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A new application of almost increasing sequences

Abstract. In this paper, a known result dealing with $|\bar{N}, p_n|_k$ summability of infinite series has been generalized to the $\varphi - |\bar{N}, p_n; \delta|_k$ summability of infinite series by using an almost increasing sequence.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants L and M such that

$$Lc_n \leq b_n \leq Mc_n$$

(see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (w_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]).

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The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2]),

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [14]),

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |w_n - w_{n-1}|^k < \infty.$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |\bar{N}, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n; \delta|_k$ summability (see [3]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $\delta = 0$, then we get $|\bar{N}, p_n|_k$ summability.

2. The known result

The following theorem is known dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 2.1 ([6])
Let (X_n) be an almost increasing sequence and let there be sequences (λ_n) and (β_n) such that

$$|\Delta \lambda_n| \leq \beta_n, \tag{1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \tag{3}$$

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} |\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{5}$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{6}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{7}$$

where (t_n) is the n-th $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. The main result

Some works dealing with generalized absolute summability methods of infinite series have been done (see [4, 5, 7, 9, 10, 11, 12, 13, 15, 16, 17]). The aim of this paper is to generalize Theorem 2.1 to $\varphi - |\bar{N}, p_n; \delta|_k$ summability, in the following form.

Theorem 3.1
Let (X_n) be an almost increasing sequence and let (φ_n) be a sequence of positive real numbers such that

$$\varphi_n p_n = O(P_n), \tag{8}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\Big(\varphi_v{}^{\delta k} \frac{1}{P_v}\Big) \quad as \quad m \to \infty. \tag{9}$$

If conditions (1)–(5) *of the Theorem 2.1 and*

$$\sum_{n=1}^{m} \varphi_n^{\delta k} \frac{|t_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{11}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta k < 1$.

We need the following lemma for the proof of Theorem 3.1.

Lemma 3.2 ([6])
Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, we have that

$$n X_n \beta_n = O(1) \quad as \quad n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{13}$$

Proof of Theorem 3.1. Let (J_n) indicate $(\bar{N}, p_n)$ means of the series $\sum a_n \lambda_n$. Then, for $n \geq 1$, we obtain

$$\bar{\Delta} J_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v$$
$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_v}{v} v a_v.$$

Applying Abel's formula, we get

$$\begin{aligned}\bar{\Delta} J_n &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{\lambda_{v+1}}{v} P_v t_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{v+1}{v} p_v \lambda_v t_v \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{v+1}{v} P_v t_v \Delta\lambda_v + \frac{n+1}{nP_n} p_n \lambda_n t_n \\ &= J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4}.\end{aligned}$$

For the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |J_{n,r}|^k < \infty \qquad \text{for } r = 1, 2, 3, 4.$$

By using Hölder's inequality and Abel's formula, we have

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |J_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \Big(\frac{p_n}{P_n P_{n-1}}\Big)^k \left(\sum_{v=1}^{n-1} P_v |t_v| \frac{|\lambda_{v+1}|}{v}\right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v |t_v| \frac{|\lambda_{v+1}|}{v}\right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v |t_v|^k \frac{|\lambda_{v+1}|}{v}\right) \\ &\quad \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \frac{|\lambda_{v+1}|}{v}\right)^{k-1} \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |t_v|^k \frac{|\lambda_{v+1}|}{v} \\ &= O(1) \sum_{v=1}^{m} P_v |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} |\lambda_{v+1}| \frac{|t_v|^k}{v} \\ &= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_{v+1}| \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{|t_r|^k}{r} + O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{|t_v|^k}{v} \\ &= O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\ &= O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by virtue of (1), (4), (5), (8)–(10) and (13).

Again, using Hölder's inequality and Abel's formula, we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |J_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} p_v |\lambda_v| |t_v|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |\lambda_v| |t_v|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} p_v |\lambda_v|^k |t_v|^k\right) \\
&\quad \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v |\lambda_v|^k |t_v|^k \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{p_v}{P_v} |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} |\lambda_v| |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \varphi_r^{\delta k-1} |t_r|^k + O(1) |\lambda_m| \sum_{v=1}^{m} \varphi_v^{\delta k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) |\lambda_m| X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

in view of (1), (4), (8), (9), (11) and (13). Also, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |J_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} P_v |t_v| |\Delta\lambda_v|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v |t_v| \beta_v\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v |t_v|^k \beta_v\right) \\
&\quad \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v\right)^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\beta_v|t_v|^k\\
&= O(1)\sum_{v=1}^{m}P_v\beta_v|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{|t_v|^k}{v}v\beta v\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{|t_r|^k}{r}+O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{|t_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)X_v+O(1)m\beta_mX_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}+O(1)m\beta_mX_m\\
&= O(1)\quad \text{as}\quad m\to\infty.
\end{aligned}
$$

by means of (1), (3), (8)–(10), (12) and (13). Finally, as in $J_{n,2}$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}|J_{n,4}|^k &= O(1)\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
&= O(1)\sum_{n=1}^{m}\varphi_n^{\delta k-1}|\lambda_n||t_n|^k\\
&= O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}
$$

in view of (1), (4), (8), (11) and (13). Thus the proof of Theorem 3.1 is completed.

4. Conclusion

If we take $\varphi_n=\frac{P_n}{p_n}$ and $\delta=0$ in Theorem 3.1, then we get Theorem 2.1. In this case, conditions (10) and (11) reduce to conditions (6) and (7), respectively. Also, the condition (8) is automatically satisfied.

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