DE GRUYTER OPEN
Ann. Univ. Paedagog. Crac. Stud. Math. 17 (2018), 127-151
DOI: 10.2478/aupcsm-2018-0010

FOLIA 233

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XVII (2018)

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Prime numbers with a certain extremal type property

Communicated by Justyna Szpond

Abstract. The convex hull of the subgraph of the prime counting function $x \to \pi(x)$ is a convex set, bounded from above by a graph of some piecewise affine function $x \to \epsilon(x)$. The vertices of this function form an infinite sequence of points $(e_k, \pi(e_k))_1^\infty$. The elements of the sequence $(e_k)_1^\infty$ shall be called *the extremal prime numbers*. In this paper we present some observations about the sequence $(e_k)_1^\infty$ and we formulate a number of questions inspired by the numerical data. We prove also two – it seems – interesting results. First states that *if the Riemann Hypothesis is true, then* $\lim \frac{e_{k+1}}{e_k} = 1$. The second, also depending on Riemann Hypothesis, describes the order of magnitude of the differences between consecutive extremal prime numbers.

1. Introduction

This paper is a revised and enlarged version of our preprint [7] and of the paper [8] (in Polish). As the preprint [7] attained some interest in the field, we decided to make it published, in spite of the fact that some conjectures stated there were have been in the meantime proved. More precisely, [7] concerns the convex hull of the graph of the function $\pi\colon [2,\infty) \to [1,\infty)$, which counts the prime numbers in the interval $[2, x]$, and which is usually defined by the formula

$$\pi(x) = \sum_{p\in\mathbb{P}, p\le x} 1 \tag{1}$$

where $\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$ denotes the set (or the sequence, if necessary) of prime numbers. Some properties related to the graph of the function π were studied in

AMS (2010) Subject Classification: 11N05.
Keywords and phrases: prime numbers, prime counting function, Riemann hypothesis.

1979 by Carl Pommerance [5] and more recently (2006) by H.L. Montgomery and S. Wagon [6]. In [7], we formulated a number of conjectures concerning the subject and we proved one of them, however assuming the Riemann Hypothesis. Quite recently McNew [2], in his PhD thesis [3], written under the supervision of C. Pommerence, found, applying similar methods as in [7], the proofs of some of the conjectures from [7] without using the Riemann Hypothesis. In the same paper [2], the author discusses the pioneering paper of Pommerence [5] and presents some further results concerning the different types of "extremal" prime numbers.

In this paper we present large parts of the original text of [7]. Some changes are necessary because of the results of McNew, which we will comment in details below and because our set of numerical data is now bigger. The paper of Pommerance [5] was perhaps the first, where the study of convexity of the graph of the function π appeared, but as far as we know, it is in [7] that the sequence $(e_k)_1^\infty$ of extremal prime numbers appears for the first time as an independent mathematical object. It was noticed by OEIS and lives there under the number A246033. The precise definition of the sequence $(e_k)_1^\infty$ will be presented in Part 1 of this paper (see also [2]), but at this moment we give an intuitive description. Setting $\pi(1) = 0$ we may consider in the plane $\mathbb{R}^2$ the set of points $\mathbb{G} = \{(n, \pi(n)) : \ n \in \mathbb{N}\}$ and its convex hull $conv(\mathbb{G})$. One may say that this convex hull is an unbounded, convex polygon, whose boundary consists of the graph of two continuous functions, namely of the constant function $\epsilon_0(x) = 0$ and of a polygonal function $x \to \epsilon(x)$. The vertices of $conv(\mathbb{G})$ form the sequence $(e_k, \pi(e_k))_0^\infty$ (where $e_0 = 1$ and $\pi(e_0) = 0$) and the sequence $(e_k)_1^\infty$ is just the sequence of extremal primes. In [7], in [2] and in this paper a number of results and a number of questions are presented, concerning the geometrical structure of $conv(\mathbb{G})$. Before formulating these results, it should be noticed, that this geometrical structure of $conv(\mathbb{G})$ is relatively easy to study numerically. Namely, we consider the bounded polygons $\mathbb{G}_x = conv(\mathbb{G}) \cap ([1, x] \times [0, \infty))$ and we count for example its number of vertices $\pi_\epsilon(x)$ or we calculate the length of its sides. Passing with x to infinity we may study the geometry of $conv(\mathbb{G})$. In [7] and [8] we presented the selected elements of (e_k) for $x \leq 10^{12}$. In [2] some data are presented for $x \leq 10^{13}$. In this paper we will present some data for $x \leq 10^{17}$. The present paper is an example of the papers, where the analysis of a large set of numerical data is a start point to some purely theoretical consideration. Another example worth mentioning is the article of A. Odlyzko, M. Rubinstein and M. Wolf about "jumping champions" [4].

One may ask what is the reason to study the sequence of extremal primes. Well, the function $x \to \epsilon(x)$ is the smallest "reasonable" function bounding the function $\pi(x)$ from above, but on the other hand, the set $(e_k)_1^\infty$ of extremal primes is very thin in comparison with the set $\mathbb{P}$ of all primes and then one may hope, that it should be easier to tame. Basing on the numerical data we formulated in [7] a number of conjectures. All these conjectures concern the distribution of extremal primes in $\mathbb{N}$. First of them states, that the set of extremal primes is a small subset of $\mathbb{P}$. More exactly

Conjecture A: *The series* $\sum_1^\infty \frac{1}{e_k}$ *is convergent.*

The second conjecture states that the set of extremal primes is not to small, since

Conjecture B: *The series* $\sum_1^\infty \frac{1}{\ln(e_k)}$ *diverges.*

These conjectures are both proved in [2]: Conjecture A is a consequence of Theorem 2.2 there and Conjecture B is stated as Corollary 2.7. Another conjecture speaks about the range of growth at infinity of the function $\pi_\epsilon(x)$ counting the extremal primes. This was mentioned in [7] and is named in the present paper *the* $\gamma/2$ *conjecture*. It states – roughly speaking – that the right range of $\pi_\epsilon(x)$ at infinity is comparable to $x^{\gamma/2}$, where γ is the Euler constant. More exactly,

Conjecture C: *There exists infimum*

$$\beta = \inf\{\alpha > 0 : \ \pi_\epsilon(x) = o(x^\alpha)\}$$

and it is positive. Moreover $\beta = \frac{\gamma}{2}$, *where* γ *is the Euler constant.*

In [2] (Theorem 2.2) it is observed, that $\beta \leq \frac{2}{3}$, but this evaluation seems to be far from the best possible. Further discussion of this conjecture will be continued in Part III.

In [7] also the following two conjectures were formulated.

Conjecture D: *For the sequence of extremal primes we have*

$$\lim \frac{e_{k+1}}{e_k} = 1.$$

Conjecture E: *For the sequence of extremal prime numbers we have*

$$e_{k+1} - e_k = o(e_k).$$

These two last Conjectures are strictly related to each other and they were proved in [7] assuming the Riemann Hypothesis. McNew proves them unconditionally (Corollary 2.8 in [2]).

Part I of this paper contains the precise definition of the extremal primes, the presentation of the selected numerical data and the formulation of the number of conjectures. The contents of the present paper is essentially the same as in [7], however there are some differences. In Part II we give the proof of Conjecture D, practically unchanged relative to the original version in [7]. In Part III we show that an important observation from [2] (Theorem 2.4) concerning Conjecture E, can be deduced from the formulas presented in [7].

2. Part I

2.1. Definition of extremal prime numbers

In [7] (as well as in [2]) one speaks about the so called *extremal prime numbers* or *convex prime numbers*, which may be defined as below. Let us observe first that

$$\begin{aligned} &conv(\{(x,y) : x \in [2,\infty),\ y \in [1,\pi(x)]\}) \\ &\qquad = conv(\{(x,y) :\ x \in \mathbb{N}, 2 \le x,\ y \in [1,\pi(x)]\}) \\ &\qquad = conv(\{(x,y) :\ 2 \le x,\ x \in \mathbb{P},\ y \in [1,\pi(x)]\}). \end{aligned}$$

It is easy to observe that there are many proper subsets $\mathbb{F} \subset \mathbb{P}$ such that

$$\begin{aligned} &conv(\{(x,y) : 2 \le x,\ x \in \mathbb{P},\ y \in [1,\pi(x)]\}) \\ &\qquad = conv(\{(x,y) :\ 2 \le x,\ x \in \mathbb{F},\ y \in [1,\pi(x)]\}). \end{aligned} \tag{2}$$

It is also not hard to observe that among the sets $\mathbb{F} \subset \mathbb{P}$ having the above property (2), there exists the smallest subset (in consequence only one) $\mathbb{E} \subset \mathbb{P}$ and this is exactly, by definition, the set of extremal primes. Clearly, $\mathbb{E}$ is infinite (see Proposition 1) and in many situations it will be more convenient to speak about the strictly increasing sequence $\mathbb{E} = (e_k)_1^\infty$ of extremal prime numbers. It should be noticed here, that in [2] the author considers the convex hull of the set $\{(n, p_n) : p_n \in \mathbb{P}\}$, which, clearly, does not make any essential difference. We present below an inductive method of finding e_{k+1} provided that e_k is known.

Proposition 1
The set $\mathbb{E}$ is infinite.

Proof. Consider the piecewise affine function $\epsilon\colon x \to \epsilon(x)$, whose vertices form exactly the set $\mathbb{E}$. Let l_k denote the straight line (the affine function) passing through the points $(e_{k-1}, \pi(e_{k-1}))$ and $(e_k, \pi(e_k))$. It follows from the definition of extremal points that the graph of the function ϵ lies below the line l_k. This gives a simple inductive method of finding the next extremal prime e_{k+1} providing that we know $e_1, e_2, \ldots, e_{k-1}, e_k$ (in fact it is sufficient to know only e_{k-1} and e_k). We can do it as follows. We consider the difference quotients of the form

$$I_k(p) = \frac{\pi(p) - \pi(e_k)}{p - e_k},$$

for $p \in \mathbb{P}$, $p > e_k$. It follows from the remark above, that for each $p > e_k$ we have

$$0 < I_k(p) < \frac{\pi(e_k) - \pi(e_{k-1})}{e_k - e_{k-1}} = I_{k-1}(e_k).$$

Using the commonly known fact

$$\lim_{p\to\infty} \frac{\pi(p)}{p} = 0,$$

we have $\lim_{p\to\infty} I_k(p) = 0$. Then there exists a finite set $\mathbb{P}_k \subset \mathbb{P}$ of primes such that $q \in \mathbb{P}_k \to q > e_k$ and such that $I_k(p) \le I_k(q)$ for $p > e_k$. We set then $e_{k+1} = \max \mathbb{P}_k$. This implies that the set $\mathbb{E}$ is infinite. Clearly, this means that

$$\lim_{k\to\infty} e_k = +\infty.$$

Let us return to the piecewise affine function $\epsilon\colon x \to \epsilon(x)$ and consider the sequence

$$\delta_k = \frac{\pi(e_{k+1}) - \pi(e_k)}{e_{k+1} - e_k},$$

i.e. δ_n is the slope of the n-th segment lying on the graph of the function ϵ. Since the function ϵ is strictly increasing and concave, the sequence $(\delta_k)_1^\infty$ is positive and strictly decreasing. Let us observe, that the sequence $(\delta_k)_1^\infty$ may be identified with the derivative of the function ϵ. Since δ_k is decreasing then the limit $\delta = \lim_{k\to\infty} \delta_k \geq 0$ exists and it must be $\delta = 0$. Indeed, suppose for instance, that $\delta > 0$. Hence for each $j \in \mathbb{N}^*$ we have $\pi(e_{j+1}) - \pi(e_j) > \delta(e_{j+1} - e_j)$. This implies (adding the above inequalities for $1 \leq j \leq k$) that for each $k \in \mathbb{N}^*$ the following inequality holds $\pi(e_{k+1}) - 1 > \delta(e_{k+1} - 2)$. But the last inequality is impossible since (once more) $\lim_{n\to\infty} \frac{\pi(n)}{n} = 0$. The property $\delta = \lim_{k\to\infty} \delta_k = 0$ makes it possible to observe that the set $\mathbb{P} \setminus \mathbb{E}$ is infinite.

Proposition 2
The set $\mathbb{P} \setminus \mathbb{E}$ is infinite.

Proof. This is almost obvious from the intuitive point of view. However, a short proof we present here is related to the very non-trivial results about *small gaps between primes*. Suppose, for the sake of contradiction, that $\mathbb{P} \setminus \mathbb{E}$ is finite. Hence for sufficiently great $a > 0$ we have $\mathbb{E} \cap [a, \infty) = \mathbb{P} \cap [a, \infty)$. In consequence we have

$$\begin{aligned} 0 = \delta = \lim_{k\to\infty} \delta_k &= \lim_{k\to\infty} \frac{\pi(e_{k+1}) - \pi(e_k)}{e_{k+1} - e_k} \\ &= \lim_{n\to\infty} \frac{\pi(p_{n+1}) - \pi(p_n)}{p_{n+1} - p_n} = \lim_{n\to\infty} \frac{1}{p_{n+1} - p_n}, \end{aligned}$$

where $p_n \in \mathbb{P}$. This would imply, that $\lim_{n\to\infty}(p_{n+1} - p_n) = \infty$. But this is impossible, because we know now from many recent results (for example of Zhang, [9]), that $\liminf(p_{n+1} - p_n) < 7 \cdot 10^7$. Since the paper of Zhang the constant $7 \cdot 10^7$ was considerably diminished.

The observations about the extremal primes made above are rather elementary. We will speak later about some deeper results. The problem with the sequence of extremal primes is in some sense similar to the problem we have with sequence of all primes and with its subsequences like for example the sequence (conjectured infinite) of twin primes. Namely, it is relatively easy to produce the consecutive elements of the sequence $(e_k)_1^\infty$ but it is rather hopeless to find an (exact) analytical formula describing the set of extremal primes. Now, it is perhaps a good moment to notice, that it is practically impossible to calculate "by hand" the elements of the sequence $\mathbb{E}$. Using Proposition 1 we may find ten or twenty first terms of the sequence $(e_k)_1^\infty$ without using computers, but for to go further we need strong calculating machines.

We have calculated more than the first 50000 extremal primes and after studying these numerical data, we can formulate a number of more or less interesting questions. It is impossible to give here the complete list of the first 50000 extremal primes, but we present below some selected data. The first forty nine terms of the sequence $\mathbb{E}$ are

n	1	2	3	4	5	6	7
e_n	2	3	7	19	47	73	113
n	8	9	10	11	12	13	14
e_n	199	283	467	661	887	1129	1327
n	15	16	17	18	19	20	21
e_n	1627	2803	3947	4297	5881	6379	7043
n	22	23	24	25	26	27	28
e_n	9949	10343	13187	15823	18461	24137	33647
n	29	30	31	32	33	34	35
e_n	34763	37663	42863	43067	59753	57797	82619
n	36	37	38	39	40	41	42
e_n	96017	102679	129643	130699	142237	155893	187477
n	43	44	45	46	47	48	49
e_n	194419	210533	211949	230393	267961	272423	284839

The list of e_k, where $k \leq 3000$ and $k \equiv 0 (\mathrm{mod}\ 100)$ and the list of e_k, where $k \leq 50000$ and $k \equiv 0 (\mathrm{mod}\ 10000)$.

e_{100}	5253173	e_{1600}	157169830847
e_{200}	67596937	e_{1700}	196062395777
e_{300}	314451367	e_{1800}	241861008029
e_{400}	883127303	e_{1900}	296478801431
e_{500}	2122481761	e_{2000}	365234091199
e_{600}	4205505103	e_{2100}	435006680401
e_{700}	7274424463	e_{2200}	524320812671
e_{800}	12251434927	e_{2300}	625382499043
e_{900}	19505255383	e_{2400}	727995116377
e_{1000}	28636137347	e_{2500}	842057152381
e_{1100}	40001601779	e_{2600}	975455207557
e_{1200}	55036621907	e_{2700}	1098339926353
e_{1300}	73753659461	e_{2800}	1234264464703
e_{1400}	97381385771	e_{2900}	1388032354369
e_{1500}	125232859691	e_{3000}	1563678255869

e_{10000}	92375151455953
e_{20000}	981254018753539
e_{30000}	3757752577836253
e_{40000}	9797619494633261
e_{50000}	20596671738838703

The examination of the sequence of the first 50000 extremal primes allows us to formulate a number of questions. First of all it seems to be interesting to say something about the density of the sequence $(e_k)_1^\infty$. Our experimental data support some conjectures. Namely,

Conjecture F: (see [7], Conjecture A in Introduction, Theorem 2.2 [2]) *The series*

$$\sum_{k=1}^{\infty} \frac{1}{e_k}$$

is convergent.

It follows from our data that

$$\sum_{k=1}^{50000} \frac{1}{e_k} \cong 1,090298\ldots.$$

Conjecture G: (see [7], Conjecture B in Introduction, Corollary 2.7 in [2]) *The series*

$$\sum_{k=1}^{\infty} \frac{1}{\ln(e_k)}$$

is divergent.

Our data gives

$$\sum_{k=1}^{50000} \frac{1}{\ln(e_k)} > 1486.$$

Let us remember that, as it was mentioned in Introduction, this two conjectures are proved in [2].

Since the set $\mathbb{E}$ of extremal prime numbers is infinite and, clearly, the problem of finding any reasonable explicit formula describing the correspondence $\mathbb{N} \ni n \to e_n$ is rather out of reach, we will define and try to study a function, which may be called *extremal primes counting function* π_ϵ. The formula for π_ϵ is analogous to formula (1). We set

$$\pi_\epsilon(x) = \sum_{p\in\mathbb{E},\ p\leq x} 1.$$

Unfortunately we know only 50000 values of $\pi_\epsilon(x)$ for $x \leq 10^{17}$. However, it seems to be possible to formulate some conjectures about π_ϵ. Clearly, $\pi_e(x) \leq \pi(x)$ and the growth of π_ϵ is much slower than the growth of π. For example, $\pi_\epsilon(x_0) = 1700$, when $x_0 = 196062395777$ and for the same x_0 we have $\pi(x_0) = 7855721212$. In particular, we may try to find the best $\alpha < 1$ such that $\pi_\epsilon(x) = o(x^\alpha)$ observing the ratio $\frac{\ln(n)}{\ln(e_n)}$ when n tends to infinity (in our case only for $n \leq 10^{17}$). Maybe only accidentally, but the best α obtained from our data is near to $\frac{\gamma}{2}$, where γ is the Euler constant. Hence we formulate.

Conjecture H: (see [7], Conjecture C in Introduction, also [2]) *There exists infimum*

$$\beta = \inf\{\alpha > 0 : \ \pi_\epsilon(x) = o(x^\alpha)\}$$

and is positive. Moreover, $\beta = \frac{\gamma}{2}$, *where* γ *is the Euler constant.*

Our numerical data support strongly also the following interesting conjecture.

Conjecture I: (see [7], Conjecture D in Introduction, Corollary 2.8 in [2]) *In the notations as above, we have:*

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = 1.$$

We will prove below, in Part II, that the Riemann Hypothesis implies Conjecture I. This conjecture is interesting itself, but also because of the following observation.

Proposition 3
If

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = 1$$

then

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1,$$

where $p_n \in \mathbb{P}$.

Proof. For each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that

$$e_{k(n)} \leq p_n < p_{n+1} \leq e_{k(n)+1}.$$

Thus

$$\frac{p_{n+1}}{p_n} \leq \frac{e_{k(n)+1}}{e_{k(n)}}$$

and the last sequence tends by our assumption to 1. Let us recall here, that $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$ implies Prime Number Theorem. This was proved by P. Erdös in his elementary proof of PNT (see [1]).

It follows directly from the definitions of the functions π and π_ϵ that $\pi(e_{k+1}) - \pi(e_k) \geq 1$ and the equality may occur. Except for trivial $e_1 = 2$ and $e_2 = 3$ we have found two such "twin extremal primes" for $k = 116$ and $k = 976$. Namely, $e_{116} = 8787901$, $e_{117} = 8787917$ and $\pi(e_{116}) = 589274$, $e_{976} = 26554262369$, $e_{977} = 26554262393$ and $\pi(e_{976}) = 1156822345$. We may state the question.

Question 4
Does there exist infinitely many $k \in \mathbb{N}$ such that $\pi(e_{k+1}) - \pi(e_k) = 1$.

Another phenomenon is related to the inequality $I_k(p) \leq I_k(p_o)$, which is described in Proposition 1. One may ask if the number of points $p > e_k$ such that $I_k(p) = I_k(p_o)$ is greater than 1. In our numerical data we have only three such examples, namely for $k = 2$ we have $I_2(5) = I_2(7)$, $I_3(13) = I_3(19)$ and also $I_4(23) = I_4(31) = I_4(43) = I_4(47) = \frac{1}{4} = \delta_4$ but in fact our programme searching next extremal primes was not written to search such exceptions.

3. Part II

As we announced in Introduction in this Part we present the original proof of Conjecture I (from [7]) depending on Riemann Hypothesis.

3.1. Definition of lenses

The gaps between extremal primes will be called *lenses*. More exactly,

Definition 5
Given a positive integer $k \in \mathbb{N}$ the lens S_k is a set

$$S_k = \{n \in \mathbb{N} : \ e_k \leq n < e_{k+1}\}.$$

The difference $e_{k+1} - e_k$ will be called *the length* of the lens S_k and will be denoted by $|S_k|$.

Since we will apply in the sequel the language of differential calculus, it will be more comfortable to work with the function $[2, \infty) \ni x \to S(x) \in [1, \infty)$, where

$$x \in [e_k, e_{k+1}) \Rightarrow S(x) = |S_k|.$$

We consider the following – well known – functions L and ε called *integral logarithm* and *error term* respectively, defined by the following formulae:

$$L(x) := Li(x) = \int_2^x \frac{1}{\ln t} dt \tag{3}$$

and

$$\varepsilon(x) = \sqrt{x} \cdot \ln x. \tag{4}$$

Together with L and ε we consider the function

$$\varphi(x) = L(x) - \varepsilon(x)$$

and for $x \in (2, \infty)$ and $h \in \mathbb{R}$,

$$l(x, h) = \varphi'(x) \cdot h + \varphi(x).$$

Clearly, all these functions are analytic at least in $(2,\infty)$. We will use the derivatives of the considered functions to the order four and we shall write y instead of $\ln x$ to present some formulas in more compact form. Hence we have

$$L^{(1)}(x) = \frac{1}{\ln x} = \frac{1}{y},$$

and further derivatives

$$L^{(2)}(x) = \frac{-1}{x \cdot y^2},$$
$$L^{(3)}(x) = \frac{y+2}{x^2 \cdot y^3},$$
$$L^{(4)}(x) = \frac{-(2 \cdot y^2 + 6y + 6)}{x^3 \cdot y^4}.$$

The derivatives of error term function, written in an analogous manner, run as follows

$$\varepsilon(x) = \sqrt{x} \cdot \ln x = \sqrt{x} \cdot y,$$
$$\varepsilon^{(1)}(x) = \frac{y+2}{2\sqrt{x}},$$
$$\varepsilon^{(2)}(x) = \frac{-y}{4x\sqrt{x}},$$
$$\varepsilon^{(3)}(x) = \frac{3y-2}{8x^2\sqrt{x}},$$
$$\varepsilon^{(4)}(x) = \frac{-15y+16}{16x^3\sqrt{x}}.$$

Let us observe, that the second derivatives of the functions L and ε are negative, so both these functions are concave. The second derivative of the function φ has the form

$$\varphi^{(2)}(x) = \frac{-4\sqrt{x} + \ln^3 x}{x\sqrt{x}\ln^2 x} = \frac{-4\sqrt{x} + y^3}{4x\sqrt{x}y^2},$$

then taking into account that

$$\lim_{x\to\infty} (-4\sqrt{x} + \ln^3 x) = -\infty$$

we can state

Proposition 6
There exists $x_0 \in (2,\infty)$ such that the function φ is concave in the interval $[x_0,\infty)$.

3.2. A remark on Taylor polynomials of considered functions

Let us fix a point $x \in (2,\infty)$. Let $T^{(3)}_{x,L}$ denote the Taylor polynomial of order three of the function L with the center at x. Hence

$$T^{(3)}_{x,L}(h) = L(x) + L^{(1)}(x) \cdot h + \frac{1}{2} \cdot L^{(2)}(x) \cdot h^2 + \frac{1}{6} \cdot L^{(3)}(x) \cdot h^3. \tag{5}$$

The remainder $R^{(3)}_x(h) = L(x+h) - T^{(3)}_{x,L}(h)$, written in the Lagrange form, is given by the formula

$$R^{(3)}_x(h) = \frac{1}{24} L^{(4)}(\xi) \cdot h^4, \tag{6}$$

where ξ is a point from the $(x, x+h)$. Since $L^{(4)} < 0$ in all its domain, we have the inequality.

Proposition 7
For each $x \in (2, \infty)$ and for each $h \in (2-x, \infty)$ the following inequality is true

$$L(x+h) \le T^{(3)}_{x,L}(h).$$

Let $T^{(3)}_{x,\varepsilon}$ denote the Taylor polynomial of order three of the function ε with the center at x, i.e.

$$T^{(3)}_{x,\varphi}(h) = \varepsilon(x) + \varepsilon^{(1)}(x) \cdot h + \frac{1}{2} \cdot \varphi^{(2)}(x) \cdot h^2 + \frac{1}{6} \cdot L^{(3)}(x) \cdot h^3. \tag{7}$$

Using an analogous argumentation as in the case of the function L we have

Proposition 8
For each $x \in (2, \infty)$ and for each $h \in (2-x, \infty)$ the following inequality is true

$$\varepsilon(x+h) \le T^{(3)}_{x,\varepsilon}(h). \tag{8}$$

In consequence, we have the inequality (true for all $h \in (2-x, \infty)$)

$$L(x+h) + \varepsilon(x+h) < T^{(3)}_{x,L}(h) + T^{(3)}_{x,\varepsilon}(h). \tag{9}$$

3.3. Definition of two functions

In this section we will define two functions $h_+ \colon (x_0, \infty) \ni x \to h_+(x) \in \mathbb{R}$ and $h_- \colon (x_0, \infty) \ni x \to h_-(x) \in \mathbb{R}$, where x_0 is the point defined in Proposition 6. First we will describe in details the definition of the function h_+. The definition of h_- will be similar.

Let us fix a point $x \in (x_0, \infty)$. Take into account the tangent line $l(x, h)$ to the graph of the function φ at the point $(x, \varphi(x))$. Its equation for $h \in \mathbb{R}$ is given by

$$l(x, h) = \varphi'(x) \cdot h + \varphi(x) = L'(x)h - \varepsilon'(x)h + L(x) - \varepsilon(x). \tag{10}$$

The tangent half-lines obtained, when we restrict ourselves in the formula (10) to $h \in [0, \infty)$ or $h \in (-\infty, 0]$ will be denoted by $l_+(x, h)$ or $l_-(x, h)$, respectively. For $h = 0$ we have the inequality

$$l(x, 0) = \varphi(x) = L(x) - \varepsilon(x) < L(x) + \varepsilon(x).$$

This means that the half-line l_+ starts from the interior point $(x, \varepsilon(x))$ of the subgraph of the function $L + \varphi$, which is a convex set. Since

$$\frac{d}{dh} L(x+h) = \frac{1}{\ln(x+h)}$$

and

$$\frac{d}{dh}\varepsilon(x+h) = \frac{\ln(x+h)+2}{2\sqrt{x+h}},$$

then

$$\lim_{h\to\infty}\frac{d}{dh}(L(x+h)+\varepsilon(x+h)) = 0.$$

On the other hand,

$$\frac{d}{dh}l(x+h) = \varphi'(x) > 0,$$

hence the half-line $l_+(x,h)$ must intersect the graph of the strictly concave function $L(x+h)+\varepsilon(x+h)$ in exactly one point. Hence we have proved the following

Proposition 9
For each $x \in (x_0,\infty)$ there exists exactly one positive number $h_+(x)$ such that

$$L(x+h_+(x)) + \varepsilon(x+h_+(x)) = \varphi'(x)\cdot h_+(x) + \varphi(x).$$

In other words, for each $x \in (x_0,\infty)$ the equation (with unknown h)

$$L(x+h)+\varepsilon(x+h) = \varphi'(x)\cdot h + \varphi(x) \tag{11}$$

has exactly one positive solution, which we will denote by $h_+(x)$. If one replaces the half-line $l_+(x,h)$, by the half line $l_-(x,h)$, then applying the same arguments as above, we obtain

Proposition 10
For each $x \in (x_0,\infty)$ there exists exactly one negative number $h_-(x)$ such that

$$L(x+h_-(x)) + \varepsilon(x+h_-(x)) = \varphi'(x)\cdot h_-(x) + \varphi(x).$$

In other words, equation (11) has exactly one negative solution, which we will denote by $h_-(x)$.

3.4. An auxiliary equation

In this paper we would like to establish the order of magnitude of the functions $x \to h_+(x)$ and $x \to h_-(x)$ (in fact of the difference $h_+(x)-h_-(x)$, when x tends to $+\infty$). Since the equation (11) is rather hard to solve, we will consider an auxiliary equation

$$T^{(3)}_{x,L}(h) + T^{(3)}_{x,\varepsilon}(h) = \varphi'(x)\cdot h + \varphi(x), \tag{12}$$

which can be written in the form

$$\begin{aligned} W_x(h) := {}& \frac{1}{6}(L^{(3)}(x)+\varepsilon^{(3)}(x))\cdot h^3 + \frac{1}{2}(L^{(2)}(x)+\varepsilon^{(2)}(x))\cdot h^2 \\ &+ 2\varepsilon^{(1)}(x)\cdot h + 2\varepsilon(x) = 0. \end{aligned} \tag{13}$$

Equation (13) is an algebraic equation of degree three. It has at least one real root. We will see that it can have (and has) more then one real root and we will be interested not only in the existence of roots of equation (13), but also on theirs signs. Let us observe, that since $W_x(0) = 2\varepsilon(x) > 0$, the number $h = 0$ cannot be a root of considered equation. Let us also observe that, in fact, equation (13) is not a single algebraic equation, but it is a one parameter family of algebraic equations, where the parameter is $x \in (x_0, \infty)$.

We will prove the following result.

Lemma 11
(i) There exists $x_+ \in (x_0, \infty)$, such that for each $x > x_+$ the equation $W_x(h) = 0$ has a positive root.
(ii) There exists $x_- \in (x_0, \infty)$, such that for each $x > x_-$ the equation $W_x(h) = 0$ has a negative root.

The proof of the lemma is done together with the proof of Proposition 16. Assume now that Lemma 11 is true. This allows us to define two new functions h_+^* and h_-^*. We will describe in details the definition of h_+^*. We set

Definition 12
Let $x \in (x_+, \infty)$. Then the set of positive roots of Equation (13) is not empty and we set

$$h_+^*(x) = \min\{h > 0 : \ W_x(h) = 0\}.$$

The relation between the functions h_+ and h_+^* is the following

Proposition 13
If Lemma 11 is true, then for $x \in (x_+, \infty)$ we have the inequality $h_+(x) < h_+^*(x)$.

Proof. Let us fix $x \in (x_+, \infty)$. In the interval $[x, x + h_+(x)]$, i.e. for $h \in [0, h_+(x)]$ the line $l(x, h)$ lies below the graph of the function $L + \varepsilon$. This follows directly from the definition of the function $h_+(x)$. Hence in this interval the line $l(x, h)$ cannot intersect the graph of the function $T^{(3)}_{x,\varepsilon} + T^{(3)}_{x,L}$ because of inequality (9). Hence the equation $W_x(h) = 0$ has no roots in the interval $h \in [0, h_+(x)]$. But this means that $h_+(x) < h_+^*(x)$, which ends the proof of Proposition 13.

Assume once more that Lemma 11 is true. We have

Definition 14
Let $x \in (x_-, \infty)$. Then the set of negative roots 13 is not empty and we set

$$h_-^*(x) = \max\{h < 0 : \ W_x(h) = 0\}.$$

The relation between the functions h_- and h_-^* is as follows

Proposition 15
If Lemma 11 is true, then for $x \in (x_-, \infty)$ we have $h_-(x) > h_-^*(x)$.

The proof of Proposition 15 is similar to the proof of Proposition 13, so we skip it.

3.5. The proof of the main result

Now we will prove the Proposition 16 formulated below. Equation (13) we are interested in, can be written in the form

$$A_3(x) \cdot h^3 + A_2(x) \cdot h^2 + A_1(x) \cdot h + A_o(x) = 0 \tag{14}$$

where, using formulas (5)–(13) we have

$$A_3(x) = \frac{1}{6}(L^{(3)}(x) + \varepsilon^{(3)}(x)) = \frac{1}{48} \cdot \frac{8\sqrt{x}(y+2) + y^3(3y-2)}{x^2\sqrt{x}y^3},$$

$$A_2(x) = \frac{1}{2}(L^{(2)}(x) + \varepsilon^{(2)}(x)) = \frac{-1}{8} \cdot \frac{4\sqrt{x} + y^3}{x\sqrt{x}y^2},$$

$$A_1(x) = \frac{y+2}{\sqrt{x}},$$

$$A_0(x) = 2\sqrt{x}y.$$

Now, taking into account the fact, that for x sufficiently large $A_3(x) > 0$, we divide equation (14) by $A_3(x)$ in order to obtain the form

$$h^3 + B_2(x) \cdot h^2 + B_1(x) \cdot h + B_0(x) = 0, \tag{15}$$

where

$$B_2(x) = \frac{A_2(x)}{A_3(x)} = -6x\frac{4\sqrt{x}y + y^4}{8\sqrt{x}y + 16\sqrt{x} + 3y^4 - 2y^3},$$

$$B_1(x) = \frac{A_1(x)}{A_3(x)} = 48x^2\frac{y^3}{8\sqrt{x}y + 16\sqrt{x} + 3y^4 - 2y^3},$$

$$B_0(x) = \frac{A_o(x)}{A_3(x)} = 96x^3\frac{y^4}{8\sqrt{x}y + 16\sqrt{x} + 3y^4 - 2y^3}.$$

For further analysis of (15) it will be convenient to use some Landau symbols. Let us recall that for a function g defined in the neighbourhood of $+\infty$ one writes $g = o(1)$ if and only if $\lim_{x\to+\infty} g(x) = 0$. Using this convention, we can write

$$B_2(x) = -6x\frac{\frac{1}{2} + o(1)}{1 + o(1)},$$

$$B_1(x) = 48x^2\frac{o(1)}{1 + o(1)},$$

$$B_0(x) = 96x^3\frac{o(1)}{1 + o(1)}.$$

This makes it possible to write 15 in the form

$$h^3 - 6x\frac{\frac{1}{2} + o(1)}{1 + o(1)}h^2 + 48x^2\frac{o(1)}{1 + o(1)}h + 96x^3\frac{o(1)}{1 + o(1)} = 0.$$

Now apply the substitution $h = \theta x$, which leads to the form

$$\theta^3 x^3 - 6x\frac{\frac{1}{2}+o(1)}{1+o(1)}\theta^2 x^2 + 48x^2\frac{o(1)}{1+o(1)}\theta x + 96x^3\frac{o(1)}{1+o(1)} = 0. \tag{16}$$

Since we work only with $x > 0$, we can divide the last equation by x^3, and obtain the following equation (with unknown θ),

$$\theta^3 - 6\frac{\frac{1}{2}+o(1)}{1+o(1)}\theta^2 + 48\frac{o(1)}{1+o(1)}\theta + 96\frac{o(1)}{1+o(1)} = 0. \tag{17}$$

Finally, taking into account the equality

$$\frac{\frac{1}{2}+o(1)}{1+o(1)} = \frac{1}{2} + o(1)$$

we can write equation (16) in the form

$$\theta^3 - 3\theta^2 + v_2(x)\theta^2 + v_1(x)\theta + v_0(x) = 0, \tag{18}$$

where $v_1(x)$, $v_2(x)$, $v_0(x)$ are three positive functions defined in a neighbourhood of $+\infty$ and tending to 0 when x tends to $+\infty$. If for a fixed x' we find a number θ' being a root of equation (17), then the number $h' = \theta' \cdot x'$ is a root of 15. It is then enough to study equation (17). We shall prove much more. Namely we have the following result.

Proposition 16
For each $\alpha > 0$ there exists a point x_2 such that for each $x > x_2$ equation (18) *has in the interval $[-\alpha, \alpha]$ exactly two roots θ_- and θ_+, and moreover $\theta_- < 0 < \theta_+$.*

Proof. Indeed, Proposition 16 is stronger than Lemma 11, where we need only the existence of a negative root and of a positive root. In Proposition 16 we prove not only that the roots exist, but also that we can find the solutions in an arbitrary open interval containing the origin. Without loss of generality, we may assume, that $\alpha \le 1$. Let us fix then a positive number $1 \ge \alpha > 0$ and choose x_2 so large, that for $x > x_2$ we have

$$v_2(x) \cdot \alpha^2 + v_1(x) \cdot \alpha + v_0(x) < 2\alpha^2 \tag{19}$$

and

$$v_2(x) \cdot \alpha^2 - v_1(x) \cdot \alpha + v_0(x) < 2\alpha^2. \tag{20}$$

Such an x_2 exists since all three functions v_2, v_1, v_0 are $o(1)$ when x tends to $+\infty$. Let us fix $x > x_2$. We rewrite equation (17)) in the form $f(\theta) = g(\theta)$, where

$$f(\theta) = \theta^3 + v_2(x) \cdot \theta^2 + v_1(x) \cdot \theta + v_0(x)$$

and

$$g(\theta) = 3 \cdot \theta^2.$$

Let us set $h(\theta) = f(\theta) - g(\theta)$ and let us consider the interval $[0, \alpha]$. We have $h(0) = f(0) - g(0) = v_0(x) > 0$ and, (since $\alpha < 1$ and using the inequality (19)) we obtain

$$h(\alpha) = f(\alpha) - g(\alpha) = \alpha^3 + v_2(x) \cdot \alpha^2 + v_1(x) \cdot \alpha + v_0(x) < \alpha^2 + 2\alpha^2 - 3\alpha^2 = 0.$$

Thus equation (17) has a root $\theta_+ \in (0, \alpha)$.

Now we will consider the interval $[-\alpha, 0]$. For $\theta = 0$ we have, as above $h(0) = v_0(x) > 0$. For $\theta = -\alpha$ we have (since $-\alpha^3 < 0$ and we have inequality (20))

$$\begin{aligned} h(-\alpha) = f(-\alpha) - g(-\alpha) &= -\alpha^3 + v_2(x) \cdot \alpha^2 - v_1(x) \cdot \alpha + v_0(x) - 3\alpha^2 \\ &< v_2 \cdot \alpha^2 - v_1(x) \cdot \alpha + v_0(x) - 3\alpha^2 < 2\alpha^2 - 3\alpha^2 < 0. \end{aligned}$$

Once more the continuity argument implies the existence of the root θ_- of the equation in the interval $(-\alpha, 0)$. Let us remark, that $\theta_- \cdot x = h^*_-(x)$ and $\theta_+ \cdot x = h^*_+(x)$. This ends the proof of Proposition 16 and hence moreover, the proof of Lemma 11.

3.6. The order of magnitude of lenses

By the results of the previous subsection, we can consider four functions h_-, h_+, h^*_- and h^*_+, which are defined in an interval (M, ∞), and such that the following inequalities hold for each $x \in (M, \infty)$,

$$h^*_-(x) < h_-(x) < 0 < h_+(x) < h^*_+(x).$$

Our aim is to establish the order of magnitude at $+\infty$ of the difference $H(x) = h_+(x) - h_-(x)$. We will prove the following result.

Proposition 17
The function H satisfies the relation

$$H(x) = o(x),$$

when x tends to $+\infty$.

Proof. This follows directly from the property formulated in Proposition 16. Indeed, it is sufficient to show separately that $h_+(x) = o(x)$ and $|h_-(x)| = o(x)$. To prove the first relation, let us fix a positive number $\epsilon > 0$. It follows from Proposition 16 (setting $\alpha = \epsilon$) that there exists $M_1 > M$, such that $x > M_1$ implies, that there exists a number $\theta < \epsilon$ (θ depending on x) such that $h^*_+(x) = \theta \cdot x$. But this means that

$$\frac{h^*_+(x)}{x} < \epsilon$$

for $x > M_1$. The proof for h^*_- is similar.

Now we can prove a theorem on the order of magnitude of the length of lenses S_k using Proposition 17. First we shall prove the following lemma about sequences tending to $+\infty$.

Lemma 18

Suppose that we have four sequences $(x_k^-)_1^\infty$, $(x_k^+)_1^\infty$, $(z_k)_1^\infty$ *and* $(e_k)_1^\infty$ *such that*

$$0 < x_k^- \le e_k < e_{k+1} \le x_k^+, \tag{21}$$

$$x_k^- \le z_k \le x_k^+, \tag{22}$$

$$\lim_{k\to\infty} e_k = +\infty, \tag{23}$$

$$\lim_{k\to\infty} \frac{x_k^+ - x_k^-}{z_k} = 0. \tag{24}$$

Then

$$\lim_{k\to\infty} \frac{e_{k+1} - e_k}{e_k} = 0.$$

Proof. From (21) and (23) we deduce that

$$\lim_{k\to\infty} x_k^+ = +\infty.$$

It must be also

$$\lim_{k\to\infty} x_k^- = +\infty.$$

Indeed, suppose that there exists an infinite subset $\mathbb{L} \subset \mathbb{N}$ and a constant $K > 0$ such that $0 \le x_n^- \le K$ for $n \in \mathbb{L}$. Then for $n \in \mathbb{L}$ we have

$$0 \le \frac{x_n^+ - K}{z_n} \le \frac{x_n^+ - x_n^-}{z_n}.$$

Hence by (24),

$$\frac{x_n^+ - K}{z_n} \to 0, \qquad n \in \mathbb{L}.$$

This implies that $\lim_{n\in\mathbb{L}} z_n = +\infty$. In consequence,

$$\lim_{n\in\mathbb{L}} \frac{x_n^+}{z_n} = 0,$$

thus there exists $n \in \mathbb{L}$ such that $x_n^+ < z_n$, but this is impossible.

From the inequality

$$\frac{x_k^+ - x_k^-}{x_k^+} \le \frac{x_k^+ - x_k^-}{z_k}$$

we deduce that

$$\lim_{k\to+\infty} \frac{x_k^-}{x_k^+} = 1$$

which gives

$$\lim_{k\to+\infty} \frac{x_k^+ - x_k^-}{x_k^-} = 0.$$

But

$$\frac{x_k^+ - x_k^-}{e_k} \le \frac{x_k^+ - x_k^-}{x_k^-},$$

then

$$\lim_{k\to\infty} \frac{x_k^+ - x_k^-}{e_k} = 0.$$

Since

$$\frac{e_{k+1} - e_k}{e_k} \leq \frac{x_k^+ - x_k^-}{e_k}$$

we have

$$\lim_{k\to\infty} \frac{e_{k+1} - e_k}{e_k} = 0$$

and this ends the proof of Lemma 18.

Lemma 19
The graph of the function π^* lies between the graphs of the functions $L - \varepsilon$ and $L + \varepsilon$, where the functions L and ε are defined by (3) and (4).

Proof. Suppose the opposite. Then there exist two consecutive prime numbers p_n and p_{n+1} such that the points $A = (p_n, n)$ and $B = (p_{n+1}, n + 1)$ lie between $L - \varepsilon$ and $L + \varepsilon$ and the segment $[A; B]$ cuts the graph of $L - \varepsilon$ or $L + \varepsilon$. But the subgraph of $L + \varepsilon$ is convex, then $[A; B]$ cuts only the graph of $L - \varepsilon$. This means, that there exists a point $x \in (p_n, p_{n+1})$ such that the point $X = (x, n)$ lies below the graph of $L - \varepsilon$. But $X = (x, \pi(x))$, then from the definition of the error term, X lies between the graphs of $L - \varepsilon$ and $L + \varepsilon$. This ends the proof of Lemma 19.

Lemma 20
Let S_k be a lens defined by the extremal prime numbers e_k and e_{k+1}. Then the straight line joining the points $U = (e_k, \pi(e_k))$ and $V = (e_{k+1}, \pi(e_{k+1}))$ cannot cut the graph of $L - \varepsilon$ in two distinct points.

Proof. This follows from the Lemma 19 since, by the definition of extremal points, the whole graph of π^* lies below the straight line joining the points U and V.

The main theorem of this section is the following.

Theorem 21
With the notations as above if the Riemann Conjecture is true, then

$$\lim_{k\to+\infty} \frac{e_{k+1}}{e_k} = 1.$$

Proof. Let U and V be as in Lemma 20. Take the straight line $l(U, V)$ joining U and V and translate it to the position l^*, where the straight line l^* is parallel to $l(U, V)$ and tangent to the graph of $L - \varepsilon$. This line l^* cuts the graph of $L + \varepsilon$ in points U^* and V^*, whose first coordinates are x_k^- and x_k^+ respectively, and the tangent point is z_k. It is not hard to check, that the sequences $(x_k^-)_1^\infty$, $(x_k^+)_1^\infty$, $(z_k)_1^\infty$ and $(e_k)_1^\infty$ satisfy the assumptions of Lemma 18. Then this ends the proof of the theorem.

We have an equivalent formulation.

Corollary 22
The length of lenses $x \to S(x)$ satisfies the equality $S(x) = o(x)$.

4. Part III

4.1. Additional remarks

In [7] we wrote: *It is natural to ask if one can prove the results like Theorem 21 or Corollary 22 without assuming the Riemann Hypothesis. Maybe this is possible, but it seems, that the method used in this paper is insufficient.* And also: *I was not able to prove Theorem 21 using $L(x) = Li(x)$ and*

$$\varepsilon(x) = O\Bigg(x \cdot \exp\left(\frac{A(\ln x)^{\frac{3}{5}}}{(\ln(\ln x))^{\frac{1}{5}}}\right)\Bigg). \tag{25}$$

(Un)fortunately it appeared, that we were to pessimistic. McNew in [2], using similar methods as in [7], but applied to the Vinogradow error term (25), proved Conjecture I without assuming the Riemann Hypothesis. In the same paper [2], he gave unconditional proofs of Conjectures F and G.

4.2. The conjecture $\gamma/2$

Among the conjectures formulated in [7], the Conjecture H seems to be the most interesting. It is related to the stronger version of the Corollary 22, which is proved in the present paper with the use of the Riemann Hypothesis, but which is true, as it was proved in [2], unconditionally. In the same paper [2] it is observed, that the Riemann Hypothesis allows us to formulate a stronger version of Corollary 22. McNew deduces this version from his proof of the theorem on the behaviour of the sequence $(e_k)_1^\infty$ of extremal primes. We will check below, that a little deeper analysis of the proof of Corollary 22 given in the present paper, leads to the same conclusion as Theorem 2.4 in [2]. Namely, we have the following.

Theorem 23
In the notation as in the previous section, there exists a constant $C > 0$ such that

$$S(x) \leq C \cdot x^{\frac{3}{4}} \cdot y^{\frac{3}{2}},$$

where $y = \ln(x)$.

Before proving this theorem, we will return to the proof of the relation $S(x) = o(x)$. As we have observed, the function $S(x)$ is controlled by the function $H(x) = h_+(x) - h_-(x)$ considered in Proposition 17. Hence to control the function $H(x)$ it is sufficient to control the function $x \to h_+^*(x)$ defined by the relation $h_+^*(x) = \theta(x) \cdot x$, where the function $\theta(x)$ satisfies the implicit equation (16))

$$\theta^3 - 3\theta^2 + v_2(x)\theta^2 + v_1(x)\theta + v_0(x) = 0. \tag{26}$$

We have proved above, that this implicit equation has a positive solution $x \to \theta(x)$, such that $\lim_{x\to\infty} \theta(x) = 0$ and this was enough for $S(x) = o(x)$.

First we will prove a proposition, which is weaker than Theorem 23 but stronger than $S(x) = o(x)$.

Proposition 24
For each $k \in \mathbb{N}$ there exists $M_k > 0$ such that

$$\theta(x) < \frac{3^{\frac{k}{2}}}{y^{\frac{k}{2}}}, \tag{27}$$

for $x > M_k$.

Proof. We will use the inductive argument. Taking into account the particular form of the coefficients $v_2(x)$, $v_1(x)$ and $v_0(x)$ we may state, that there exist the polynomials $U_2(y)$, $U_1(y)$ and $U_0(y)$ (with respect to $y = \ln(x)$) such that the following inequality holds

$$3\theta^2 < \theta^3 + \frac{\frac{6}{y} + \frac{1}{\sqrt{x}}U_2(y)}{1 + \frac{2}{y}}\theta^2 + \frac{\frac{1}{\sqrt{x}}U_1(y)}{1 + \frac{2}{y}}\theta + \frac{\frac{1}{\sqrt{x}}U_0(y)}{1 + \frac{2}{y}}. \tag{28}$$

Clearly, we may assume, that $0 < \theta < 1$ (for x sufficiently large), and in consequence, that $\theta^3 < \theta^2$. Hence, we may deduce from (28) that

$$2\theta^2 < \frac{\frac{6}{y} + \frac{1}{\sqrt{x}}U(y)}{1 + \frac{2}{y}},$$

where $U(y)$ is a polynomial with respect to $y = \ln(x)$. Hence, for x large enough, we have the inequality

$$2\theta^2 < \frac{6}{y},$$

which gives the inequality from (27) for $k = 1$.

Assume now, that there exists $M_k > 0$ such that for $x > M_k$ we have

$$\theta(x) < \frac{3^{\frac{k}{2}}}{y^{\frac{k}{2}}}.$$

We will prove that there exists $M_{k+1} > 0$ such that for $x > M_{k+1}$ we have

$$\theta(x) < \frac{3^{\frac{k+1}{2}}}{y^{\frac{k+1}{2}}}.$$

We return once more to inequality (28). Using $\theta < 1$ we can obtain from this inequality that

$$2\theta^2 < \frac{6}{y+2}\theta^2 + \frac{1}{\sqrt{x}}U_3(y),$$

where $U_3(y)$ is a polynomial with respect to $y = \ln(x)$. Hence, for sufficiently large $x > M_{k+1}$, we have the inequality

$$2\theta^2 < \frac{6}{y}\theta^2.$$

Since, clearly we may assume that $M_{k+1} > M_k$ then it follows from the inductive assumption, that for $x > M_{k+1}$ we have

$$2\theta^2 < \frac{6}{y}\frac{3^k}{y^k},$$

and thus

$$\theta^2 < \frac{3^{k+1}}{y^{k+1}}.$$

This ends the proof of Proposition 24.

It appears, that one can "squeeze" much more from equation (26) in order to obtain the proof of Theorem 23 presented below.

Proof of Theorem 23. We return to inequality (28). Namely,

$$3\theta^2 < \theta^3 + \frac{\frac{6}{y} + \frac{1}{\sqrt{x}}U_2(y)}{1+\frac{2}{y}}\theta^2 + \frac{\frac{1}{\sqrt{x}}U_1(y)}{1+\frac{2}{y}}\theta + \frac{\frac{1}{\sqrt{x}}U_0(y)}{1+\frac{2}{y}}.$$

To obtain Proposition 24 we used only the fact that the functions $U_2(y)$, $U_1(y)$ and $U_0(y)$ are polynomials. Clearly, one may find many polynomials, U_2, U_1, U_0 for which inequality (28) holds. In particular, one may choose the above polynomials to have all the degree 3 and not to big coefficients. More exactly, there exists a constant $M > 0$ such that for $x > M$ we get $\theta(x) < 1$ and

$$3\theta^2 < \theta^3 + \frac{6}{y+2}\theta^2 + \frac{1}{\sqrt{x}}\frac{y^3}{1+\frac{2}{y}}\theta^2 + \frac{1}{\sqrt{x}}\frac{8y^3}{1+\frac{2}{y}}\theta + \frac{1}{\sqrt{x}}\frac{12y^3}{1+\frac{2}{y}}.$$

Taking into account the fact that $\theta < 1$ and $\theta^3 < \theta^2$ we obtain the inequality

$$2\theta^2 < \frac{6}{y+2}\theta^2 + \frac{1}{\sqrt{x}}\left(\frac{y^3}{1+\frac{2}{y}} + \frac{8y^3}{1+\frac{2}{y}}\right)\theta + \frac{1}{\sqrt{x}}\frac{12y^3}{1+\frac{2}{y}}.$$

Let us consider now the term

$$\frac{9y^4}{y+2}\theta.$$

Applying Proposition 24 we find $M' > M > 0$ such that for $x > M'$ there is $\theta(x) \le \frac{3}{y}$. Thus

$$\frac{9y^4}{y+2}\theta < \frac{3^3}{y}\frac{y^4}{y+2}$$

for $x > M'$. Hence there exists $M'' > M'$ such that for $x > M''$ we have

$$\frac{9y^4}{y+2}\theta < \frac{y^4}{y+2},$$

which gives the inequality

$$2\theta^2 < \frac{6}{y+2}\theta^2 + \frac{1}{\sqrt{x}}\frac{10y^4}{y+2},$$

or equivalently, the inequality

$$\Big(2 - \frac{6}{y+2}\Big)\theta^2 < \frac{1}{\sqrt{x}}\frac{y}{y+2}10y^3,$$

which may be rewritten in the form

$$\theta^2 < 5\frac{y}{y-1}\frac{1}{\sqrt{x}}y^3.$$

Finally, we choose a constant $K > M''$ such that $\frac{y}{y-1} < \frac{5}{4}$ and we obtain the inequality

$$\theta < \frac{5}{2}\frac{1}{x^{\frac{1}{4}}}y^{\frac{3}{2}},$$

which is valid for $x > K$. Clearly, it is enough to prove the Theorem 23 since $h_+^*(x) = \theta(x) \cdot x$.

Theorem 23 brings us some information about Conjecture H. Since the sum of lenses contained in the interval $[2, x]$ equals – roughly speaking – x and their number is of order x^α, then $x^\alpha \cdot x^{\frac{3}{4}}$ must be as large as x. Hence $\alpha \geq \frac{1}{4}$. This observation is mentioned in [2] (Corollary 2.6). On the other hand, there is a strong numerical argument supporting the conjecture $\alpha = \gamma/2$. We will present these numerical data in a future paper, however, in the next subsection, we give some tables and some graphs for to illustrate what we mean by the term "supporting argument".

4.3. Some more numerical data

At present we know the exact values of the sequence e_k for $k \leq 8 \cdot 10^4$. The tables inserted below contain some selected data, which one may use to confirm (or – if one prefers – to disprove) the $\frac{\gamma}{2}$ conjecture formulated as follows. Let us denote $\beta_k = \frac{\ln(k)}{\ln(e_k)}$. Then the $\gamma/2$ conjecture say simply that $\lim \beta_k = \gamma/2$. The presented data seem to be promising with respect to the $\gamma/2$ conjecture. On the other hand, the examples like the conjecture of Mertens, show that one should be careful. The fact, that the constant γ is present in many theorems of analytical theory of numbers is, perhaps, an additional argument for optimists.

Below we present also two pictures. First of them shows the convex hull of the graph of the function $\pi(x)$ for $1 \leq x \leq 113$. The second shows the behaviour of the ratio $\pi_e(x)/x^{\frac{\gamma}{2}}$ depending on $\log(x)$.

The list of e_k, where $2000 < k \leq 5000$ and $k \equiv 0 (\mathrm{mod}\ 100)$.

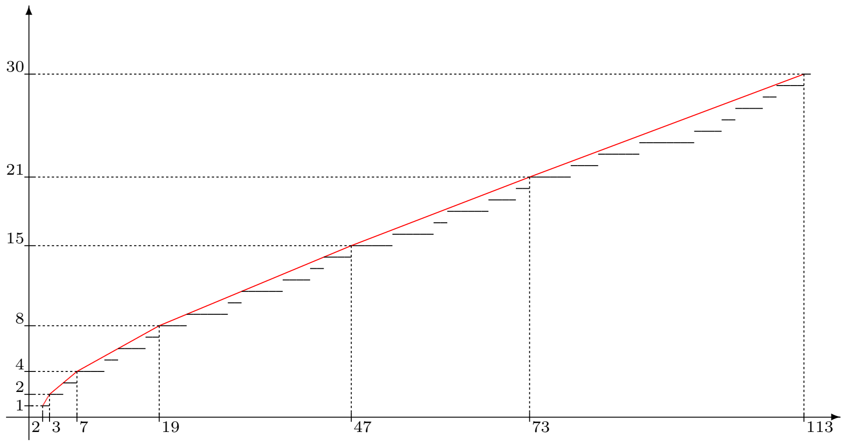
e_{2000}	366475869647	e_{3500}	2629506983759
e_{2100}	435449834927	e_{3600}	2913566116711
e_{2200}	526631656829	e_{3700}	3190893571937
e_{2300}	625382499043	e_{3800}	3510196910639
e_{2400}	727995116377	e_{3900}	3833525419133
e_{2500}	842057152381	e_{4000}	4199423202899
e_{2600}	975455207557	e_{4100}	4572918641341
e_{2700}	1098339926353	e_{4200}	4955275213949
e_{2800}	1234264464703	e_{4300}	5341974321851
e_{2900}	1388032354369	e_{4400}	5816608130917
e_{3000}	1563678255869	e_{4500}	6325581587071
e_{3100}	1746099699947	e_{4600}	6803401026713
e_{3200}	1940953406761	e_{4700}	7330968666577
e_{3300}	2143710526487	e_{4800}	7891749045409
e_{3400}	2407357435771	e_{4900}	8431057440089
		e_{5000}	9007738703933

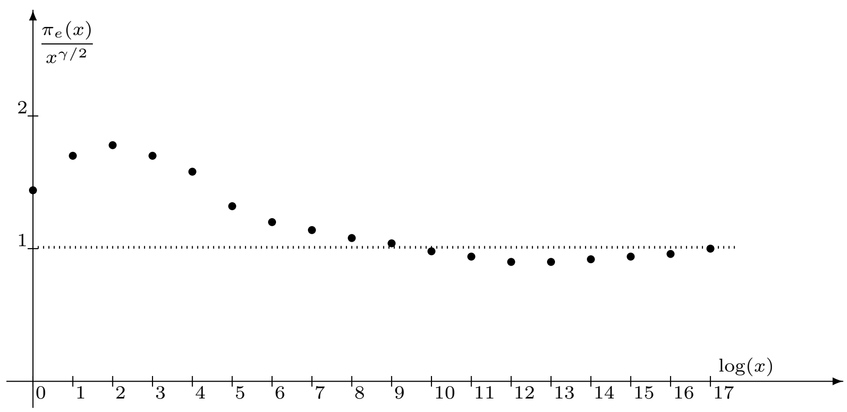
The list of e_k, where $5000 \leq k \leq 80000$ and $k \equiv 0 (\mathrm{mod}\ 5000)$.

e_{5000}	8993279276101	e_{45000}	14561650764869701
e_{10000}	92375151455953	e_{50000}	20596671738838703
e_{15000}	365792669405717	e_{55000}	27207858885194953
e_{20000}	981254018753539	e_{60000}	37527564754591409
e_{25000}	206980315408291	e_{65000}	48947619329037853
e_{30000}	3757752577836253	e_{70000}	62377984224294623
e_{35000}	6306717938948543	e_{75000}	78299477848810957
e_{40000}	9797619494633261	e_{80000}	97052934098045459

The next table illustrates the behaving of the ratio $\frac{\pi_e(x)}{x^{\gamma/2}}$.

$\pi_e(x)$	x	$\frac{\pi_e(x)}{x^{\gamma/2}}$	$\pi_e(x)$	x	$\frac{\pi_e(x)}{x^{\gamma/2}}$
1	2	1,457	414	1017804913	1,041
4	19	1,710	757	10016844407	0,984
7	113	1,788	1410	100124651999	0,944
13	1129	1,711	2622	1000519435087	0,902
23	10343	1,596	5151	10005000431033	0,912
37	102679	1,324	10242	100054690967381	0,933
66	1021487	1,217	20113	1000045596177333	0,943
122	10716313	1,141	40241	10000581581252813	0,970
224	102611477	1,091	80748	100009811119192067	1,002





Acknowledgement. I would like to thank to dr Witold Jarnicki and dr Rafał Kurczab for their inestimable help in the process of producing the set of data presented in this paper.

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Received: September 12, 2018; final version: January 13, 2019;
available online: February 8, 2019.