Ann. Univ. Paedagog. Crac. Stud. Math. 18 (2019), 123-136
DOI: 10.2478/aupcsm-2019-0009

FOLIA 277

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XVIII (2019)

Adara M. Blaga, Kanak Kanti Baishya and Nihar Sarkar

Ricci solitons in a generalized weakly (Ricci) symmetric D-homothetically deformed Kenmotsu manifold

Abstract. The object of the present paper is to investigate the nature of Ricci solitons on D-homothetically deformed Kenmotsu manifold with generalized weakly symmetric and generalized weakly Ricci symmetric curvature restrictions.

1. Introduction

In this paper, we consider an almost contact metric manifold

$$(M^{2n+1}, \phi, \xi, \eta, g)$$

that consists of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η called respectively the structure endomorphism, the characteristic vector field and the contact form. Let the symbols ∇ and ∇^* stand for the Levi-Civita connection and the D-homothetically deformed connection respectively. Also, let R, S, Q, r, $\bar{R}$ and R^*, S^*, Q^*, r^*, $\bar{R}^*$ respectively stand for Riemannian curvature tensor, Ricci tensor, Ricci operator, scalar curvature, Riemannian curvature tensor of type $(0,4)$ with respect to ∇ and ∇^* respectively. In the recent paper [4], the present author has introduced a new type of space called generalized weakly symmetric manifold. In Section 3 of this paper we extend this concept to a D-homothetically deformed structure of a $(2n+1)$-dimensional Kenmotsu manifold.

According to [4], a Riemannian manifold is said to be *generalized weakly symmetric* if it satisfies the condition

AMS (2010) Subject Classification: 53C15, 53C25.
Keywords and phrases: Kenmotsu manifold, D-homothetic deformation, generalized weakly symmetric manifold, generalized weakly Ricci symmetric manifold, Ricci solitons.

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,W) = {} & \alpha_1(X)\bar{R}(Y,U,V,W) + \beta_1(Y)\bar{R}(X,U,V,W) \\ & + \beta_1(U)\bar{R}(Y,X,V,W) + \delta_1(V)\bar{R}(Y,U,X,W) \\ & + \delta_1(W)\bar{R}(Y,U,V,X) + \alpha_2(X)\bar{G}(Y,U,V,W) \\ & + \beta_2(Y)\bar{G}(X,U,V,W) + \beta_2(U)\ \bar{G}(Y,X,V,W) \\ & + \delta_2(V)\ \bar{G}(Y,U,X,W) + \delta_2(W)\ \bar{G}(Y,U,V,X),\end{aligned}$$

where

$$\bar{G}(Y,U,V,W) = g(U,V)g(Y,W) - g(Y,V)g(U,W)$$

and α_i, β_i, δ_i are non-zero 1-forms defined by $\alpha_i(X) = g(X,\sigma_i)$, $\beta_i(X) = g(X,\varrho_i)$, and $\delta_i(X) = g(X,\pi_i)$ for $i = 1,2$. The magnificence of such generalized weakly symmetric manifold is that it has the flavour of

(i) locally symmetric space [14] (for $\alpha_i = \beta_i = \delta_i = 0$),

(ii) recurrent space [25] (for $\alpha_1 \neq 0$, $\alpha_2 = \beta_i = \delta_i = 0$),

(iii) generalized recurrent space [16] (for $\alpha_i \neq 0$, $\beta_i = \delta_i = 0$),

(iv) pseudo symmetric space [15] (for $\frac{\alpha_1}{2} = \beta_1 = \delta_1 = H_1 \neq 0$, $\alpha_2 = \beta_2 = \delta_2 = 0$),

(v) generalized pseudo symmetric space [2] (for $\frac{\alpha_i}{2} = \beta_i = \delta_i = H_i \neq 0$),

(vi) semi-pseudo symmetric space [24] (for $\alpha_i = \beta_2 = \delta_2 = 0, \beta_1 = \delta_1 \neq 0$),

(vii) generalized semi-pseudo symmetric space [3] (for $\alpha_i = 0, \beta_i = \delta_i \neq 0$),

(viii) almost pseudo symmetric space [15] (for $\alpha_1 = H_1 + K_1, \beta_1 = \delta_1 = H_1 \neq 0$ and $\alpha_2 = \beta_2 = \delta_2 = 0$),

(ix) almost generalized pseudo symmetric space ([7], [8]) ($\alpha_i = H_i + K_i, \beta_i = \delta_i = H_i \neq 0$),

(x) weakly symmetric space [23] (for α_1, β_1, $\delta_1 \neq 0$, $\alpha_2 = \beta_2 = \delta_2 = 0$).

A Riemannian manifold is said to be *generalized weakly Ricci symmetric* [4] if it satisfies the condition

$$\begin{aligned}(\nabla_X S)(Y,Z) = {} & A_1(X)S(Y,Z) + B_1(Y)S(X,Z) + D_1(Z)S(Y,X) \\ & + A_2(X)g(Y,Z) + B_2(Y)g(X,Z) + D_2(Z)g(Y,X),\end{aligned}$$

where A_i, B_i and D_i are non-zero 1-forms defined by $A_i(X) = g(X,\theta_i)$, $B_i(X) = g(X,\phi_i)$ and $D_i(X) = g(X,\pi_i)$ for $i = 1,2$. The beauty of the generalized weakly Ricci symmetric manifold is that it has the flavor of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo Ricci symmetric, generalized pseudo Ricci symmetric, semi-pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric [15], almost generalized pseudo Ricci symmetric and weakly Ricci symmetric space as special cases.

Ricci solitons were introduced by Hamilton [18]. An important topic in contact metric geometry is the study of Ricci flow and Ricci solitons. A Riemannian manifold admits a Ricci soliton [19] if there exists a smooth vector field V (called the potential vector field) such that

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0,$$

where $\mathcal{L}_V$ denotes the Lie derivative along V and λ is a real number. A Ricci soliton is said to be expanding, steady or shrinking according to λ is positive, zero and negative respectively. Ricci solitons has now become a popular topic for many mathematicians, for details we refer ([17], [22], [5]).

An η-Ricci soliton (V, λ, μ) is a generalization of a Ricci soliton defined as ([9], [12], [6])

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g + \mu\eta \otimes \eta = 0,$$

where $\mathcal{L}_V$ denotes the Lie derivative along V and λ and μ are real numbers.

We organize the present paper as follows: after Introduction, in Section 2, we briefly recall some known results for Kenmotsu manifolds and D-homothetic deformations on a Kenmotsu manifold and give some properties of the deformed Kenmotsu manifold. In Section 3, we discuss the properties of a D-homothetically deformed Kenmotsu manifold under generalized weakly symmetric and generalized weakly Ricci symmetric curvature restrictions equipped with Ricci solitons and η^*-Ricci solitons. We determine a necessary condition for which the Ricci solitons of these types in such a manifold are shrinking, steady and expanding.

2. Preliminaries

According to the definition of Blair [13], an *almost contact structure* (ϕ, ξ, η) on a $(2n+1)$-dimensional Riemannian manifold satisfies the following conditions

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \operatorname{rank} \phi = n - 1.$$

Moreover, if g is a Riemannian metric on M^{2n+1} satisfying

$$\begin{aligned} g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ g(X, \xi) &= \eta(X), \\ g(\phi X, Y) &= -g(X, \phi Y) \end{aligned}$$

for any vector fields X, Y on M^{2n+1}, then the manifold M^{2n+1} [13] is said to admit an *almost contact metric structure* (ϕ, ξ, η, g).

Definition 2.1 ([20])
If in an almost contact metric structure (ϕ, ξ, η, g) on M^{2n+1} the Levi-Civita connection ∇ of g satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y on M^{2n+1}, then the structure is called *Kenmotsu*.

Proposition 2.2 ([20])
If $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Kenmotsu manifold, then for any vector fields X, Y, Z on M^{2n+1} the following relations hold

$$\begin{aligned} \nabla_X \xi &= X - \eta(X)\xi, \\ (\nabla_X \eta)Y &= g(X, Y) - \eta(X)\eta(Y), \\ S(X, \xi) &= -2n\eta(X), \\ \eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \\ R(\xi, X)Y &= \eta(Y)X - g(X, Y)\xi, \\ R(X, Y)\xi &= \eta(X)Y - \eta(Y)X. \end{aligned}$$

Definition 2.3 ([1])
If a contact metric manifold M^{2n+1} with the almost contact metric structure (ϕ, ξ, η, g) is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$, where

$$\phi^* = \phi, \quad \xi^* = \frac{1}{p}\xi, \quad \eta^* = p\eta, \quad g^* = pg + p(p-1)\eta \otimes \eta \tag{1}$$

and p is a positive constant, then the transformation is called a *D-homothetic deformation.*

The relation between the Levi-Civita connections ∇ of g and ∇^* of g^* is given by [1],

$$\nabla^*_X Y = \nabla_X Y + \frac{p-1}{p} g(\phi X, \phi Y)\xi \tag{2}$$

for any vector fields X, Y on M^{2n+1}.

In view of (1), (2) and definition of Riemannian curvature tensor, Ricci tensor, scalar curvature, we get the following

Proposition 2.4 ([11])
If a Kenmotsu structure (ϕ, ξ, η, g) on M^{2n+1} is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$ under a D-homothetic deformation, then R, R^*, S, S^*, r and r^* are related by

$$\begin{aligned} R^*(X, Y)Z &= R(X, Y)Z + \frac{p-1}{p}[g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y], \\ S^*(X, Y) &= S(X, Y) + 2n\frac{p-1}{p} g(\phi X, \phi Y), \\ r^* &= \frac{1}{p}r + 2n(2n+1)\frac{p-1}{p^2} \end{aligned}$$

for any vector fields X, Y, Z on M^{2n+1}.

Now we shall bring out some properties of a D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} as follows

Proposition 2.5
If under a D-homothetic deformation, a Kenmotsu structure (ϕ, ξ, η, g) on M^{2n+1} is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$, then for any vector fields X, Y, Z on M^{2n+1}, we have

$$\begin{aligned} \phi^{*2} &= -I + \eta^* \otimes \xi^*, \\ \eta^*(\xi^*) &= 1, \\ \phi^*\xi^* &= 0, \\ \eta^* \circ \phi^* &= 0, \end{aligned} \tag{3}$$

$$\begin{aligned} g^*(\phi^* X, \phi^* Y) &= g^*(X,Y) - \eta^*(X)\eta^*(Y), \\ g^*(X, \xi^*) &= \eta^*(X), \end{aligned} \tag{4}$$

$$\nabla^*_X \xi^* = \frac{1}{p}[X - \eta^*(X)\xi^*], \tag{5}$$

$$(\nabla^*_X \eta^*)Y = \frac{1}{p}[g^*(X,Y) - \eta^*(X)\eta^*(Y)], \tag{6}$$

$$S^*(X, \xi^*) = -\frac{2n}{p^2}\eta^*(X), \tag{7}$$

$$\begin{aligned} \eta^*(R^*(X,Y)Z) &= \frac{1}{p^2}[g^*(X,Z)\eta^*(Y) - g^*(Y,Z)\eta^*(X)], \\ R^*(\xi^*, X)Y &= \frac{1}{p^2}[\eta^*(Y)X - g^*(X,Y)\xi^*], \end{aligned} \tag{8}$$

$$R^*(X,Y)\xi^* = \frac{1}{p^2}[\eta^*(X)Y - \eta^*(Y)X]. \tag{9}$$

Notice that using (6) and (7), we obtain

$$(\nabla^*_X S^*)(Y, \xi^*) = \frac{1}{p}\Big[- \frac{2n}{p^2} g^*(X,Y) - S^*(X,Y)\Big] \tag{10}$$

for any vector fields X and Y on M^{2n+1}.

3. Solitons in a generalized weakly (Ricci) symmetric D-homothetically deformed Kenmotsu manifold

3.1. Generalized weakly symmetric deformed Kenmotsu manifold

Definition 3.1
A D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is said to be generalized weakly symmetric if it satisfies the condition

$$\begin{aligned}(\nabla^*_X \bar{R}^*)(Y,U,V,W) &= \alpha^*_1(X)\bar{R}^*(Y,U,V,W) + \beta^*_1(Y)\bar{R}^*(X,U,V,W)\\ &+ \beta^*_1(U)\bar{R}^*(Y,X,V,W) + \delta^*_1(V)\bar{R}^*(Y,U,X,W)\\ &+ \delta^*_1(W)\bar{R}^*(Y,U,V,X) + \alpha^*_2(X)\bar{G}^*(Y,U,V,W)\\ &+ \beta^*_2(Y)\bar{G}^*(X,U,V,W) + \beta^*_2(U)\ \bar{G}^*(Y,X,V,W)\\ &+ \delta^*_2(V)\ \bar{G}^*(Y,U,X,W) + \delta^*_2(W)\ \bar{G}^*(Y,U,V,X),\end{aligned} \tag{11}$$

where

$$\bar{G}^*(Y,U,V,W) = g^*(U,V)g^*(Y,W) - g^*(Y,V)g^*(U,W) \tag{12}$$

and α^*_i, β^*_i, δ^*_i are non-zero 1-forms defined by $\alpha^*_i(X) = g^*(X,\sigma_i)$, $\beta^*_i(X) = g^*(X,\varrho_i)$ and $\delta^*_i(X) = g^*(X,\pi_i)$ for $i = 1,2$.

Now, contracting Y over W in both sides of (11) and using (12), we get

$$\begin{aligned}(\nabla^*_X S^*)(U,V) &= \alpha^*_1(X)S^*(U,V) + \beta^*_1(U)S^*(X,V) + \beta^*_1(R^*(X,U)V)\\ &+ \delta^*_1(R^*(X,V)U) + \delta^*_1(V)S^*(X,U) + \beta^*_2(X)g^*(U,V)\\ &- \beta^*_2(U)g^*(X,V) + \delta^*_2(X)g^*(U,V) - \delta^*_2(V)g^*(X,U)\\ &+ 2n[\alpha^*_2(X)g^*(U,V) + \beta^*_2(U)g^*(X,V) + \delta^*_2(V)g^*(X,U)].\end{aligned}$$

Now setting $V = \xi^*$ and using (7), (8), (9) in the foregoing equation we obtain

$$\begin{aligned}(\nabla^*_X S^*)(U,\xi^*) &= \frac{1}{p^2}\{[\delta^*_1(\xi^*) + (2n-1)p^2\delta^*_2(\xi^*)]g^*(X,U)\\ &- [2n\alpha^*_1(X) + \beta^*_1(X) + \delta^*_1(X)\\ &- p^2(2n\alpha^*_2(X) + \beta^*_2(X) + \delta^*_2(X))]\eta^*(U)\\ &- (2n-1)[\beta^*_1(U) - p^2\beta^*_2(U)]\eta^*(X)\} + \delta^*_1(\xi^*)S^*(X,U).\end{aligned} \tag{13}$$

Now from (10) and (13) we get

$$\begin{aligned}-\frac{2n}{p}g^*(X,Y) - pS^*(X,Y) &= [\delta^*_1(\xi^*) + (2n-1)p^2\delta^*_2(\xi^*)]g^*(X,Y)\\ &- [2n\alpha^*_1(X) + \beta^*_1(X) + \delta^*_1(X)\\ &- p^2(2n\alpha^*_2(X) + \beta^*_2(X) + \delta^*_2(X))]\eta^*(Y)\\ &- (2n-1)[\beta^*_1(Y) - p^2\beta^*_2(Y)]\eta^*(X)\\ &+ p^2\delta^*_1(\xi^*)S^*(X,Y).\end{aligned} \tag{14}$$

Now putting successively $X = Y = \xi^*$, $Y = \xi^*$ and $X = \xi^*$, we get respectively that

$$\alpha^*_1(\xi^*) + \beta^*_1(\xi^*) + \delta^*_1(\xi^*) = p^2[\alpha^*_2(\xi^*) + \beta^*_2(\xi^*) + \delta^*_2(\xi^*)], \tag{15}$$

$$\begin{aligned}2n\alpha^*_1(X) + \beta^*_1(X) + \delta^*_1(X) + (2n-1)[\beta^*_1(\xi^*) + \delta^*_1(\xi^*)]\eta^*(X)\\ = p^2\{2n\alpha^*_2(X) + \beta^*_2(X) + \delta^*_2(X)\\ + (2n-1)[\beta^*_2(\xi^*) + \delta^*_2(\xi^*)]\eta^*(X)\}\end{aligned} \tag{16}$$

and

$$\begin{aligned}2n[\alpha_1^*(\xi^*)+\delta_1^*(\xi^*)]\eta^*(Y)+\beta_1^*(\xi^*)\eta^*(Y)+(2n-1)\beta_1^*(Y)\\ =p^2\{2n[\alpha_2^*(\xi^*)+\delta_2^*(\xi^*)]\eta^*(Y)\\ +\beta_2^*(\xi^*)\eta^*(Y)+(2n-1)\beta_2^*(Y)\}.\end{aligned} \tag{17}$$

Using (15), we get from (16) and (17) respectively

$$\begin{aligned}2n\alpha_1^*(X)+\beta_1^*(X)+\delta_1^*(X)-(2n-1)\alpha_1^*(\xi^*)\eta^*(X)\\ =p^2[2n\alpha_2^*(X)+\beta_2^*(X)+\delta_2^*(X)\\ -(2n-1)\alpha_2^*(\xi^*)\eta^*(X)]\end{aligned} \tag{18}$$

and

$$\beta_1^*(Y)-\beta_1^*(\xi^*)\eta^*(Y)=p^2[\beta_2^*(Y)-\beta_2^*(\xi^*)\eta^*(Y)]. \tag{19}$$

By virtue of (15), (18) and (19), the equation (14)yields

$$\begin{aligned}p^2[1+p\delta_1^*(\xi^*)]S^*(X,Y)=-[2n+p\delta_1^*(\xi^*)+(2n-1)p^3\delta_2^*(\xi^*)]g^*(X,Y)\\ +(2n-1)p[p^2\delta_2^*(\xi^*)-\delta_1^*(\xi^*)]\eta^*(X)\eta^*(Y).\end{aligned} \tag{20}$$

Thus we can state the following

Theorem 3.2
Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a Kenmotsu manifold and let $(\phi^*,\xi^*,\eta^*,g^*)$ be a D-homothetically deformed structure of (ϕ,ξ,η,g). If the deformed structure is generalized weakly symmetric and $\delta_1^*(\xi^*)\neq-\frac{1}{p}$, then the Ricci curvature tensor S^* of the deformed manifold satisfies (20).

Proposition 3.3
If $\delta_1^*(\xi^*)\neq-\frac{1}{p}$, then the scalar curvature of a generalized weakly symmetric deformed Kenmotsu structure $(\phi^*,\xi^*,\eta^*,g^*)$ of a manifold M^{2n+1} is given by

$$r^*=-\frac{2n}{p^2}\frac{2n+1+2p\delta_1^*(\xi^*)+(2n-1)p^3\delta_2^*(\xi^*)}{1+p\delta_1^*(\xi^*)}.$$

3.2. Generalized weakly Ricci symmetric deformed Kenmotsu manifold

Definition 3.4
A D-homothetically deformed Kenmotsu structure $(\phi^*,\xi^*,\eta^*,g^*)$ of a manifold M^{2n+1} is said to be generalized weakly Ricci symmetric if it satisfies the condition

$$\begin{aligned}(\nabla_X^*S^*)(Y,Z)=A_1^*(X)S^*(Y,Z)+B_1^*(Y)S^*(X,Z)+D_1^*(Z)S^*(Y,X)\\ +A_2^*(X)g^*(Y,Z)+B_2^*(Y)g^*(X,Z)+D_2^*(Z)g^*(Y,X),\end{aligned} \tag{21}$$

where A_i^*, B_i^* and D_i^* are non-zero 1-forms defined by $A_i^*(X)=g^*(X,\theta_i)$, $B_i^*(X)=g^*(X,\phi_i)$ and $D_i^*(X)=g^*(X,\pi_i)$ for $i=1,2$.

Now using (7) and (10) in (21) for $Z = \xi^*$ we obtain

$$\begin{aligned} -\frac{2n}{p} g^*(X,Y) - pS^*(X,Y) = &-[2nA_1^*(X) - p^2 A_2^*(X)]\eta^*(Y) \\ &- [2nB_1^*(Y) - p^2 B_2^*(Y)]\eta^*(X) \\ &+ p^2 D_1^*(\xi^*) S^*(X,Y) + p^2 D_2^*(\xi^*) g^*(X,Y). \end{aligned} \tag{22}$$

Now putting successively $X = Y = \xi^*$, $Y = \xi^*$ and $X = \xi^*$ we get respectively that

$$2n[A_1^*(\xi^*) + B_1^*(\xi^*) + D_1^*(\xi^*)] = p^2[A_2^*(\xi^*) + B_2^*(\xi^*) + D_2^*(\xi^*)], \tag{23}$$

$$\begin{aligned} 2n\{A_1^*(X) + [B_1^*(\xi^*) + D_1^*(\xi^*)]\eta^*(X)\} \\ = p^2\{A_2^*(X) + [B_2^*(\xi^*) + D_2^*(\xi^*)]\eta^*(X)\} \end{aligned} \tag{24}$$

and

$$\begin{aligned} 2n\{[A_1^*(\xi^*) + D_1^*(\xi^*)]\eta^*(Y) + B_1^*(Y)\} \\ = p^2\{[A_2^*(\xi^*) + D_2^*(\xi^*)]\eta^*(Y) + B_2^*(Y)\}. \end{aligned} \tag{25}$$

Using (23), we get from (24) and (25) respectively

$$2n[A_1^*(X) - A_1^*(\xi^*)\eta^*(X)] = p^2[A_2^*(X) - A_2^*(\xi^*)\eta^*(X)] \tag{26}$$

and

$$2n[B_1^*(Y) - B_1^*(\xi^*)\eta^*(Y)] = p^2[B_2^*(Y) - B_2^*(\xi^*)\eta^*(Y)]. \tag{27}$$

By virtue of (23), (26) and (27), the equation (22) yields

$$\begin{aligned} p^2[1 + pD_1^*(\xi^*)]S^*(X,Y) = &-[2n + p^3 D_2^*(\xi^*)]g^*(X,Y) \\ &+ p[p^2 D_2^*(\xi^*) - 2nD_1^*(\xi^*)]\eta^*(X)\eta^*(Y). \end{aligned} \tag{28}$$

Theorem 3.5
Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(\phi^*, \xi^*, \eta^*, g^*)$ be a D-homothetically deformed structure of (ϕ, ξ, η, g). If the deformed structure is generalized weakly Ricci symmetric and $D_1^*(\xi^*) \neq -\frac{1}{p}$, then the Ricci curvature tensor S^* of the deformed manifold satisfies (28).

Proposition 3.6
If $D_1^*(\xi^*) \neq -\frac{1}{p}$, then the scalar curvature of a generalized weakly Ricci symmetric deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by

$$r^* = -\frac{2n}{p^2}\frac{2n + 1 + pD_1^*(\xi^*) + p^3 D_2^*(\xi^*)}{1 + pD_1^*(\xi^*)}.$$

3.3. Solitons in the deformed manifold

Notice that in [21], H. G. Nagaraja, D. L. K. Kumar, and V. S. Prasad and in [11] the present authors have also studied some properties of Ricci solitons on Kenmotsu manifolds under D-homothetic deformations.

Ricci solitons in the deformed manifold with potential vector field $V = \xi^*$

Assume now that in a D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the pair (ξ^*, λ) defines a Ricci soliton, that is

$$\frac{1}{2}\mathcal{L}_{\xi^*}g^* + S^* + \lambda g^* = 0$$

for λ a real number. Using the definition of the Lie derivative and (5) in the above equation, we get

$$S^*(X,Y) = -\Big(\frac{1}{p} + \lambda\Big)g^*(X,Y) + \frac{1}{p}\eta^*(X)\eta^*(Y), \tag{29}$$

for any vector fields X, Y on M^{2n+1}. Then (7) and (29) imply $\lambda = \frac{2n}{p^2}$ and the above equation is equivalent to

$$S^*(X,Y) = -\Big(\frac{1}{p} + \frac{2n}{p^2}\Big)g^*(X,Y) + \frac{1}{p}\eta^*(X)\eta^*(Y). \tag{30}$$

Remark 3.7
The Ricci soliton on a D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ is expanding.

Comparing (20) and (30), we obtain

$$(p + 2n - 1)\delta_1^*(\xi^*) - (2n-1)p^2\delta_2^*(\xi^*) + 1 = 0. \tag{31}$$

This leads to the following

Theorem 3.8
Assume that a Kenmotsu structure (ϕ, ξ, η, g) on M^{2n+1} is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$ under a D-homothetic deformation which is a generalized weakly symmetric space. If the pair $(\xi^*, \lambda = \frac{2n}{p^2})$ defines a Ricci soliton on the deformed structure, then the 1*-forms must verify* (31).

Corollary 3.9
There exists no Ricci soliton on the D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ when the deformed structure is locally symmetric space or recurrent space or generalized recurrent space.

Comparing (28) and (30), we obtain

$$(p + 2n)D_1^*(\xi^*) - p^2 D_2^*(\xi^*) + 1 = 0. \tag{32}$$

Now we can state the following

Theorem 3.10
Assume that a Kenmotsu structure (ϕ, ξ, η, g) on M^{2n+1} is transformed into $(\phi^*, \xi^*, \eta^*, g^*)$ under a D-homothetic deformation which is generalized weakly Ricci symmetric space. If the pair $(\xi^*, \lambda = \frac{2n}{p^2})$ defines a Ricci soliton on the deformed structure, then the 1*-forms must verify* (32).

Corollary 3.11
There exists no Ricci soliton on the D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ when the deformed structure is Ricci symmetric space or Ricci recurrent space or generalized Ricci recurrent space.

Ricci solitons in the deformed manifold with a potential vector field collinear with ξ^*

Assume now that in a D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the pair (V, λ) defines a Ricci soliton

$$\frac{1}{2}\mathcal{L}_V g^* + S^* + \lambda g^* = 0,$$

for λ a real number and $V = f\xi^*$, where f is a smooth function on M^{2n+1}. Using the definition of the Lie derivative and (5)) in the above equation, we get

$$\begin{aligned} S^*(X,Y) = &-\Big(\frac{f}{p}+\lambda\Big)g^*(X,Y) + \frac{f}{p}\eta^*(X)\eta^*(Y) \\ &- \frac{1}{2}[(Xf)\eta^*(Y) + (Yf)\eta^*(X)], \end{aligned} \tag{33}$$

for any vector fields X, Y on M^{2n+1}.

Now setting $Y = \xi^*$ and using (3), (4) and (7) we get

$$Xf = \Big(\frac{4n}{p^2} - 2\lambda - \xi^* f\Big)\eta^*(X). \tag{34}$$

Again replacing X by ξ^* in the above equation and using (3) we obtain

$$\xi^* f = \frac{2n}{p^2} - \lambda.$$

Using this in (34), we get

$$Xf = \Big(\frac{2n}{p^2} - \lambda\Big)\eta^*(X).$$

Substituting this in (33) we have

$$S^*(X,Y) = -\Big(\frac{f}{p}+\lambda\Big)g^*(X,Y) + \Big(\frac{f}{p} - \frac{2n}{p^2} + \lambda\Big)\eta^*(X)\eta^*(Y). \tag{35}$$

Comparing (20) and (35) we obtain

$$p^2[1 + p\delta_1^*(\xi^*)]\lambda = 2n - pf - p(pf-1)\delta_1^*(\xi^*) + (2n-1)p^3\delta_2^*(\xi^*). \tag{36}$$

This leads to the following

Theorem 3.12
If $\delta_1^*(\xi^*) \neq -\frac{1}{p}$, then the Ricci soliton $(V = f\xi^*, \lambda)$ of a generalized weakly symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by (36).

Corollary 3.13
If $\delta_1^*(\xi^*) \neq -\frac{1}{p}$, then the Ricci soliton $(V = f\xi^*, \lambda)$ of a generalized weakly symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is shrinking, steady or expanding according as

$$(pf-1)p\delta_1^*(\xi^*) \gtreqless (2n-1)p^3\delta_2^*(\xi^*) + 2n - pf.$$

Comparing (28) and (35) we obtain

$$p^2[1+pD_1^*(\xi^*)]\lambda = 2n - pf - p^2 fD_1^*(\xi^*) + p^3 D_2^*(\xi^*). \tag{37}$$

Now we can state the following

Theorem 3.14
If $D_1^*(\xi^*) \neq -\frac{1}{p}$, then the Ricci soliton $(V = f\xi^*, \lambda)$ of a generalized weakly Ricci symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by (37).

Corollary 3.15
If $D_1^*(\xi^*) \neq -\frac{1}{p}$, then the Ricci soliton $(V = f\xi^*, \lambda)$ of a generalized weakly Ricci symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is shrinking, steady or expanding according as

$$p^2 fD_1^*(\xi^*) \gtreqless p^3 D_2^*(\xi^*) + 2n - pf.$$

η^*-Ricci solitons in the deformed manifold with a potential vector field $V = \xi^*$

Assume now that in a D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the triple (ξ^*, λ, μ) defines an η^*-Ricci soliton [10], that is

$$\frac{1}{2}\mathcal{L}_{\xi^*} g^* + S^* + \lambda g^* + \mu\eta^* \otimes \eta^* = 0,$$

for λ and μ real numbers. Using the definition of the Lie derivative and (5) in the above equation, we get

$$S^*(X,Y) = -\Big(\frac{1}{p} + \lambda\Big) g^*(X,Y) + \Big(\frac{1}{p} - \mu\Big)\eta^*(X)\eta^*(Y), \tag{38}$$

for any vector fields X, Y on M^{2n+1}. Then (7) and (38) imply $\lambda + \mu = \frac{2n}{p^2}$.

Comparing (20) and (38), we obtain

$$p^2[1+p\delta_1^*(\xi^*)]\lambda = 2n - p - (p-1)p\delta_1^*(\xi^*) + (2n-1)p^3\delta_2^*(\xi^*), \tag{39}$$

$$p[1+p\delta_1^*(\xi^*)]\mu = 1 + (p+2n-1)\delta_1^*(\xi^*) - (2n-1)p^2\delta_2^*(\xi^*). \tag{40}$$

This leads to the following

Theorem 3.16
If $\delta_1^*(\xi^*) \neq -\frac{1}{p}$, then the η^*-Ricci soliton (ξ^*, λ, μ) of a generalized weakly symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by (39) *and* (40).

Comparing (28) and (38) we obtain

$$p^2[1+pD_1^*(\xi^*)]\lambda = 2n - p - p^2 D_1^*(\xi^*) + p^3 D_2^*(\xi^*), \tag{41}$$

$$p[1+pD_1^*(\xi^*)]\mu = 1 + (p+2n)D_1^*(\xi^*) - p^2 D_2^*(\xi^*). \tag{42}$$

Now we can state the following

Theorem 3.17
If $D_1^*(\xi^*) \neq -\frac{1}{p}$, then the η^*-Ricci soliton (ξ^*, λ, μ) of a generalized weakly Ricci symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by (41) *and* (42)*.*

η^*-Ricci solitons in the deformed manifold with a potential vector field collinear with ξ^*

Assume now that in a D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} the triple (V, λ, μ) defines an η^*-Ricci soliton, that is

$$\frac{1}{2}\mathcal{L}_V g^* + S^* + \lambda g^* + \mu\eta^* \otimes \eta^* = 0,$$

for λ and μ real numbers and $V = f\xi^*$, where f is a smooth function on M^{2n+1}. Using the definition of the Lie derivative and (5) in the above equation, we get

$$\begin{aligned} S^*(X,Y) = &-\Big(\frac{f}{p} + \lambda\Big)g^*(X,Y) + \Big(\frac{f}{p} - \mu\Big)\eta^*(X)\eta^*(Y) \\ &- \frac{1}{2}[(Xf)\eta^*(Y) + (Yf)\eta^*(X)] \end{aligned} \tag{43}$$

for any vector fields X, Y on M^{2n+1}.

Now setting $Y = \xi^*$ and using (3), (4) and (7) we get

$$Xf = \Big(\frac{4n}{p^2} - 2\lambda - 2\mu - \xi^* f\Big)\eta^*(X). \tag{44}$$

Again replacing X by ξ^* in the above equation and using (3) we obtain

$$\xi^* f = \frac{2n}{p^2} - \lambda - \mu.$$

Use this in (44) we get

$$Xf = \Big(\frac{2n}{p^2} - \lambda - \mu\Big)\eta^*(X).$$

Substituting this in (43) we have

$$S^*(X,Y) = -\Big(\frac{f}{p} + \lambda\Big)g^*(X,Y) + \Big(\frac{f}{p} - \frac{2n}{p^2} + \lambda\Big)\eta^*(X)\eta^*(Y). \tag{45}$$

Comparing (20) and (45) we obtain

$$p^2[1 + p\delta_1^*(\xi^*)]\lambda = 2n - pf - p(pf-1)\delta_1^*(\xi^*) + (2n-1)p^3\delta_2^*(\xi^*), \tag{46}$$

which leads to the following

Theorem 3.18
If $\delta_1^*(\xi^*) \neq -\frac{1}{p}$, then the η^*-Ricci soliton $(V = f\xi^*, \lambda, \mu = \frac{2n}{p^2} - (\xi^* f) - \lambda)$ of a generalized weakly symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by (46).

Comparing 28 and 45 we get

$$p^2[1 + pD_1^*(\xi^*)]\lambda = 2n - pf - p^2 f D_1^*(\xi^*) + p^3 D_2^*(\xi^*). \tag{47}$$

Hence

Theorem 3.19
If $D_1^*(\xi^*) \neq -\frac{1}{p}$, then the η^*-Ricci soliton $(V = f\xi^*, \lambda, \mu = \frac{2n}{p^2} - (\xi^* f) - \lambda)$ of a generalized weakly Ricci symmetric D-homothetically deformed Kenmotsu structure $(\phi^*, \xi^*, \eta^*, g^*)$ of a manifold M^{2n+1} is given by (47).

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Adara M. Blaga
Department of Mathematics
West University of Timişoara
Romania
E-mail: adarablaga@yahoo.com

Kanak Kanti Baishya, Nihar Sarkar
Department of Mathematics
Kurseong College
Kurseong Darjeeling
India
E-mail: kanakkanti.kc@gmail.com
math.nihar222@gmail.com

Received: December 18, 2018; final version: April 19, 2019;
available online: July 22, 2019.