we note down yet another proof of the logarithmic concavity of the Lebesgue measure.

Attila Gilányi *On a functional equation arising from the comparison of utility representations*

Joint work with János Aczél and Che Tat Ng.

In this talk the functional equation $F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)]$ is solved for real valued functions defined on intervals, assuming that F_2 is positive valued and strictly monotonic, and that F_3 is continuous. The equation with these assumptions arises from the comparison of utility representations characterized under the assumptions of separability, homogeneity and segregation (cf. [3]). It has been encountered before by A. Lundberg [2] and by J. Aczél, Gy. Maksa, C.T. Ng and Zs. Páles [1] under various conditions.

[1] J. Aczél, Gy. Maksa, C.T. Ng, Zs. Páles, *A functional equation arising from ranked additive and separable utility*, Proc. Amer. Math. Soc. **129** (2000), 989-998.

[2] A. Lundberg, *On the functional equation* $f(\lambda((x) + g(y)) = \mu(x) + h(x + y)$, Aequationes Math. **16** (1977), 21-30.

[3] C.T. Ng, R.D. Luce, J. Aczél, *Functional characterization of basic properties of utility representations*, Monatsh. Math. **135** (2002), 305-319.

Roland Girgensohn *A recursive sequence*

Joint work with Jonathan Borwein.

The following problem appeared in the American Mathematical Monthly in 2002. Let

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \frac{1}{2} + \frac{1}{3}, \\ a_3 &= \frac{1}{3} + \frac{1}{7} + \frac{1}{4} + \frac{1}{13}, \\ a_4 &= \frac{1}{4} + \frac{1}{13} + \frac{1}{8} + \frac{1}{57} + \frac{1}{5} + \frac{1}{21} + \frac{1}{14} + \frac{1}{183}, \end{aligned}$$

and continue the sequence, constructing a_{n+1} by replacing each fraction $\frac{1}{d}$ in the expression for a_n with $\frac{1}{(d+1)} + \frac{1}{(d^2+d+1)}$. Compute $\lim\limits_{n\to\infty} a_n$.

In the talk we will show how this problem can be solved using functional equations, and we will give generalizations.

Grzegorz Guzik *Cocycles and continuous iteration semigroups of triangular mappings*

In the comprehensive paper [2] M.C. Zdun found a form of all continuous iteration semigroups of continuous selfmappings of a compact interval. We use some results of W. Jarczyk, J. Matkowski and the present author on continuous solutions of the so called cocycle equation (see [1]) for construction of some continuous iteration semigroups of triangular functions mapping the product of two compact intervals into itself.

[1] G. Guzik, W. Jarczyk, J. Matkowski, *Cocycles of continuous iteration semigroups*, Bull. Pol. Acad. Sci. **51**(2)(2003), 185-197.

[2] M.C. Zdun, *Continuous and differentiable iteration semigroups*, Prace Nauk. Uniw. Śl. w Katowicach **308**, Katowice 1979.

Gabriella Hajdu *An extension theorem for a Matkowski–Sutô type problem for weighted quasi-arithmetic means*

Joint work with Zoltán Daróczy.

Let $I \subset \mathbb{R}$ be an interval, $0 < \lambda < 1$, $\mu \neq 0, 1$. We consider the following generalized Matkowski–Sutô type problem.

$$\mu A_\varphi(x, y; \lambda) + (1 - \mu) A_\psi(x, y; \lambda) = \lambda x + (1 - \lambda) y \qquad (x, y \in I),$$

where φ, ψ are continuous strictly monotone real functions on I and

$$A_\varphi(x, y; \lambda) := \varphi^{-1}(\lambda \varphi(x) + (1 - \lambda) \varphi(y))$$

denotes the weighted quasi-arithmetic mean generated by φ.

The solutions of the equation, under some regularity conditions, were given by Daróczy and Páles.

Our aim is to prove that if K is a proper subinterval of I then the solutions of the equation can be uniquely extended from K to the whole interval I.

Attila Házy *On approximately t-convexity*

Joint work with Zsolt Páles.

A real valued function f defined on an open convex set D is called (ε, p, t)-convex if it satisfies

$$f\left(tx + (1 - t)y\right) \leq tf(x) + (1 - t)f(y) + \sum_{i=0}^{k} \varepsilon_i |x - y|^{p_i} \qquad \text{for } x, y \in D,$$

where $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \ldots, p_k) \in [0, 1[^{k+1}$ and $t \in]0, 1[$ are fixed parameters. The main result of the paper states that if f is locally bounded from above at a point of D and (ε, p, t)-convex then it satisfies the convexity-type inequality

$$f\big(sx + (1 - s)y\big) \leq sf(x) + (1 - s)f(y) + \sum_{i=0}^{k} \varepsilon_i \phi_{p_i}(s) |x - y|^{p_i}$$

for $x, y \in D$, $s \in [0, 1]$, where $\phi_{p_i} \colon [0, 1] \longrightarrow \mathbb{R}$ is defined by

$$\phi_{p_i}(s) = \max\left\{\frac{1}{(1 - t)^{p_i} - (1 - t)}; \frac{1}{t^{p_i} - t}\right\} \big(s(1 - s)\big)^{p_i}.$$

The particular case $k = 0$, $p = 0$ of this result is due to Páles [4], the case $k = 0$, $p = 0$ and $t = \frac{1}{2}$ was investigated by Nikodem and Ng [3], the specialization $k = 0$, $\varepsilon_0 = 0$ yields the celebrated theorem of Bernstein and Doetsch [1]. The case $k = 1$, $\varepsilon = (\varepsilon_0, \varepsilon_1)$, $p = (1, 0)$ and $t = \frac{1}{2}$ was investigated in Házy and Páles [2].

[1] F. Bernstein, G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Annalen **76** (1915), 514-526.

[2] A. Házy and Zs. Páles, *Approximately midconvex functions*, Bull. London Math. Soc. (2002), to appear.

[3] C.T. Ng, K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 103-108.

[4] Zs. Páles, *Bernstein–Doetsch-type results for general functional inequalities*, Rocznik Nauk.-Dydakt. **204** Prace Mat. **17** (2000), Dedicated to Professor Zenon Moszner on his 70th birthday, 197-206.

Witold Jarczyk *Improving regularity of some functions by Grosse-Erdmann's theorems*

Joint work with Karol Baron.

Making use of a theorem of K.-G. Grosse-Erdmann we prove a result providing the effect "measurability implies continuity". It can be applied to functions satisfying a pretty large class of equalities of the form

$$\varphi(s+t) = \Phi(s, \gamma(t)),$$

among others to functional equations. In particular, we get a slight generalization of a theorem of M.C. Zdun concerning the continuity of Lebesgue measurable solutions of the translation equation

$$F(s+t, x) = F(t, F(s, x)).$$

We also generalize some results of G. Guzik, among others that one dealing with the regularity of solutions of the cocycle equation

$$G(s+t, x) = G(s, x)G(t, F(s, x)).$$

Hans-Heinrich Kairies *Spectral properties of a sum type operator*

Joint work with Karol Baron.

The eigenspaces of the sum type operator $F\colon D_n \longrightarrow D_n$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$$

can be characterized as solution sets of a Schröder equation and (depending on the structure of the domain D_n) of other linear iterative equations, which are simultaneously satisfied.

Some particular cases are discussed in detail.

Zoltán Kaiser *Stability of the monomial functional equation in normed spaces over fields with valuation*

S.M. Ulam's problem was to give conditions for the existence of a linear mapping near an approximately linear mapping. The first solution for this problem was given by D.H. Hyers in real Banach spaces. Th.M. Rassias and Z. Gajda gave a generalized solution to Ulam's problem. A. Gilányi investigated the stability of the monomial functional equation in the same sense. This talk extends their results to a more general setting when we consider Banach spaces over fields of characteristic zero with valuation.

Zygfryd Kominek *On a problem of Rădulescu*

Vincentiu Rădulescu in American Mathematical Monthly [1] posed the following problem.

Let $g\colon (0,\infty) \longrightarrow (0,\infty)$ *be a continuous function satisfying the condition*

$$\lim_{x\to\infty} \frac{g(x)}{x^{1+\alpha}} = \infty, \tag{1}$$

where $\alpha > 0$ *is a given constant. Let* $f\colon \mathbb{R} \longrightarrow (0,\infty)$ *be twice differentiable function for which there exist* x_0 *and* $a > 0$ *such that*

$$f''(x) + f'(x) > ag(f(x)) \qquad \text{for every } x \geq x_0. \tag{2}$$

Prove that $\lim_{x\to\infty} f(x)$ *exists and is finite, and evaluate the limit.*

We prove that if a solution does exist then $\lim_{x\to\infty} f(x) = 0$.

[1] V. Rădulescu, *Problem 11024*, Amer. Math. Monthly, **110** no. 6 (June–July 2003), 543.

Dorota Krassowska *Measurable solutions of a pair of linear functional inequalities of iterative type*

Suppose that a Lebesgue measurable function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the system of inequalities

$$f(x+a) \leq f(x) + \sum_{j=0}^{k} \alpha_j x^j; \quad f(x+b) \leq f(x) + \sum_{j=0}^{k} \beta_j x^j, \qquad x \in \mathbb{R}, \tag{1}$$

where $k \in \mathbb{N}$, $a, b, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k \in \mathbb{R}$ are fixed.

Assuming that

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\alpha_k}{a} - \frac{\beta_k}{b} = 0,$$

(where $\mathbb{Q}$ stands for the set of all rational numbers) we show that the function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$,

$$g(x) := f(x) - \frac{\alpha_k}{a(k+1)} x^{k+1}, \qquad x \in \mathbb{R},$$

satisfies system (1) with the same a, b and some uniquely determined polynomials of the order not greater than $k-1$.

The Lebesgue measurable solutions of (1) in the case $\alpha_0 = \alpha_1 = \cdots = \alpha_k = \beta_0 = \beta_1 = \cdots = \beta_k = 0$ were considered by J. Brzdęk in [1].

Our result and the result of Brzdęk, under some algebraic conditions of involved constants, allow to characterize the Lebesgue measurable solutions of (1) as the functions which coincide with polynomials almost everywhere.

[1] J. Brzdęk, *On functions satisfying some inequalities*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 277-281.

Károly Lajkó *A special case of the generalized Hosszú equation on an interval*

The functional equation

$$F(xy) + G((1-x)y) = H(x) + K(y) \tag{1}$$

plays an important role in solving of the generalized Hosszú functional equation, introduced by I. Fenyő,

$$f(r_0 + (r_1 x + r_2)(r_3 y + r_4)) + g(s_0 + (s_1 x + s_2)(s_3 y + s_4)) = h(x) + k(y). \tag{2}$$

In this presentation we consider equation (1) for the unknown functions $F, G, H, K\colon]0,1[\longrightarrow \mathbb{R}$ on the restricted domain $D = \{(x,y) \mid x, y \in]0,1[\}$. We have found the general solution of (1) on D.

Piroska Lakatos *Zeros of Coxeter and reciprocal polynomials*

We give sufficient conditions (linear inequalities in the coefficients) for reciprocal polynomials to have all their zeros on the unit circle.

We apply these results for the construction of Salem and PV numbers as well as to get estimates for the spectral radius of Coxeter transformation.

Zbigniew Leśniak *On boundary and limit orbits of a flow on the plane*

We present some properties of an equivalence relation defined for a given flow of the plane which have no fixed points. In particular, we observe that each point belonging to the first prolongational limit set of the plane is contained in the union of boundaries of equivalence classes of the relation, which implies that each limit orbit is a boundary orbit. The main result says that in the strip between two orbits lying in different equivalence classes there exists a point such that its first prolongational limit set contains the intersection of boundaries of the two classes.

László Losonczi *Inequalities for the coefficients of some reciprocal polynomials*

We characterize reciprocal polynomials all of whose zeros are on the unit circle. Using this characterization theorem we obtain bounds for the coefficients of such reciprocal polynomials.

If all zeros of the complex reciprocal polynomial

$$p(z) = \sum_{k=0}^{m} A_k z^k \qquad (A_k \in \mathbb{C},\ A_0 = 1,\ A_k = A_{m-k} \text{ for } k = 0, 1, \ldots, m)$$

of degree m ($m \in \mathbb{N}$) are on the unit circle then A_k are real and

$$|A_k| \le \binom{m}{k} \qquad (k = 0, 1, \ldots, m).$$

Here equality holds for all reciprocal polynomials if $k = 0, m$ and for the polynomials $p(z) = (z \pm 1)^m$ equality holds for all $k = 0, \ldots, m$.

We also show how other necessary conditions can be obtained from the characterization theorem.

Grażyna Łydzińska *On iteration semigroups of set-valued functions*

We present some conditions under which a family of set-valued functions, naturally occuring in iteration theory, fulfils one of the following conditions

$$F(s+t, x) \subset F\big(t, F(s, x)\big), \tag{C}$$

$$F\big(t, F(s, x)\big) \subset F(s+t, x) \tag{E}$$

for every $x \in X,\ s, t \in (0, \infty)$ (where X is an arbitrary set). Moreover, we compare the above conditions and answer the question whether either (C) or (E) implies that F is an iteration semigroup:

$$F\big(t, F(s, x)\big) = F(s+t, x)$$

for every $x \in X,\ s, t \in (0, \infty)$.

Janusz Morawiec *On a property of continuous solutions of the dilation equation*

Assume N is a positive integer, $c_0, \ldots, c_N$ are positive reals summing up to 2 and let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and compactly supported solution of the dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n).$$

We show that either $f = 0$ or $f|_{(0,N)} > 0$ or $f|_{(0,N)} < 0$.

Jacek Mrowiec *A counterexample to the stability property for δ-Jensen-convex functions defined on a convex set $D \subset \mathbb{R}^n$*

We give an example of a δ-Jensen-convex function f defined on a convex subset of $\mathbb{R}^n$ such that there is no Jensen-convex function uniformly close to f.

A function f defined on a convex subset D of a linear space X is said to be *δ-Jensen-convex* if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \delta, \qquad x, y \in D$$

where $\delta \ge 0$ is a given constant. If $\delta = 0$, a function f is said to be Jensen-convex.

Anna Mureńko *On some conditional generalizations of the Gołąb–Schinzel equation*

Joint work with Janusz Brzdęk.

We show connections between solutions $f\colon (0,\infty) \longrightarrow \mathbb{R}$, $g\colon [0,\infty) \longrightarrow \mathbb{R}$ and $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the conditional equations

$$\begin{aligned} \text{if } x + f(x)y > 0, &\text{ then } f(x+f(x)y) = f(x)f(y), \\ \text{if } x + g(x)y \ge 0, &\text{ then } g(x+g(x)y) = g(x)g(y), \\ \text{if } x > 0,\ y > 0, &\text{ then } h(x+h(x)y) = h(x)h(y), \end{aligned}$$

respectively, and solutions $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the Gołąb–Schinzel equation

$$F(x+F(x)y) = F(x)F(y).$$

In particular, we describe the solutions of the conditional equations that are continuous at a point, Lebesgue measurable or Baire measurable (i.e. have the Baire property).

Adam Najdecki *On a certain characterization of continuous functions*

Joint work with Jacek Tabor.

Let $f\colon X \longrightarrow Y$, where X is a Banach space and Y is a Hausdorff topological space. We show that if $f \circ \gamma$ is continuous for every curve $\gamma\colon [0,1] \longrightarrow X$ of class C^∞, then f is continuous.

Shin-ichi Nakagiri *Functional equations arising from the Cauchy-Riemann equations*

Joint work with Shigeru Haruki (Okayama University of Science).

We consider some functional equations arising from the Cauchy–Riemann equations, and certain related functional equations. First, we propose a new functional equation of the form

$$f(x+t,y) - f(x-t,y) = -i[f(x,y+t) - f(x,y-t)] \tag{1}$$

over a divisible Abelian group. This equation (1) is a discrete version of the Cauchy–Riemann equations, and we determine the general and regular solutions of (1). For the related functional equation of the form

$$f(x+t,y) - f(x,y) = -i[f(x,y+t) - f(x,y)] \tag{2}$$

it was shown in J. Aczél and S. Haruki [1], and S. Haruki [2] that (2) does not lead essentially beyond a linear function. However, for the functional equation of the form

$$f(x+t,y+t) - f(x-t,y-t) = -i[f(x-t,y+t) - f(x+t,y-t)] \tag{3}$$

such a result has not been obtained. We show that (3) is equivalent to (1), equations (1), (2) and (3) satisfy the Haruki functional equation, and that the general solutions of (3) are given by quadratic functions. Further we propose and solve partial differential-difference type and nonsymmetric type functional equations which are also arising from the Cauchy-Riemann equations.

[1] J. Aczél, S. Haruki, *Partial difference equations analogous to the Cauchy–Riemann equations*, Funkcial. Ekvac. **24** (1981), 95-102.

[2] S. Haruki, *Partial difference equations analogous to the Cauchy–Riemann equations II*, Funkcial. Ekvac. **29** (1986), 237-241.

Kazimierz Nikodem *Convexity triplets and t-convex functions*

Joint work with Zsolt Páles.

For a function $f\colon I \longrightarrow \mathbb{R}$ we denote by $C(f)$ the set of all triplets (x,y,z) at which the second order divided difference $f[x,y,z] \geq 0$. Some properties of the set $C(f)$ and their application to t-convex functions are presented. A characterization of t-convex functions in terms of a second order generalized derivative is also given.

Jolanta Olko *Metric on the space of multimeasures*

We define the metric on the space of set-valued measures, generated by the Fortet–Mourier norm on the space of signed measures. The properties of this metric space are studied.

Iwona Pawlikowska *A method used in solving functional equations stemming from MVTs*

Let X, Y be two linear spaces over a field $\mathbb{K} \subset \mathbb{R}$ and let K be a convex balanced set with $0 \in \operatorname{alg\,int} K$. Fix $N, M \in \mathbb{N} \cup \{0\}$ and a, $b \in \mathbb{K}$, $b \neq 0$. We denote by $I = \{(\alpha,\beta) \in \mathbb{K} \times \mathbb{K} : \ |\alpha| + |\beta| \leq 1\}$ and $I^+ = \{(\alpha,\beta) \in I : \ \beta \neq 0\}$. Assume that $I_0, \ldots, I_M$ are finite subsets of I^+. We prove the following lemma: *if functions $\varphi_i\colon K \longrightarrow SA^i(X;Y)$, $i \in \{0,\ldots,N\}$ and*

$\psi_{j,(\alpha,\beta)}\colon K \longrightarrow SA^j(X;Y)$, $(\alpha,\beta)\in I_j$, $j\in\{0,\dots,M\}$ *satisfy the equation*

$$\sum_{i=0}^{N}\varphi_i(x)((ax+by)^i)=\sum_{i=0}^{M}\sum_{(\alpha,\beta)\in I_i}\psi_{i,(\alpha,\beta)}(\alpha x+\beta y)((ax+by)^i)$$

for every x, $y\in K$ *then there exists a* $p\in\mathbb{N}$ *such that* φ_N *is a local polynomial function on* $\frac{1}{p}K$ *of order at most equal to*

$$\sum_{i=0}^{M}\operatorname{card}\left(\bigcup_{k=i}^{M} I_k\right)-1.$$

This outcome in case of $N=0$ is an extension of Corollary 2 from [1]. We also generalize some results of W.H. Wilson [4], L. Székelyhidi [3] and M. Sablik [2]. Then we use this lemma to solve functional equations derived from Mean Value Theorems.

[1] Z. Daróczy, Gy. Maksa, *Functional equations on convex sets*, Acta Math. Hungarica, **68**(3) (1995), 187-195.

[2] M. Sablik, *Taylor's theorem and functional equations*, Aequationes Math. **60** (2000), 258-267.

[3] L. Székelyhidi, *Convolution type functional equations on topological Abelian groups*, World Scientific Singapore – New Jersey – London – Hong Kong, 1991.

[4] W.H. Wilson, *On a Certain General Class of Functional Equations*, Amer. J. Math. **40** (1918), 263-282.

Zsolt Páles *On higher-order convexity and Wright-convexity*

Joint work with Attila Gilányi.

Motivated by the notions of convexity, t-convexity, Wright-convexity and t-Wright convexity, higher-order analogoues of these concepts are introduced and several results known for the classical situation are extended to the higher-order setting.

Themistocles M. Rassias *On new properties of isometric mappings*

This talk is concerned with results on new properties of isometric mappings in the spirit of the Mazur–Ulam theorem and the Aleksandrov problem of conservative distances.

Some old and new problems will be discussed.

[1] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel, Berlin, 1998.

[2] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

Maciej Sablik *Applications of a method for characterizing polynomials*

A lemma proved in [3] turns out to be a useful tool in dealing with equations characterizing polynomial functions. We illustrate its application by examples taken from [1] and [2].

[1] T. Riedel, M. Sablik, *A different version of Flett's Mean Value Theorem and an associated functional equation.* Acta Math. Sinica **20** (2004), 1073-1078.

[2] T. Riedel, M. Sablik, A. Sklar, *Polynomials and divided differences.* Publ. Math. Debrecen **66**/3-4 (2005), 313-326.

[3] M. Sablik, *Functional equations and Taylor's theorem*, Aequationes Math. **60** (2000), 258-267.

Fulvia Skof *About a functional equation related to d'Alembert and other classical equations*

The known d'Alembert equation $f(x+y)+f(x-y)=2f(x)f(y)$ forces each of its solutions f to be an even function. An interesting generalized form of this equation, admitting also non even solutions, is the following one,

$$f(x+y)+f(x-y)=f(x)[f(y)+f(-y)], \tag{1}$$

which is the matter of the present study, where f is assumed to be a complex valued function from a linear space (or from a commutative group, or a suitable more general algebraic structure).

Equation (1) turns out to have a rather wide class of solutions, which contains the classes of functions satisfying either the d'Alembert equation, or the exponential Cauchy equation or the Jensen equation (with $f(0)=1$); when f is split into its even and odd components, further interesting trigonometric equations, widely studied by other authors, are involved too.

On such ground, the general table of the solutions of (1) can be easily drawn without any regularity assumption on f.

Then, some remarks are added about the similar generalization of the Wilson equation, namely

$$f(x+y)+f(x-y)=f(x)[g(y)+g(-y)],$$

although equation (1) seems to be a more interesting one.

The variety of solutions of (1), satisfying more than one equation, allows us to observe how it can happen that the solutions of a given functional equation on some "bounded restricted domains" may be locally very different from the restriction of the solution of the same equation given over the whole space. Simple examples support the above remark.

Wilhelmina Smajdor *Local analytic solutions of some functional equations*

Joint work with Andrzej Smajdor.

We determine all analytic solutions of the functional equations

$$\left|f\left(re^{i\theta}\right)\right|^2 + |f(1)|^2 = |f(r)|^2 + \left|f\left(e^{i\theta}\right)\right|^2,$$
$$\left|f\left(re^{i\theta}\right)\right| = |f(r)|,$$
$$\left|f\left(re^{i\theta}\right)\right| = \left|f\left(e^{i\theta}\right)\right|,$$

in the domains

$$\{z \in \mathbb{C} :\ 1 - \varepsilon < |z| < 1 + \varepsilon\}$$

and

$$\{re^{i\theta} \in \mathbb{C} :\ 1 - \varepsilon < r < r + \varepsilon,\ -\delta < \theta < \delta\},$$

where $0 < \varepsilon \leq 1$ and $0 < \delta \leq \pi$.

Paweł Solarz *On some iterative roots with a rational rotation number*

Let S^1 be the unit circle with the positive orientation, i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $F\colon S^1 \longrightarrow S^1$ is a homeomorphism such that $F^n = \mathrm{id}_{S^1}$, where $n \in \mathbb{N}$, $n \geq 2$, is the minimal such a number. If F preserves orientation, then for every integer $m \geq 2$ there exist infinitely many orientation-preserving roots of F, i.e., solutions of the following functional equation

$$G^m(z) = F(z), \qquad z \in S^1. \tag{1}$$

However, if $\gcd(m, n) > 1$, each of these roots depends on an arbitrary function. Otherwise, there also exists a solution such that $G = F^l$ for some $l \in \mathbb{N}$.

Orientation-reversing solutions of (1) either, if F preserves orientation, do not exist or, if F reverses orientation, are equal to F.

Peter Volkmann *A characterization of quasimonotonicity*

It is known that quasimonotonicity of a continuous function can be characterized by means of differential inequalities. Using this, Karol Baron and me give a characterization by means of functional inequalities.

Eugeniusz Wachnicki *Sur la monotonie des suites en moyenne*

Travail commun avec Zbigniew Powązka.

F. Leja a introduit différentes notions de monotonie des suites en moyenne et il a démontré que toute suite monotone en moyenne au sens considéré par lui est convergente. Nous présentons les cas plus généraux que F. Leja en considérant la moyenne quasi-arithmétique et quasi-arithmétique pondérée.

Janusz Walorski *On homeomorphic solutions of the Schröder equation in Banach spaces*

Let X be a Banach space, $f: X \longrightarrow X$ be a homeomorphism and $A: X \longrightarrow X$ be a continuous linear operator.

We establish conditions, different from that of Grobman–Hartman, under which there exists a homeomorphism $\varphi : X \longrightarrow X$ which solves the Schröder equation

$$\varphi(f(x)) = A\varphi(x)$$

and such that $\varphi - \mathrm{id}_X$ is bounded.

Szymon Wąsowicz *Separation by functions belonging to Haar spaces*

Joint work with Mircea Balaj (Oradea, Romania).

In 1996 M. Balaj gave the necessary and sufficient condition for two functions f, g mapping a real interval I into $\mathbb{R}$ to be separated by a polynomial (cf. [2, Theorem 2]). We generalize this result dealing with functions belonging to Haar spaces instead of polynomials. Recall that if D is a set containing at least n elements, then a linear subspace $\mathcal{H}_n(D)$ of $\mathbb{R}^D$ is called an *n-dimensional Haar space on D*, if for any n distinct elements $x_1, x_2, \dots, x_n$ of D and any $y_1, y_2, \dots, y_n \in \mathbb{R}$ there exists the unique function $h \in \mathcal{H}_n(D)$ such that $h(x_j) = y_j$, $j = 1, \dots, n$.

Let $\Phi_1, \dots, \Phi_{n+1}$ be multifunctions defined on D, which values are compact real intervals. We give conditions under which at least one multifunction Φ_{i_0} admits a selection belonging to $\mathcal{H}_n(D)$. Next we consider $n+1$ pairs of functions $f_i, g_i: I \longrightarrow \mathbb{R}$, $i = 1, \dots, n+1$, and we give the conditions under which at least one pair of functions f_{i_0}, g_{i_0} can be separated by a function belonging to $\mathcal{H}_n(I)$. As a consequence we obtain an extended version of Theorem 2 of [2] and some result on the stability of Hyers–Ulam type for polynomials.

[1] M. Balaj, Sz. Wąsowicz, *Haar spaces and polynomial selections*, Math. Pannonica **14**/1 (2003), 63-70.

[2] Sz. Wąsowicz, *Polynomial selections and separation by polynomials*, Studia Math. **120** (1996), 75-82.

Marek Cezary Zdun *On a limit formula for regular iteration groups in Banach space*

Let X be a real Banach space, $U \subset X$ be an open set, $f: U \longrightarrow U$ be a diffeomorphism and $0 \in U$ be a unique globally attractive fixed point of f. Assume that there exists a linear bounded operator $A: X \longrightarrow X$ such that $f'(0) = \exp A$. We give some conditions which imply the existence of the following limit

$$f_t(x) := \lim_{n\to\infty} f^{-n}((\exp tA)f^n(x)), \qquad x \in U,\ t \geq 0$$

and the property that the family of mappings $\{f_t,\ t \geq 0\}$ yields a C^1 iteration semigroup of f.

An application for C^1 iterative roots of f is given. In particular we consider the case $X = \mathbb{R}^n$.

Marek Żołdak *Nonhomogeneous iterative equation*

Using the Banach fixed point theorem we investigate the existence and uniqueness of Lipschizian solutions of the nonhomogeneous iterative equation

$$\sum_{i=1}^{\infty} a_i f^i(x) = F(x) \qquad \text{for } x \in I,$$

where I is compact, convex subset in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with nonempty interior, $F\colon I \longrightarrow I$ – is a given Lipschitz function, $f\colon I \longrightarrow I$ – is an unknown function, a_i for $i = 1, 2, \ldots$ – are real constants.

Problems and Remarks

1. Problem.
In describing the 1-periodic solutions of a certain Schröder equation, the classes

$$K_\alpha := 2^{\mathbb{Z}}\alpha + \mathbb{D}$$

($\alpha \in \mathbb{R}$, $\mathbb{D}$ – the set of dyadic rationals) play an important role.

It is an open problem to find an explicit description of the set

$$S := \{\alpha \in \mathbb{R} :\ K_\alpha = K_{-\alpha}\}.$$

It is known that $\mathbb{D} \subset S$, $(\mathbb{R} \setminus \mathbb{Q}) \cap S = \emptyset$, $S \neq \mathbb{Q}$.

Hans-Heinrich Kairies

2. Problem.
Let $(G, +)$ be an abelian group. The operation $\circ : G^2 \to G$ is called *translative* (*quasicommutative*) if

$$(x+z) \circ (y+z) = x \circ y + z \qquad (\ x \circ (y \circ z) = y \circ (x \circ z)\ )$$

for all $x, y, z \in G$. Denote the set of all translative and quasicommutative operations $\circ\colon G^2 \longrightarrow G$ by $T(G)$.

Determine $T(\mathbb{Z})$ and $T(\mathbb{Q})$.

Zoltán Daróczy

3. Remark. (To *the problem of Z. Daróczy*)
Let us note that among the elements of $T(\mathbb{Z})$, apart from those being restrictions of translative and quasicommutative operations $\circ$ defined in $\mathbb{R} \times \mathbb{R}$ we also have

$$x \circ y = \begin{cases} y & \text{if } y - x \in 2\mathbb{Z}, \\ y + 2 & \text{if } y - x \in 2\mathbb{Z} + 1. \end{cases}$$

Maciej Sablik

4. Problem.
Define the Takagi function $T: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T(x) := \sum_{k=0}^{\infty} \frac{\operatorname{dist}(2^k, \mathbb{Z})}{2^k}.$$

Prove (or disprove) that it satisfies the following approximate Jensen-convexity inequality

$$T\left(\frac{x+y}{2}\right) \leq \frac{T(x) + T(y)}{2} + \frac{1}{2}|x - y| \qquad (x, y \in \mathbb{R}).$$

Zsolt Páles

5. Problems and Remark. *Functional-integral equations stemming from Steffensen's inequality* (presented by B. Choczewski).
The Steffensen inequality reads:
If $f: [a, b] \longrightarrow [0, +\infty)$ is a decreasing function and $g: [a, b] \longrightarrow [0, 1]$, and both are integrable, then

$$\int_{b-c}^{b} f(t)\,dt \leq \int_{a}^{b} f(t)g(t)\,dt \leq \int_{a}^{a+c} f(t)\,dt, \qquad c = \int_{a}^{b} g(t)\,dt. \tag{S}$$

The question when the medial term in (S) is the arithmetic mean of side terms leads to the formula:

$$\int_{a}^{a+c} f(t)\,dt + \int_{b-c}^{b} f(t)\,dt = 2\int_{a}^{b} f(t)g(t)\,dt. \tag{E}$$

Study different functional equations which may stem from (E): the numbers a, b, c may be treated as constants, or variables, or some of them may be functions of some others.

The author studied the functional equation of type (E):

$$\int_{x}^{H(xy+x+y)} f(t)\,dt + \int_{H(xy-x-y)}^{y} f(t)\,dt = 2\int_{x}^{y} f(t)g(t)\,dt \tag{C}$$

with three unknown functions f, g and H (selfmappings of reals). Assuming that they satisfy (E) and are analytic the author derived from (C) the following relations:

$$\begin{cases} H'(x)f(H(x)) = \dfrac{\alpha x+\beta}{x+1}; & \alpha,\beta\in\mathbb{R},\ x\neq -1 \\ f(x)(1-2g(x)) = \dfrac{2(\beta-\alpha x^2)}{1-x^2}, & |x|\neq 1. \end{cases}$$

Ilie Corovei (Cluj-Napoca, Romania)

6. Problem.
A function $f\colon I\longrightarrow\mathbb{R}$ is called $\left(\frac{1}{3},\frac{2}{3}\right)$*-convex* if it satisfies

$$f\left(\frac{x+2y}{3}\right)\leq\frac{f(x)+2f(y)}{3}$$

for all $x,y\in I$ with $x<y$.

Clearly, the $\frac{1}{3}$-convexity implies $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity.

Question 1. Does $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity imply Jensen convexity?

Question 2. Is $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity a localizable property?

Question 3. Can $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity be characterized by the nonnegativity of the 2nd order derivative

$$\underline{\delta}^2_{\left(\frac{1}{3},\frac{2}{3}\right)}f(p)=\liminf_{\substack{(x,y)\to(p,p)\\ x\leq p\leq y}} 2\cdot\left[x,\frac{x+2y}{3},y;f\right]$$

where $[x,y,z;f]$ is the 2nd order divided difference of f.

Zsolt Páles

7. Remark.
There are homeomorphisms $f\colon(0,\infty)\longrightarrow(0,\infty)$ with $f(x)<x$, $x>0$, for which $d:=\lim_{x\to 0^+}\frac{f(x)}{x}\in(0,1)$ and such that

(a) $\frac{f(x)}{x}-d=O\left(|\log x|^{1+\nu}\right)$, as $x\to 0$, where $\nu>0$;

(b) $\frac{f(x)}{x}-d$ is of bounded variation in $x\in(0,\infty)$;

(c) $\varphi_f(x|1):=\lim_{n\to\infty}\frac{f^n(x)}{f^n(1)}$ is a non-bijective continuous function.

In particular there are examples of f satisfying the above conditions without any regular iterative root.

For given $d\in(0,1)$, $(a_n)_{n\in\mathbb{Z}}\subset(0,1)^{\mathbb{Z}}$, such a function can be built as follows:

$$f(x)=\begin{cases} d^{n+1} & \text{for } x=d^n,\ n\in\mathbb{N}, \\ d^{n+1}(1-(1-d)a_{n+1}) & \text{for } x=d^n(1-(1-d)a_n) \end{cases}$$

and then as a function piecewise linear between the indicated points. The only additional requirement is that $\lim_{n\to\infty} a_n = 0$.

The details are in preparation for publication elsewhere.

Joachim Domsta

8. Remark and Problem.

S.-M. Jung [7] investigated the Hyers–Ulam stability of the orthogonality equation for a class of mappings defined on a closed ball in $\mathbb{R}^3$:

THEOREM 1 ([7])
Let $D \subset \mathbb{R}^3$ be a closed ball of radius $d > 0$ and with center at the origin. If a mapping $T\colon D \longrightarrow D$ satisfies the conditions:

$$T(0) = 0 \qquad \textit{and} \qquad |\langle Tx, Ty\rangle - \langle x, y\rangle| \le \varepsilon \tag{1}$$

for some ε such that $0 \le \varepsilon < \min\left\{\frac{1}{4}, \frac{d^2}{17}\right\}$ and for all $x, y \in D$, then there exists an isometry $I\colon D \longrightarrow D$ that satisfies, for any $x \in D$,

$$\|Tx - Ix\| \le \begin{cases} 16\sqrt{\varepsilon} & \textit{for } d < \frac{\sqrt{17}}{2}, \\ (6d+3)\sqrt{\varepsilon} & \textit{for } d \ge \frac{\sqrt{17}}{2}. \end{cases}$$

Jung's estimate was improved in [8] for the upper bound of the norm of the difference $Tx - Ix$ by proving the following:

THEOREM 2 ([8])
Under assumptions of Theorem 1,

$$\|Tx - Ix\| \le \begin{cases} 13\sqrt{\varepsilon} & \textit{for } d < \frac{\sqrt{17}}{2}, \\ (4.5d+3.5)\sqrt{\varepsilon} & \textit{for } d \ge \frac{\sqrt{17}}{2} \end{cases} \tag{2}$$

for any $x \in D$.

In 1994, J. Chmieliński [1] was the first to prove the Hyers–Ulam stability of the orthogonality functional equation.

THEOREM 3 ([1])
Let E be a real Hilbert space with dimension greater than 1. If a mapping $T\colon E \longrightarrow E$ satisfies the property

$$|\langle Tx, Ty\rangle - \langle x, y\rangle| \le \varepsilon, \qquad \textit{for all } x, y \in E,$$

then there exists a unique isometry $I\colon E \longrightarrow E$ such that

$$\|Tx - Ix\| \le \sqrt{\varepsilon}, \qquad \textit{for all } x \in E.$$

This estimate is the best possible. Chmieliński [2] proved the superstability of the orthogonality functional equation in the case $E = \mathbb{R}^n$ for $n \geq 2$. A similar result was also obtained by Jung [6]. In 1997, Chmieliński [3] proved the Hyers–Ulam stability of the orthogonality equation for complex inner product spaces. Furthermore, Chmieliński and Jung [4] investigated the Hyers–Ulam–Rassias stability of the orthogonality equation for mappings defined on restricted domains of Hilbert spaces.

The interested reader is referred to the book [5] for an extensive account on stability results for functional equations.

Problem: Determine the best coefficients of $\sqrt{\varepsilon}$ in (2).

[1] J. Chmieliński, *On the Hyers–Ulam stability of the generalized orthogonality equation in real Hilbert spaces*, in: *Stability of mappings of Hyers–Ulam Type* (eds. Th.M. Rassias and J. Tabor), Hadronic Press, Palm Harbor, FL, 1994, 31-41.

[2] J. Chmieliński, *On the superstability of the generalized orthogonality equation in Euclidean spaces*, Ann. Math. Sil. **8**(1994), 127-140.

[3] J. Chmieliński, *On the stability of isometric operators for Hilbert spaces*, in: *Inner Product Spaces and Applications* (ed. Th.M. Rassias), Longman, 1997, 15-21.

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