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Akbar Zada[1] *and Hira Waheed*

Stability analysis of implicit fractional differential equation with anti–periodic integral boundary value problem

Abstract. In this manuscript, we study the existence, uniqueness and various kinds of Ulam stability including Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability of the solution to an implicit nonlinear fractional differential equations corresponding to an implicit integral boundary condition. We develop conditions for the existence and uniqueness by using the classical fixed point theorems such as Banach contraction principle and Schaefer's fixed point theorem. For stability, we utilize classical functional analysis. The main results are well illustrated with an example.

1. Introduction

A fractional order differential equation is a generalization of the integer order differential equation. The idea of fractional calculus has been introduced at the end of sixteenth century (1695). Fractional calculus is a generalization of ordinary differentiation and integration up to arbitrary order (non-integer). The advantages of fractional derivative become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids, rocks and in many other fields, see [17, 32]. Fractional derivative is used as a global operator for modelling of various processes and physical systems which arises in subjects like physics, dynamics, fluid mechanics, control theory, chemistry, mathematical biology, etc., see [6, 10, 12, 14, 13, 16, 7, 38]. It turns

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[1] Corresponding author.
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out that fractional differential equations (FDEs) can describe real world problems more accurately comparing with integer order differential equations. Due to its importance and large number of applications, this area has attracted attention of many mathematicians and researchers in the last few decades. Also, rich material on theoretical aspects and analytic methods for solving fractional order models, attracts the researchers. More specifically, FDEs with an implicit boundary condition are applicable in different fields of applied sciences, including population dynamics, thermo-elasticity, blood flow, underground water flow, chemical engineering and so on, see [2, 3, 25, 28].

Now we want to discuss another aspect of qualitative theory which is the notion of stability analysis. In fields such as numerical analysis, optimization theory, and nonlinear analysis, stability is very important. Various kinds of stability have been investigated, for instance exponential, Lyapunov, asymptotic stability etc., see [18, 41, 27]. In this manuscript, we will discuss Hyers–Ulam stability (HUS). The mentioned stability was first pointed out by Ulam [24] in 1940, which was properly formulated by Hyers [11] in 1941, for problems of functional equations in Banach space [15, 20]. Afterwards, the results were generalized and extended by many researchers, for details we refer the reader to [1, 4, 18, 22, 23, 29, 31, 30, 33, 36, 37, 39, 40, 35, 34, 26]. The aforesaid stability is rarely studied for FDEs and specially for fractional boundary value problems. We study approximate solutions and investigate how close are these solutions to the actual solution of the concerned system or systems. Many approaches can be used for this purpose, but HUS approach seems to be the most important approach. Moreover, a fractional order system may have additional attractive features over the integer order system. Let us recall the following example from [19], showing more stable system in the aforementioned (fractional order and integer order) systems.

Example 1.1
Consider the following two equations with the initial condition $u(0)$,

$$\frac{d}{dt}u(t) = \upsilon t^{\upsilon-1}, \tag{1.1}$$

$$ {}_0^cD_t^p u(t) = \upsilon t^{\upsilon-1}, \qquad 0 < p < 1, \tag{1.2}$$

where $\upsilon \in (0,1)$. Then the analytical solutions of (1.1) and (1.2) are $t^\upsilon + u(0)$ and $\frac{\upsilon\Gamma(\upsilon)t^{\upsilon+p-1}}{\Gamma(\upsilon+p)} + u(0)$, respectively. Clearly, the integer order system (1.1) is unstable for any $0 < \upsilon < 1$, but the fractional order dynamic system (1.2) is stable for each $0 < \upsilon \leq 1-p$. Thus the fractional order system has better features than the integer order system.

Benchohra and Lazreg in [8], investigated the existence theory and different kinds of stability in the sense of Ulam for the following nonlinear implicit FDE:

$$\begin{cases} {}^cD^p y(t) = f(t, y(t), {}^cD^p y(t)) & \text{for all } t \in J,\ 0 < p \leq 1, \\ \quad y(0) = y_0, \end{cases}$$

where ${}^cD^p$ is the Caputo fractional derivative, $f\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $y_0 \in \mathbb{R}$, and $J = [0, T]$, $T > 0$.

Recently, Zeeshan *et al.* [5] studied the above problem with different boundary conditions, particularly they modified it to the following:

$$\begin{cases} D^p u(t) = f(t, u(t), D^p u(t)) \qquad \text{for all } t \in J = [0,T],\ T > 0,\ p \in (1,2], \\ D^{p-2} u(0^+) = \gamma D^{p-2} u(T^-), \\ D^{p-1} u(0^+) = \beta D^{p-1} u(T^-), \end{cases}$$

where D^p is the Riemann–Liouville derivative of fractional order, $f\colon [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, and $\beta, \gamma \neq 1$.

In this manuscript, we study the following class of implicit FDE with implicit integral boundary condition:

$$\begin{cases} {}^cD^\omega p(t) = \mathcal{G}(t, p(t), {}^cD^\omega p(t)) + \displaystyle\int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), {}^cD^\omega p(s)) ds \\ \qquad\qquad \text{for all } t \in \mathcal{X} = [0,T],\ T > 0,\ 0 < \omega \leq 1, \\ p(0) = -\displaystyle\int_0^T \frac{(T-\xi)^{\omega-1}}{\Gamma(\omega)} \mathcal{F}(\xi, p(\xi), {}^cD^\omega p(\xi)) d\xi, \end{cases} \tag{1.3}$$

where the notation ${}^cD^\omega$ is used for Caputo fractional derivative of order $0 < \omega \leq 1$, $\mathcal{G}$, $\mathcal{F}$, $f\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, δ and σ are real constants greater than zero.

Using classical fixed point theorems of Banach and Schaefer's, we derive necessary conditions for the existence, uniqueness and stability of the concerned class of FDE, given in (1.3).

The manuscript is structured as follows: In section 2, we present some basic materials needed to prove our main results. In section 3, we set up some appropriate conditions for the existence and uniqueness of the solutions of the proposed system (1.3) by applying some standard fixed point principles. In section 4, we built up conditions for different kinds of Ulam stability to the solution of the proposed system (1.3). An example illustrating our results is given in section 5.

2. Preliminaries

Let $\mathcal{X} = [0,T]$, we represent the space of all continuous functions $\mathcal{C}(\mathcal{X}, \mathbb{R})$ by $\mathcal{A}$, i.e. $\mathcal{A} = \{p\colon \mathcal{X} \to \mathbb{R};\ p \in \mathcal{C}(\mathcal{X}, \mathbb{R})\}$. Clearly, $\mathcal{A}$ is a Banach space with the norm defined by $\|p\| := \sup\{\|p(t)\|,\ t \in \mathcal{X}\}$. We recall the following definitions from [14].

Definition 2.1
Let $\omega > 0$, then the Riemann–Liouville integral of a function $\mathcal{G} \in L^1([0,T], \mathbb{R})$ is defined by

$$I^\omega \mathcal{G}(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \mathcal{G}(s) ds,$$

provided the integral on the right is point-wise defined on $(0, \infty)$.

Definition 2.2
If $\omega > 0$, then the Caputo fractional derivative of a function $\mathcal{G} \in \mathcal{C}^{(n)}((0,\infty),\mathbb{R})$ is defined by

$$^{c}D^{\omega}\mathcal{G}(t) = \frac{1}{\Gamma(n-\omega)}\int_0^t (t-s)^{n-\omega-1}\mathcal{G}^{(n)}(s)ds,$$

provided the integral on the right is point-wise defined on $(0,\infty)$, where $n = [\omega]+1$ and $[\omega]$ represents the integer part of ω.

Lemma 2.3
For $\omega > 0$ equation $^{c}D^{\omega}\mathcal{G}(t) = 0$ has a solution of the form

$$\mathcal{G}(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_{i-1}t^{i-1},$$

where r_{i-1} are real numbers and $i = 1, 2, \ldots, n$.

Here we mention that in this paper the definitions of stability have been adopted from [21].

Definition 2.4
Problem (1.3) is HUS if there is a real number $C_{\mathcal{G},f} > 0$ such that for each $\epsilon > 0$ and each solution $q \in \mathcal{A}$ of

$$\left|^{c}D^{\omega}q(t) - \mathcal{G}(t,q(t),{}^{c}D^{\omega}q(t)) - \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s,q(s),{}^{c}D^{\omega}q(s))ds\right| \leq \epsilon \quad \text{for all } t \in \mathcal{X}, \tag{2.1}$$

there exists a solution $p \in \mathcal{A}$ of (1.3) with

$$|q(t) - p(t)| \leq C_{\mathcal{G},f}\epsilon \qquad \text{for all } t \in \mathcal{X}.$$

Definition 2.5
Problem (1.3) is generalized HUS (GHUS) if there is a function $\digamma_{\mathcal{G},f} \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\digamma_{\mathcal{G},f}(0) = 0$ such that for each solution $q \in \mathcal{A}$ of (2.1) there exists a solution $p \in \mathcal{A}$ of (1.3) with

$$|q(t) - p(t)| \leq \digamma_{\mathcal{G},f}(\epsilon) \qquad \text{for all } t \in \mathcal{X}.$$

Definition 2.6
Problem (1.3) is Hyers–Ulam–Rassias stable (HURS) with respect to a function $\psi \in \mathcal{C}(\mathcal{X},\mathbb{R}_+)$ if there is a real number $C_{\mathcal{G},f,\psi} > 0$ such that for each solution $q \in \mathcal{A}$ of

$$\left|^{c}D^{\omega}q(t) - \mathcal{G}(t,q(t),{}^{c}D^{\omega}q(t)) - \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s,q(s),{}^{c}D^{\omega}q(s))ds\right| \leq \epsilon\psi(t) \quad \text{for all } t \in \mathcal{X}, \tag{2.2}$$

there is a solution $p \in \mathcal{A}$ to (1.3) with

$$|q(t) - p(t)| \leq C_{\mathcal{G},f}\varepsilon\psi(t) \qquad \text{for all } t \in \mathcal{X}.$$

Definition 2.7

Problem (1.3) is generalized HURS (GHURS) with respect to a function $\psi \in \mathcal{C}(\mathcal{X}, \mathbb{R}_+)$ if there is a real number $C_{\mathcal{G},f,\psi} > 0$ such that for each solution $q \in \mathcal{A}$ of

$$\left| {}^{c}D^{\omega} q(t) - \mathcal{G}(t, q(t), {}^{c}D^{\omega} q(t)) - \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, q(s), {}^{c}D^{\omega} q(s)) ds \right| \le \psi(t) \quad \text{for all } t \in \mathcal{X}, \tag{2.3}$$

there exists a solution $p \in \mathcal{A}$ to (1.3) with

$$|q(t) - p(t)| \le C_{\mathcal{G},f,\psi} \psi(t) \qquad \text{for all } t \in \mathcal{X}.$$

Remark 2.8

It is clear that

(i) Definition 2.4 implies Definition 2.5;

(ii) Definition 2.6 implies Definition 2.7.

Remark 2.9

A function $q \in \mathcal{A}$ is a solution of (2.1) if and only if there exists a function $\Psi \in \mathcal{A}$ (depending on q) such that

(i) $|\Psi(t)| \le \epsilon$ for all $t \in \mathcal{X}$;

(ii) ${}^{c}D^{\omega} q(t) = \mathcal{G}(t, q(t), {}^{c}D^{\omega} q(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, q(s), {}^{c}D^{\omega} q(s)) ds + \Psi(t)$ for all $t \in \mathcal{X}$.

Remark 2.10

A function $q \in \mathcal{A}$ is a solution of (2.3) if and only if there exists a function $\Psi \in \mathcal{A}$ (depending on q) such that

(i) $|\Psi(t)| \le \epsilon\psi(t)$ for all $t \in \mathcal{X}$;

(ii) ${}^{c}D^{\omega} q(t) = \mathcal{G}(t, q(t), {}^{c}D^{\omega} q(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, q(s), {}^{c}D^{\omega} q(s)) ds + \Psi(t)$ for all $t \in \mathcal{X}$.

Theorem 2.11 (Schaefer's fixed point theorem [9, 22])

Let $\mathcal{A}$ be a Banach space, $\mathcal{T} \colon \mathcal{A} \to \mathcal{A}$ is a completely continuous operator and $\mathcal{E} = \{p \in \mathcal{A} : \ p = \xi \mathcal{T} p, \ 0 < \xi < 1\}$ is bounded, then $\mathcal{T}$ has at least one fixed point in $\mathcal{A}$.

3. Existence and uniqueness results

In this section, we set up some adequate conditions for the existence and uniqueness of solution to (1.3).

LEMMA 3.1
The system

$$\begin{cases} {}^cD^\omega p(t) = \mathcal{G}(t, p(t), {}^cD^\omega p(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), {}^cD^\omega p(s))ds \\ \qquad\qquad \text{for all } t \in \mathcal{X},\ 0 < \omega \leq 1, \\ p(0) = -\int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)} \mathcal{F}(s, p(s), {}^cD^\omega p(s))ds, \end{cases} \tag{3.1}$$

has a solution p given by

$$p(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \alpha(s)ds - \frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1} \mathcal{F}(s, p(s), {}^cD^\omega p(s))ds, \tag{3.2}$$

where $\alpha \in \mathcal{A}$ and it is given by

$$\alpha(t) = \mathcal{G}(t, p(t), {}^cD^\omega p(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), {}^cD^\omega p(s))ds.$$

Proof. Let

$${}^cD^\omega p(s) = \alpha(t).$$

Using Lemma 2.3 we have

$$p(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \alpha(t)ds + r_0. \tag{3.3}$$

Applying the given condition, we obtain

$$r_0 = -\frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1} \beta(s)ds, \tag{3.4}$$

where

$$\beta(t) = \mathcal{F}(t, p(t), {}^cD^\omega p(t)).$$

Putting (3.4) in (3.3), we get (3.2).

COROLLARY 3.2
In view of Lemma 3.1, problem (3.1) *has the following solution*

$$p(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \alpha(s)ds - \frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1} \beta(s)ds,$$

where

$$\alpha(t) = \mathcal{G}(t, p(t), {}^cD^\omega p(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), {}^cD^\omega p(s))ds$$

and

$$\beta(t) = \mathcal{F}(t, p(t), {}^cD^\omega p(t)).$$

We use the following notation for convenience

$$\begin{aligned}\mathrm{v}(t) &= \mathcal{G}(t, p(t), {}^{c}D^{\omega}p(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), {}^{c}D^{\omega}p(s))ds,\\ &= \mathcal{G}(t, p(t), \mathrm{v}(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), \mathrm{v}(s))ds\end{aligned}$$

and

$$\mathrm{z}(t) = \mathcal{F}(t, p(t), {}^{c}D^{\omega}p(t)) = \mathcal{F}(t, p(t), \mathrm{z}(t)).$$

Now, in order to study (1.3) using the fixed point theory, we consider an operator $\mathcal{T}\colon \mathcal{A} \to \mathcal{A}$ defined by

$$(\mathcal{T}p)(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1}\mathrm{v}(s)ds - \frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1}\mathrm{z}(s)ds, \tag{3.5}$$

where $\mathrm{v}, \mathrm{z} \in \mathcal{A}$.

The following hypotheses will be used in further results:

(H1) $\mathcal{G}, \mathcal{F}, f\colon \mathcal{X} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions;

(H2) there exist constants $\mathcal{N}_1 > 0$ and $0 < \mathcal{N}_2 < 1$ such that for each $t \in \mathcal{X}$ and for all $\sigma, \overline{\sigma}, \theta, \overline{\theta} \in \mathbb{R}$, the following relation holds

$$|\mathcal{G}(t, \sigma, \theta) - \mathcal{G}(t, \overline{\sigma}, \overline{\theta})| \leq \mathcal{N}_1|\sigma - \overline{\sigma}| + \mathcal{N}_2|\theta - \overline{\theta}|;$$

(H3) there exist constants $\mathcal{N}_3 > 0$ and $0 < \mathcal{N}_4 < 1$ such that for each $t \in \mathcal{X}$ and for all $\sigma, \overline{\sigma}, \theta, \overline{\theta} \in \mathbb{R}$, the following relation holds

$$|\mathcal{F}(t, \sigma, \theta) - \mathcal{F}(t, \overline{\sigma}, \overline{\theta})| \leq \mathcal{N}_3|\sigma - \overline{\sigma}| + \mathcal{N}_4|\theta - \overline{\theta}|;$$

(H4) there exist constants $\mathcal{N}_5 > 0$ and $0 < \mathcal{N}_6 < 1$ such that for each $t \in \mathcal{X}$ and for all $\sigma, \overline{\sigma}, \theta, \overline{\theta} \in \mathbb{R}$, the following relation holds

$$|f(t, \sigma, \theta) - f(t, \overline{\sigma}, \overline{\theta})| \leq \mathcal{N}_5|\sigma - \overline{\sigma}| + \mathcal{N}_6|\theta - \overline{\theta}|;$$

(H5) there exist bounded functions $l, m, n \in \mathcal{C}(\mathcal{X}, \mathbb{R}^+)$ such that

$$|\mathcal{G}(t, \sigma(t), \theta(t))| \leq l(t) + m(t)\|\sigma\| + n(t)\|\theta\|$$

with $n^* = \sup_{t\in\mathcal{X}} n(t) < 1$;

(H6) there exist bounded functions $b, c, e \in \mathcal{C}(\mathcal{X}, \mathbb{R}^+)$ such that

$$|\mathcal{F}(t, \sigma(t), \theta(t))| \leq b(t) + c(t)\|\sigma\| + e(t)\|\theta\|$$

with $e^* = \sup_{t\in\mathcal{X}} n(t) < 1$;

(H7) there exist bounded functions $i, j, k \in \mathcal{C}(\mathcal{X}, \mathbb{R}^+)$ such that

$$|f(t, \sigma(t), \theta(t))| \leq i(t) + j(t)\|\sigma\| + k(t)\|\theta\|$$

with $k^* = \sup_{t\in\mathcal{X}} n(t) < 1$.

Theorem 3.3
If the hypotheses (H1)–(H4) *and the inequality*

$$\frac{T^{\omega}}{\Gamma(\omega+1)}\Big(\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5T^{\sigma}}{\sigma\Gamma(\delta)\big(1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}\big)}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big)<1 \tag{3.6}$$

are satisfied, then (3.1) *has a unique solution.*

Proof. Consider the operator $\mathcal{T}$ defined in (3.5). We have to show that (3.1) has a unique solution. We use the Banach contraction mapping principle. Consider for $p,q\in\mathcal{A}$,

$$\begin{aligned}|\mathcal{T}p(t)-\mathcal{T}q(t)|\leq&\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}|\mathrm{v}(s)-\overline{\mathrm{v}}(s)|ds\\&+\frac{1}{\Gamma(\omega)}\int_0^T(t-s)^{\omega-1}|\mathrm{z}(s)-\overline{\mathrm{z}}(s)|ds,\end{aligned}\tag{3.7}$$

where $\overline{\mathrm{v}},\overline{\mathrm{z}}\in\mathcal{A}$ are given by

$$\overline{\mathrm{v}}(t)=\mathcal{G}(t,q(t),\overline{\mathrm{v}}(t))+\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\delta)}f(s,q(s),\overline{\mathrm{v}}(s))ds$$

and

$$\overline{\mathrm{z}}(t)=\mathcal{F}(t,q(t),\overline{\mathrm{z}}(t)).$$

Using (H2)–(H4) we have

$$\begin{aligned}|\mathrm{v}(t)-\overline{\mathrm{v}}(t)|=&\Big|\mathcal{G}(t,p(t),\mathrm{v}(t))+\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\delta)}f(s,p(s),\mathrm{v}(s))ds\\&-\mathcal{G}(t,q(t),\overline{\mathrm{v}}(t))-\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\delta)}f(s,q(s),\overline{\mathrm{v}}(s))ds\Big|\\\leq&|\mathcal{G}(t,p(t),\mathrm{v}(t))-\mathcal{G}(t,q(t),\overline{\mathrm{v}}(t))|\\&+\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\delta)}|f(s,p(s),\mathrm{v}(s))-f(s,q(s),\overline{\mathrm{v}}(s))|ds\\\leq&\mathcal{N}_1|p(t)-q(t)|+\mathcal{N}_2|\mathrm{v}(t)-\overline{\mathrm{v}}(t)|\\&+\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\delta)}\Big(\mathcal{N}_5|p(s)-q(s)|+\mathcal{N}_6|\mathrm{v}(s)-\overline{\mathrm{v}}(s)|\Big)ds\\=&\mathcal{N}_1|p(t)-q(t)|+\mathcal{N}_2|\mathrm{v}(t)-\overline{\mathrm{v}}(t)|\\&+\frac{t^{\sigma}}{\sigma\Gamma(\delta)}\mathcal{N}_5|p(t)-q(t)|+\frac{t^{\sigma}}{\sigma\Gamma(\delta)}\mathcal{N}_6|\mathrm{v}(t)-\overline{\mathrm{v}}(t)|.\end{aligned}$$

Thus

$$\begin{aligned}|\mathrm{v}(t)-\overline{\mathrm{v}}(t)|\leq&\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)}}|p(t)-q(t)|\\&+\frac{\mathcal{N}_5t^{\sigma}}{\sigma\Gamma(\delta)\big(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)}\big)}|p(t)-q(t)|.\end{aligned}\tag{3.8}$$

Similarly,

$$|\mathrm{z}(t) - \overline{\mathrm{z}}(t)| \leq \frac{\mathcal{N}_3}{1-\mathcal{N}_4}|p(t) - q(t)|. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7) we have

$$\begin{aligned}|\mathcal{T}p(t) - \mathcal{T}q(t)| \leq \Big[&\frac{t^{\omega}}{\Gamma(\omega+1)}\Big(\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathcal{N}_5 t^{\sigma}}{\sigma\Gamma(\delta)\big(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)}\big)}\Big)\\ &+ \frac{T^{\omega}}{\Gamma(\omega+1)}\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big]|p(t)-q(t)|.\end{aligned}$$

Since $t \in [0,T] \Rightarrow t \leq T \Rightarrow t^{\omega} \leq T^{\omega}$ we get

$$\begin{aligned}|\mathcal{T}p(t) - \mathcal{T}q(t)| \leq \Big[&\frac{T^{\omega}}{\Gamma(\omega+1)}\Big(\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathcal{N}_5 T^{\sigma}}{\sigma\Gamma(\delta)\big(1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}\big)}\Big)\\ &+ \frac{T^{\omega}}{\Gamma(\omega+1)}\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big]|p(t)-q(t)|.\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{T}p - \mathcal{T}q\|_{\mathcal{A}} \leq \frac{T^{\omega}}{\Gamma(\omega+1)}\Big(&\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}}\\ &+ \frac{\mathcal{N}_5 T^{\sigma}}{\sigma\Gamma(\delta)\big(1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}\big)} + \frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big)\|p-q\|_{\mathcal{A}}.\end{aligned}$$

Moreover,

$$\frac{T^{\omega}}{\Gamma(\omega+1)}\Big(\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathcal{N}_5 T^{\sigma}}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{T^{\sigma}}{\sigma\Gamma(\delta)})} + \frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big) < 1,$$

Therefore, by the Banach contraction principle, $\mathcal{T}$ has a unique fixed point. Thus (3.1) has a unique solution.

Theorem 3.4
Under the hypotheses (H1)–(H7), *problem* (3.1) *has at least one solution.*

Proof. We begin with recalling the Schaefer's fixed point theorem and consider the predefined operator $\mathcal{T}$. The proof accomplishes in four steps.
Step 1: We claim that $\mathcal{T}$ is continuous. Consider a sequence $\{p_n\}$ in $\mathcal{A}$ such that $p_n \to p \in \mathcal{A}$. For $t \in \mathcal{X}$ we have

$$\begin{aligned}|\mathcal{T}p_n(t) - \mathcal{T}p(t)| \leq &\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}|\mathrm{v}_n(s) - \mathrm{v}(s)|ds\\ &+ \frac{1}{\Gamma(\omega)}\int_0^T (T-s)^{\omega-1}|\mathrm{z}_n(s) - \mathrm{z}(s)|ds,\end{aligned}$$

where $\mathrm{v}_n, \mathrm{z}_n \in \mathcal{A}$ are given by

$$\mathrm{v}_n = \mathcal{G}(t, p_n(t), \mathrm{v}_n(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), \mathrm{v}_n(t))ds$$

and

$$\mathrm{z}_n = \mathcal{F}(t, p_n(t), \mathrm{z}_n(t)).$$

Hence by (H2)–(H4) we obtain

$$\begin{aligned}|\mathrm{v}_n(t) - \mathrm{v}(t)| \leq & \frac{\mathcal{N}_1}{1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}} |p_n(t) - p(t)| \\ & + \frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)\big(1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}\big)} |p_n(t) - p(t)|.\end{aligned}$$

Similarly,

$$|\mathrm{z}_n(t) - \mathrm{z}(t)| \leq \frac{\mathcal{N}_3}{1 - \mathcal{N}_4} |p_n(t) - p(t)|.$$

Thus

$$\begin{aligned}|\mathcal{T}p_n(t) - \mathcal{T}p(t)| \leq & \Big[\frac{t^\omega}{\Gamma(\omega+1)}\Big(\frac{\mathcal{N}_1}{1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)\big(1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}\big)}\Big) \\ & + \frac{T^\omega}{\Gamma(\omega+1)} \frac{\mathcal{N}_3}{1 - \mathcal{N}_4}\Big] |p(t) - q(t)|.\end{aligned}$$

Since for each $t \in \mathcal{X}$ the sequence $p_n \to p$ as $n \to \infty$, we have, by Lebesgue dominated convergence theorem,

$$|\mathcal{T}p_n(t) - \mathcal{T}p(t)| \to 0 \qquad \text{as } n \to \infty,$$

or

$$\|\mathcal{T}p_n - \mathcal{T}p\| \to 0 \qquad \text{as } n \to \infty.$$

Which implies that $\mathcal{T}$ is continuous on $\mathcal{X}$.
Step 2: In this step we claim that bounded sets in $\mathcal{A}$ are mapped into bounded sets in $\mathcal{A}$ by $\mathcal{T}$. Next for each $p \in \varepsilon_k = \{p \in \mathcal{A} : \ \|p\| \leq k\}$ we have to prove $\|\mathcal{T}(p)\| \leq N$ with some $N > 0$. For $t \in \mathcal{X}$, we have

$$\begin{aligned}|\mathcal{T}p(t)| &= \Big|\frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \mathrm{v}(s)ds - \frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1} \mathrm{z}(s)ds\Big| \\ &\leq \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} |\mathrm{v}(s)|ds - \frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1} |\mathrm{z}(s)|ds,\end{aligned} \tag{3.10}$$

where $\mathrm{v}, \mathrm{z} \in \mathcal{A}$ are given by

$$\mathrm{v}(t) = \mathcal{G}(t, p(t), \mathrm{v}(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), \mathrm{v}(s))ds$$

and

$$z(t) = \mathcal{F}(t, p(t), z(t)).$$

By (H5) and (H7) we have

$$v(t) = \mathcal{G}(t, p(t), v(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), v(s)) ds,$$

thus

$$\begin{aligned}
|v(t)| &= \Big|\mathcal{G}(t, p(t), v(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), v(s)) ds\Big| \\
&\le |\mathcal{G}(t, p(t), v(t))| + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} |f(s, p(s), v(s))| ds \\
&\le l(t) + m(t)|p| + n(t)|v(t)| \\
&\quad + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\sigma-1} (i(s) + j(s)|p| + k(s)|v(s)|) ds \\
&\le l^* + m^* \|p\|_{\mathcal{A}} + n^* \|v\|_{\mathcal{A}} + (i^* + j^* \|p\|_{\mathcal{A}} + k^* \|v\|_{\mathcal{A}}) \frac{t^\sigma}{\sigma \Gamma(\delta)},
\end{aligned}$$

where

$$\begin{aligned}
l^* &= \sup_{t \in \mathcal{X}} l(t), & m^* &= \sup_{t \in \mathcal{X}} m(t), & n^* &= \sup_{t \in \mathcal{X}} n(t) < 1, \\
i^* &= \sup_{t \in \mathcal{X}} i(t), & j^* &= \sup_{t \in \mathcal{X}} j(t), & k^* &= \sup_{t \in \mathcal{X}} k(t) < 1.
\end{aligned}$$

Thus

$$|v(t)| \le \|v\|_{\mathcal{A}} \le \frac{l^* + m^* \|p\|_{\mathcal{A}}}{1 - n^* - k^* \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{i^* + j^* \|p\|_{\mathcal{A}}}{1 - n^* - k^* \frac{t^\sigma}{\sigma \Gamma(\delta)}} \frac{t^\sigma}{\sigma \Gamma(\delta)} =: \hbar.$$

Similarly, by (H6) we obtain

$$|z(t)| \le \frac{b^* + c^* k}{1 - e^*} =: \hbar^*,$$

where $\hbar$ and $\hbar^*$ are positive constants. Thus from (3.10) we have

$$\|\mathcal{T} p\|_{\mathcal{A}} = \frac{T^\omega}{\Gamma(\omega+1)} (\hbar + \hbar^*) =: N.$$

Step 3: We claim that a bounded set is mapped into equi–continuous set of $\mathcal{A}$ by $\mathcal{T}$. Take $t_1, t_2 \in \mathcal{X}$ such that $t_1 < t_2$ and assume that ε_k is a bounded set as in the previous step. Then for $p \in \varepsilon_k$ we have

$$|\mathcal{T} p(t_2) - \mathcal{T} p(t_1)| = \Big|\frac{1}{\Gamma(\omega)} \int_0^{t_2} (t_2 - s)^{\omega-1} v(s) ds - \frac{1}{\Gamma(\omega)} \int_0^{t_1} (t_1 - s)^{\omega-1} v(s) ds\Big|.$$

In Step 2 we obtained that

$$|\mathrm{v}(t)| \leq \|\mathrm{v}\|_{\mathcal{A}} \leq \frac{l^* + m^*\|p\|_{\mathcal{A}}}{1 - n^* - k^* \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{i^* + j^*\|p\|_{\mathcal{A}}}{1 - n^* - k^* \frac{t^\sigma}{\sigma\Gamma(\delta)}} \frac{t^\sigma}{\sigma\Gamma(\delta)} =: \hbar.$$

Thus

$$|\mathcal{T}p(t_2) - \mathcal{T}p(t_1)| \leq \hbar \Big| \frac{1}{\Gamma(\omega)} \int_0^{t_2} (t_2 - s)^{\omega - 1} ds - \frac{1}{\Gamma(\omega)} \int_0^{t_1} (t_1 - s)^{\omega - 1} ds \Big|. \quad (3.11)$$

We see that the right hand side of (3.11) tends to zero as $t_1 \to t_2$. Therefore, as a conclusion from Step 1–Step 3 and the Arzela–Ascoli theorem, $\mathcal{T} \colon \mathcal{A} \to \mathcal{A}$ is a completely continuous mapping.

Step 4: Define

$$\mathcal{L} = \{p \in \mathcal{A} : \ p = \varpi(\mathcal{T}p) \text{ for some } 0 < \varpi < 1\}.$$

We need to show that $\mathcal{L}$ is bounded. Let $p \in \mathcal{L}$, then for some $0 < \mathcal{L} < 1$ with $p = \mathcal{L}(\mathcal{T}p)$ we have

$$\begin{aligned} |p(t)| &= \Big| \frac{\varpi}{\Gamma(\omega)} \int_0^t (t - s)^{\omega - 1} \mathrm{v}(s) ds - \frac{\varpi}{\Gamma(\omega)} \int_0^T (T - s)^{\omega - 1} \mathrm{z}(s) ds \Big| \\ &\leq \Big| \frac{\varpi}{\Gamma(\omega)} \int_0^t (t - s)^{\omega - 1} \mathrm{v}(s) ds \Big| + \Big| \frac{\varpi}{\Gamma(\omega)} \int_0^T (T - s)^{\omega - 1} \mathrm{z}(s) ds \Big| \end{aligned}$$

or

$$|p(t)| \leq \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega - 1} |\mathrm{v}(s)| ds + \frac{1}{\Gamma(\omega)} \int_0^T (T - s)^{\omega - 1} |\mathrm{z}(s)| ds. \quad (3.12)$$

By (H5)–(H7) we get that

$$|\mathrm{v}(t)| \leq \|\mathrm{v}\|_{\mathcal{A}} \leq \frac{l^* + m^*\|p\|_{\mathcal{A}}}{1 - n^* - k^* \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{i^* + j^*\|p\|_{\mathcal{A}}}{1 - n^* - k^* \frac{t^\sigma}{\sigma\Gamma(\delta)}} \frac{t^\sigma}{\sigma\Gamma(\delta)} =: \hbar$$

and

$$|\mathrm{z}(t)| \leq \|\mathrm{z}\|_{\mathcal{A}} \leq \frac{b^* + c^* k}{1 - e^*} =: \hbar^*.$$

Thus from (3.12)

$$\begin{aligned} |p(t)| &\leq \frac{\hbar}{\Gamma(\omega)} \int_0^t (t - s)^{\omega - 1} ds + \frac{\hbar^*}{\Gamma(\omega)} \int_0^T (T - s)^{\omega - 1} ds \\ &\leq \frac{T^\omega}{\Gamma(\omega + 1)} (\hbar + \hbar^*) =: N, \end{aligned}$$

i.e $|p(t)| \leq N$. This shows that the set $\mathcal{L}$ is bounded. Therefore, by the Schaefer's fixed point theorem, $\mathcal{T}$ has at least one fixed point. This confirms at least one exact solution of (3.1).

4. Ulam stability results

In this section, we are analyzing the HUS, GHUS, HURS and GHURS of the considered anti–periodic integral boundary value problem (1.3).

THEOREM 4.1
If the hypotheses (H1)–(H4) *along with* (3.6) *are satisfied, then* (3.1) *is HUS as well as GHUS.*

Proof. Let q be an approximate solution of (2.1) and let p be the unique exact solution of the following problem

$$\begin{cases} {}^{c}D^{\omega}p(t) = \mathcal{G}(t,p(t),{}^{c}D^{\omega}p(t)) + \displaystyle\int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s,p(s),{}^{c}D^{\omega}p(s))ds; \\ \qquad\qquad \text{for all } \ t\in\mathcal{X},\ 0<\omega\leq 1, \\ p(0) = -\displaystyle\int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)} \mathcal{F}(s,p(s),{}^{c}D^{\omega}p(s))ds. \end{cases}$$

By Lemma 3.1 we have

$$p(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\mathrm{v}(s)ds - \frac{1}{\Gamma(\omega)}\int_0^T (T-s)^{\omega-1}\mathrm{z}(s)ds,$$

where $\mathrm{v},\mathrm{z}\in\mathcal{A}$ are given by

$$\mathrm{v}(t) = \mathcal{G}(t,p(t),\mathrm{v}(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s,p(s),\mathrm{v}(s))ds$$

and

$$\mathrm{z}(t) = \mathcal{F}(t,p(t),\mathrm{z}(t)).$$

Since we have assumed that q is a solution to (2.1), by Remark 2.9 we have

$$\begin{cases} {}^{c}D^{\omega}q(t) = \mathcal{G}(t,q(t),{}^{c}D^{\omega}q(t)) + \displaystyle\int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s,q(s),{}^{c}D^{\omega}q(s))ds + \Psi(t) \\ \qquad\qquad \text{for } 0<\omega\leq 1, \\ q(0) = -\displaystyle\int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)} \mathcal{F}(s,q(s),{}^{c}D^{\omega}q(s))ds. \end{cases} \tag{4.1}$$

Clearly, the solution of (4.1) will be

$$q(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\mathrm{v}(s)ds + \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\Psi(s)ds - \frac{1}{\Gamma(\omega)}\int_0^T (T-s)^{\omega-1}\mathrm{z}(s)ds,$$

where $\overline{\mathrm{v}}, \overline{\mathrm{z}} \in \mathcal{A}$ are given as

$$\overline{\mathrm{v}}(t) = \mathcal{G}(t, q(t), \overline{\mathrm{v}}(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), \overline{\mathrm{v}}(s)) ds$$

and

$$\overline{\mathrm{z}}(t) = \mathcal{F}(t, q(t), \overline{\mathrm{z}}(t)).$$

For each $t \in \mathcal{X}$, we have

$$\begin{aligned} |q(t) - p(t)| \le & \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} |\overline{\mathrm{v}}(s) - \mathrm{v}(s)| ds \\ & + \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} |\Psi(s)| ds \\ & + \frac{1}{\Gamma(\omega)} \int_0^T (T-s)^{\omega-1} |\overline{\mathrm{z}}(s) - \mathrm{z}(s)| ds. \end{aligned} \tag{4.2}$$

By (H2)–(H4) we get

$$\begin{aligned} |\overline{\mathrm{v}}(t) - \mathrm{v}(t)| \le & \frac{\mathcal{N}_1}{1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}} |q(t) - p(t)| \\ & + \frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)\big(1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}\big)} |q(t) - p(t)| \end{aligned}$$

and

$$|\overline{\mathrm{z}}(t) - \mathrm{z}(t)| \le \frac{\mathcal{N}_3}{1 - \mathcal{N}_4} |q(t) - p(t)|.$$

Using part (i) of Remark 2.9 in (4.2) we get

$$\begin{aligned} & |q(t) - p(t)| \\ & \quad \le \frac{t^\omega}{\Gamma(\omega+1)} \Big[\frac{\mathcal{N}_1}{1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)\big(1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}\big)} \Big] |q(t) - p(t)| \\ & \qquad + \frac{t^\omega}{\Gamma(\omega+1)} |\psi(t)| + \frac{T^\omega}{\Gamma(\omega+1)} \frac{\mathcal{N}_3}{1 - \mathcal{N}_4} |q(t) - p(t)| \\ & \quad \le \frac{T^\omega |q(t) - p(t)|}{\Gamma(\omega+1)} \Big[\frac{\mathcal{N}_1}{1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)\big(1 - \mathcal{N}_2 - \mathcal{N}_6 \frac{t^\sigma}{\sigma\Gamma(\delta)}\big)} + \frac{\mathcal{N}_3}{1 - \mathcal{N}_4} \Big] \\ & \qquad + \frac{T^\omega}{\Gamma(\omega+1)} \epsilon. \end{aligned}$$

Thus

$$\|q-p\|_{\mathcal{A}} \leq \frac{\frac{\epsilon T^{\omega}}{\Gamma(\omega+1)}}{1-\frac{T^{\omega}}{\Gamma(\omega+1)}\left[\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5 t^{\sigma}}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)})}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\right]},$$

i.e

$$\|q-p\|_{\mathcal{A}} \leq \epsilon C_{\mathcal{G},f},$$

where

$$C_{\mathcal{G},f} = \frac{\frac{T^{\omega}}{\Gamma(\omega+1)}}{1-\frac{T^{\omega}}{\Gamma(\omega+1)}\left[\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5 t^{\sigma}}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^{\sigma}}{\sigma\Gamma(\delta)})}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\right]}.$$

Therefore, (3.1) is HUS. Furthermore, if we setiing $\digamma_{\mathcal{G}}(\epsilon) = C_{\mathcal{G}}(\epsilon)$, $\digamma(0) = 0$, we see that (3.1) is GHUS.

For the proof of our next result we assume that:

(H8) there exists a nondecreasing function $\psi \in \mathcal{C}(\mathcal{X}, \mathbb{R}_+)$ and a constant $\mathcal{L}_{\psi} > 0$ such that

$$I^{\omega}\psi(t) \leq \mathcal{L}_{\psi}\psi(t) \qquad \text{for all } t \in \mathcal{X}.$$

Theorem 4.2
Assume (H1)–(H8) *along with* (3.6) *are satisfied, then* (3.1) *is HURS and consequently it is GHURS.*

Proof. Let q be an approximate solution of (2.3) and p be the unique solution of the following problem

$$\begin{cases} {}^cD^{\omega}p(t) = \mathcal{G}(t, p(t), {}^cD^{\omega}p(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), {}^cD^{\omega}p(s))ds \\ \qquad\qquad \text{for all } t \in \mathcal{X},\ 0 < \omega \leq 1, \\ p(0) = -\int_0^T \frac{(T-s)^{\omega-1}}{\Gamma(\omega)} \mathcal{F}(s, p(s), {}^cD^{\omega}p(s))ds. \end{cases}$$

By Lemma 3.1 we have

$$p(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\mathrm{v}(s)ds - \frac{1}{\Gamma(\omega)}\int_0^T (T-s)^{\omega-1}\mathrm{z}(s)ds,$$

where $\mathrm{v}, \mathrm{z} \in \mathcal{A}$ are given by

$$\mathrm{v}(t) = \mathcal{G}(t, p(t), \mathrm{v}(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s, p(s), \mathrm{v}(s))ds$$

and

$$\mathrm{z}(t) = \mathcal{F}(t, p(t), \mathrm{z}(t)).$$

From the proof of Theorem 4.1 it follows that for each $t \in \mathcal{X}$ we have

$$\begin{aligned}|q(t)-p(t)| \leq &\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}|\overline{\mathrm{v}}(s)-\mathrm{v}(s)|ds\\ &+\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}|\Psi(s)|ds \\ &+\frac{1}{\Gamma(\omega)}\int_0^T (T-s)^{\omega-1}|\overline{\mathrm{z}}(s)-\mathrm{z}(s)|ds.\end{aligned} \tag{4.3}$$

By (H2)–(H4) we get

$$\begin{aligned}|\overline{\mathrm{v}}(t)-\mathrm{v}(t)| \leq &\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)}}|q(t)-p(t)|\\ &+\frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)})}|q(t)-p(t)|\end{aligned}$$

and

$$|\overline{\mathrm{z}}(t)-\mathrm{z}(t)| \leq \frac{\mathcal{N}_3}{1-\mathcal{N}_4}|q(t)-p(t)|.$$

Thus using the last two inequalities and part (i) of Remark 2.10 in (4.3) we have

$$\begin{aligned}&|q(t)-p(t)|\\ &\leq \frac{t^\omega}{\Gamma(\omega+1)}\Big[\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)\big(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)}\big)}\Big]|q(t)-p(t)|\\ &+\frac{t^\omega}{\Gamma(\omega+1)}|\psi(t)|+\frac{T^\omega}{\Gamma(\omega+1)}\frac{\mathcal{N}_3}{1-\mathcal{N}_4}|q(t)-p(t)|\\ &\leq \frac{T^\omega|q(t)-p(t)|}{\Gamma(\omega+1)}\Big[\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)})}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big]\\ &+\frac{T^\omega}{\Gamma(\omega+1)}\epsilon\mathcal{L}_\psi\psi(t).\end{aligned}$$

Thus

$$\|q-p\|_{\mathcal{A}} \leq \frac{\epsilon\mathcal{L}_\psi\psi(t)}{1-\frac{T^\omega}{\Gamma(\omega+1)}\Big[\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)})}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big]},$$

i.e

$$\|q-p\|_{\mathcal{A}} \leq C_{\mathcal{G},f}\epsilon,$$

where

$$C_{\mathcal{G},f} = \frac{\mathcal{L}_\psi\psi(t)}{1-\frac{T^\omega}{\Gamma(\omega+1)}\Big[\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5 t^\sigma}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{t^\sigma}{\sigma\Gamma(\delta)})}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big]}.$$

Therefore, (3.1) is HURS. Along the same lines it is easy to check that the problem under consideration is GHURS.

5. EXAMPLE

In this section, we are illustrating our theoretical results by an example.

Example 5.1

$$\begin{cases} {}^cD^{\frac{1}{2}}p(t) = \dfrac{7+|p(t)|+|{}^cD^{\frac{1}{2}}\,{}^cD^{\frac{1}{2}}p(t)|}{105e^{t+3}(1+|p(t)|+|{}^cD^{\frac{1}{2}}p(t)|)} \\ \qquad\qquad + \dfrac{1}{\Gamma(\frac{5}{2})}\displaystyle\int_0^1 (t-s)^{\frac{3}{2}}\Big(\dfrac{s\sin|p(s)|+\sin|{}^cD^{\frac{1}{2}}p(s)|}{50}\Big)ds, \quad t\in[0,1], \\ p(0) = -\dfrac{1}{\Gamma(\frac{1}{2})}\displaystyle\int_0^1 (t-s)^{\frac{1}{2}}\Big(\dfrac{s\sin|p(s)|+\sin|{}^cD^{\frac{1}{2}}p(s)|}{50}\Big)ds. \end{cases} \tag{5.1}$$

From the anti–periodic integral problem (5.1), we see that $\omega=\frac{1}{2}$, $T=1$, $\delta=\sigma=\frac{5}{2}$. Set

$$\begin{aligned} \mathcal{G}(t,\acute{\sigma},\theta) &= \frac{7+|\acute{\sigma}|+|\theta|}{105e^{t+3}(1+|\acute{\sigma}|+|\theta|)}, \qquad \acute{\sigma}\in\mathcal{C}(\mathcal{X},\mathbb{R}). \\ f(t,\acute{\sigma},\theta) &= \frac{t\sin|\acute{\sigma}|+\sin|\theta|}{50}, \\ \mathcal{F}(t,\acute{\sigma},\theta) &= \frac{t\sin|\acute{\sigma}|+\sin|\theta|}{50}. \end{aligned}$$

Clearly, the functions $\mathcal{G}$, f, $\mathcal{F}$ are continuous. For each $\acute{\sigma},\overline{\acute{\sigma}}\in\mathcal{A}$, $\theta,\overline{\theta}\in\mathbb{R}$ and $t\in[0,1]$ we have

$$|\mathcal{G}(t,\acute{\sigma},\theta)-\mathcal{G}(t,\overline{\acute{\sigma}},\overline{\theta})| \le \frac{|\acute{\sigma}-\overline{\acute{\sigma}}|+|\theta-\overline{\theta}|}{105e^3},$$

which satisfies (H2) with $\mathcal{N}_1=\mathcal{N}_2=\frac{1}{105e^3}$.

Observe that

$$|f(t,\acute{\sigma},\theta)-f(t,\overline{\acute{\sigma}},\overline{\theta})| \le \frac{|\acute{\sigma}-\overline{\acute{\sigma}}|+|\theta-\overline{\theta}|}{50},$$

satisfies (H4) with $\mathcal{N}_5=\mathcal{N}_6=\frac{1}{50}$ and

$$|\mathcal{F}(t,\acute{\sigma},\theta)-\mathcal{F}(t,\overline{\acute{\sigma}},\overline{\theta})| \le \frac{|\acute{\sigma}-\overline{\acute{\sigma}}|+|\theta-\overline{\theta}|}{50},$$

satisfies (H3) with $\mathcal{N}_3=\mathcal{N}_4=\frac{1}{50}$. Hence

$$\frac{T^\omega}{\Gamma(\omega+1)}\Big(\frac{\mathcal{N}_1}{1-\mathcal{N}_2-\mathcal{N}_6\frac{T^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathcal{N}_5T^\sigma}{\sigma\Gamma(\delta)(1-\mathcal{N}_2-\mathcal{N}_6\frac{T^\sigma}{\sigma\Gamma(\delta)})}+\frac{\mathcal{N}_3}{1-\mathcal{N}_4}\Big)\approx 0.09906.$$

We see, all the required conditions of Theorem 3.3 are fulfilled, hence (5.1) has at least one solution. Also by letting $\psi(t) = |t|$ for all $t \in \mathcal{X}$ we have

$$I^{\frac{1}{2}}\psi(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{(\frac{1}{2}-1)}|s|ds = \frac{4^{\frac{3}{2}}}{3\sqrt{\pi}} \leq \frac{2t}{\sqrt{\pi}}.$$

Hence (H8) is satisfied with $\mathcal{L}_\psi = \frac{2}{\sqrt{\pi}}$. Therefore, by Theorem 4.2 the given problem is HURS and consequently is GHURS.

6. CONCLUSION

We have derived some necessary conditions for the existence, uniqueness and different kinds of stability in the sense of Ulam for the solution of implicit FDE with an implicit integral boundary condition. We have successfully obtained some appropriate and sufficient conditions which guarantee the uniqueness, existence of at least one solution by means of the Banach contraction principle and the Arzela–Ascoli theorem and its Hyers–Ulam stability analysis to a class of nonlinear implicit FDE with an implicit anti–periodic integral boundary condition. For the justification, we have presented an example which supported the main theoretical results.

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Akbar Zada
Department of Mathematics
University of Peshawar
Peshawar 25000
Pakistan
E-mail: zadababo@yahoo.com;
akbarzada@uop.edu.pk

Hira Waheed
Department of Mathematics
University of Peshawar
Peshawar 25000
Pakistan
E-mail: hirams2017@gmail.com

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