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Initial boundary value problem for a mechanical system with local stroke change of stiffness

Abstract. The aim of this paper is to present a new method of solving the initial boundary value problem for a mechanical system with local stroke change of stiffness. The method is based on the theory of distributions.

1. Introduction

We consider the small vibrations of an Euler beam of the length l in its symmetry plane, with stroke change of stiffness of the beam described by the function $\beta\colon \langle 0,l\rangle \longrightarrow \bar{\mathbb{R}}$ defined as follows

$$\beta(x) = \begin{cases} EJ, & x \in \langle 0, x_1) \cup (x_2, l\rangle, \\ 0, & x \in \{x_1, x_2\} \\ +\infty, & x \in (x_1, x_2). \end{cases} \tag{1}$$

The two joints of the beam are located respectively at the points with abscissae x_1 and x_2, and their small vibrations are given by (see [3], [4], [5])

$$\left[\frac{\partial U}{\partial x}(x_i^+, t) - \frac{\partial U}{\partial x}(x_i^-, t)\right]\delta''_{x_i}; \qquad i = 1, 2.$$

The main idea of this formula comes from the paper [3] and is based on a sequential approach. The function $U(x,t)$ is the deflection at the point $x \in \langle 0, l\rangle$ at the moment t, δ''_{x_i} denotes the second derivative of the Dirac distribution δ_{x_i} concentrated at the point x_i.

The small vibrations of the system under consideration are described by the equation

$$\frac{\partial^2}{\partial x^2}\left[\beta(x)\frac{\partial^2 U}{\partial x^2} + \alpha_0 J\frac{\partial^3 U}{\partial x^2 \partial t}\right] + h(x)\frac{\partial^2 U}{\partial t^2} = f(x,t), \tag{2}$$

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where $\alpha_0 J = \text{const}$; $h(x)$ denotes the distribution of the beam masses; $f(x,t)$ is the distribution of the external forces applied to the beam in its symmetry plane.

Since there are the joints respectively at points x_1 and x_2, the function

$$U(.,t) \in C^0(\langle 0,l\rangle) \cap C^4(\langle 0,l\rangle \setminus \{x_1,x_2\})$$

and it is linear in the interval (x_1,x_2) with respect to the absolute stiffness of this element.

We could describe the vibration of our beam by the methods of classical mathematical analysis but this requires taking into consideration the vibrations of three elements of a beam $\langle 0,x_1)$, (x_1,x_2) and $(x_1,l\rangle$. It is both arduous and labour-concerning.

The purpose of this paper is to present a new method of determining the beam vibrations. The point of the matter is that the real situation is modelled as follows. The absolutely stiff part of the beam we describe as the only point of mass located in x_1 (i.e., we let $x_2 = x_1 = \frac{1}{2}(x_1+x_2)$) which bears all dynamical reactions that appear in this stiff part of the beam (this needs a distributional description). The method gives us the possibility to find the discontinuous at x_1 solutions of the substitute beam. It is achieved by introducing δ'''_{x_1} into the initial boundary problem substituting the real problem. Then we return to the real problem by the connection of the points x_1 and x_2 fitting the segment $y(x,t) = p(t)x + q(t)$ in the interval $\langle x_1,x_2\rangle$ to make continuous the solution of the real problem.

The mass of the stiff element of the beam and its dynamical reaction located at the point x_1 are analytically characterized by $\rho F(x_2 - x_1)$ – the mass of the absolutely stiff part of the beam (x_1,x_2) and by $\frac{1}{12}\rho_1 F_1(x_2-x_1)^2$ – its moment of inertia, computed with respect to the middle point $\frac{1}{2}(x_1+x_2)$ of this part of the beam. The symbols ρ, ρ_1, F, F_1 stand for the densities and for the cross-section areas of the parts $\langle 0,x_1) \cup (x_2,l\rangle$ and (x_1,x_2) of the beam, respectively. The equation for the function $W(x,t), x \in \langle 0,l\rangle \setminus (x_1,x_2)$ representing vibrations of the substitute beam is of form (10), cf. Section 3.

The knowledge of $W(x,t)$, the solution of the initial boundary problem of the substitute beam, and of the fact that it is impossible to bend the absolutely stiff element (its vibrations are planar) provide the possibility of construction of the function $U(x,t)$, the solution of the real beam in the interval $\langle 0,l\rangle$, via the formula

$$U(x,t) = \begin{cases} W(x,t), & x \in \langle 0,x_1) \cup (x_2,l\rangle \\ a(t)x + b(t), & x \in (x_1,x_2). \end{cases} \tag{3}$$

Since $W(x_1^-,t) = a(t)x_1 + b(t)$, $W(x_2^+,t) = a(t)x_2 + b(t)$ the continuity at x_1 and x_2 yields

$$a(t) = \frac{W(x_2^+, t) - W(x_1^-, t)}{x_2 - x_1}, \qquad b(t) = \frac{x_1 W(x_2^+, t) - x_2 W(x_1^-, t)}{x_2 - x_1}. \tag{4}$$

2. Preliminaries

Let us define the internal damping of the beam

$$\alpha(x) = \begin{cases} \alpha_0\,, & x \in \langle 0, x_1) \cup (x_2, l\rangle \\ 0, & x \in \{x_1, x_2\} \\ \alpha_1\,, & x \in (x_1, x_2) \end{cases} \tag{5}$$

and the stiffness of the beam

$$\beta(x) = \begin{cases} EJ, & x \in \langle 0, x_1) \cup (x_2, l\rangle \\ 0, & x \in \{x_1, x_2\} \\ M, & x \in (x_1, x_2) \end{cases}$$

Here E denotes the Young modulus, J is the axial moment of inertia, M is defined as the stiffness (we assume that $M = \text{const}$).

One edgepoint of the beam is fixed while the other is slidable.

Now let au assume that there are no external forces and therefore, the small transversal vibrations of the beam under consideration are described by the formula

$$\frac{\partial^2}{\partial x^2}\left(\beta(x)\frac{\partial^2 U}{\partial x^2} + \alpha(x)\frac{\partial^3 U}{\partial x^2 \partial t}\right) + h(x)\frac{\partial^2 U}{\partial t^2} = 0 \tag{6}$$

$U(.,t) \in C^0(\langle 0, l\rangle) \cap C^4(\langle 0, x_1) \cup (x_2, l\rangle)$, $U(x,t) = ax + b$, $x \in (x_1, x_2)$, a, b-const.; $h(x) = \rho F + \rho_1 F_1 (x_2 - x_1)\delta_{x_1}$.

The constants a and b are chosen so that $U(.,t) \in C^0(\langle 0, l\rangle)$.

The boundary conditions are

$$U(0,t) = 0, \qquad U(l,t) = 0, \qquad \frac{\partial U}{\partial x}(0,t) = 0, \qquad \frac{\partial U}{\partial x}(l,t) = 0. \tag{7}$$

The initial conditions are as follows

$$\begin{aligned} U(x,0) = \varphi_1(x), \qquad & \frac{\partial U}{\partial t}(x,0) = \varphi_2(x) \qquad \text{for } x \in \langle 0, l\rangle \\ \varphi_1(0) = \varphi_1(l), \qquad & \varphi_2(0) = \varphi_2(l). \end{aligned} \tag{8}$$

Let us assume *the physical conditions*:

$$\frac{\partial^2 U}{\partial x^2}(x_i^-, t) = \frac{\partial^2 U}{\partial x^2}(x_i^+, t) = 0, \qquad i = 1, 2. \tag{9}$$

3. Method of generalized functions

The initial boundary problem (3), (4), (5), (6) of the vibrations of the substitute beam is described by the formula

$$
\begin{aligned}
&(a) \qquad EJ\frac{\partial^4 W}{\partial x^4} + \alpha_0 J\frac{\partial^5 W}{\partial x^4 \partial t} + (\rho F + \rho_1 F_1 (x_2 - x_1)\delta_{x_1})\frac{\partial^2 W}{\partial t^2} \\
&(b) \qquad\qquad + \gamma_1 \left(\frac{\partial^3 W(x_1^+, t)}{\partial x \partial t^2} - \frac{\partial^3 W(x_1^-, t)}{\partial x \partial t^2}\right)\delta''_{x_1} \\
&(c) \qquad\qquad + \frac{\rho_1 F_1 (x_2 - x_1)^2}{12}\left(\frac{\partial^2 W(x_1^+, t)}{\partial t^2} - \frac{\partial^2 W(x_1^-, t)}{\partial t^2}\right)\delta'_{x_1} \\
&(d) \qquad\qquad + \gamma_2 \left(\frac{\partial^5 W(x_1^+, t)}{\partial x^3 \partial t^2} - \frac{\partial^5 W(x_1^-, t)}{\partial x^3 \partial t^2}\right)\delta'''_{x_1} \\
&\qquad\qquad = 0
\end{aligned} \tag{10}
$$

where

(a) the coefficients at δ_{x_1} and δ'_{x_1} are used to describe the dynamics of the part (x_1, x_2); (see [4])

(b) δ''_{x_1} refers to the joint of the substitute beam; (see [5])

(d) δ'''_{x_1} leads to the discontinuous solution;

(c) δ'_{x_1} characterizes the pair of the forces in x_1; (see [4]),

and γ_1 and γ_2 are the parameters that fit to obtain the continuous solution of the problem and to make the units matching.

We are using the Fourier method to solve the eigenproblem associated to the substitute problem under consideration.

Let us assume (with the constant p and q)

$$W(x,t) = X(x)T(t), \tag{11}$$

$$a(t) = pT(t), \qquad b(t) = qT(t).$$

Substituting (11) into (6), after some calculations, we obtain

$$\frac{\ddot{T}}{T + \frac{\lambda_0}{E}\dot{T}} = \frac{-EJX^{IV}}{D} = -\omega^2,$$

where

$$\begin{aligned}D &= (\rho F + \rho_1 F_1 \delta_{x_1}(x_2 - x_1))X \\ &+ \gamma_1 (X'(x_1^+) - X'(x_1^-))\delta''_{x_1} \\ &+ \gamma_2 (X'''(x_1^+) - X'''(x_1^-))\delta'''_{x_1} \\ &+ \frac{1}{12}\rho_1 F_1 (x_2 - x_1)^2 (X(x_1^+) - X(x_1^-))\delta'_{x_1}\end{aligned}$$

and the constant $-\omega^2$ is negative to obtain positive eigenvalues.

Hence

$$\ddot{T} + \frac{\alpha_0 \omega^2}{E}\dot{T} + \omega^2 T = 0, \tag{12}$$

($\dot{} = \frac{d}{dt}$) and

$$\begin{aligned}X^{IV} - \lambda^4 X &= \frac{\rho_1 F_1 (x_2 - x_1)\omega^2}{EJ} X \delta_{x_1} \\ &+ \frac{\rho_1 F_1 (x_2 - x_1)^2 \omega^2}{12 EJ}(X(x_1^+) - X(x_1^-))\delta'_{x_1} \\ &+ \frac{\gamma_1 \omega^2}{EJ}(X'(x_1^+) - X'(x_1^-))\delta''_{x_1} \\ &+ \frac{\gamma_2 \omega^2}{EJ}(X'''(x_1^+) - X'''(x_1^-))\delta'''_{x_1},\end{aligned} \tag{13}$$

where $\lambda^4 = \frac{\rho F \omega^2}{EJ}$.

4. The solution of equation (13)

We write, for short,

$$\xi := \frac{x_1 + x_2}{2} = x_1 = x_2, \qquad \Theta := \frac{\omega^2}{2EJ}, \qquad \eta := \frac{x_2 - x_1}{2}.$$

The general solution of (13) is given by (we omit the standard calculations)

$$\begin{aligned}X(x) &= P\cos\lambda x + Q\sin\lambda x + R\ \mathrm{ch}\lambda x + S\ \mathrm{sh}\lambda x \\ &+ \frac{4\rho_1 F_1 \eta \Theta}{\lambda^4} X(\xi) H(x - \xi)\left[\mathrm{sh}\lambda(x-\xi) - \sin\lambda(x-\xi)\right] \\ &+ \frac{\rho_1 F_1 \eta^2 \Theta}{3\lambda^2}\left(X'(x_1^+) - X'(x_1^-)\right) H(x-\xi)\left[\mathrm{ch}\lambda(x-\xi) - \cos\lambda(x-\xi)\right] \\ &+ \frac{\gamma_1 \Theta}{\lambda}\left(X(x_1^+) - X(x_1^-)\right) H(x-\xi)\left[\mathrm{sh}\lambda(x-\xi) + \sin\lambda(x-\xi)\right] \\ &+ \gamma_2 \Theta\left(X'''(x_2^+) - X'''(x_2^-)\right) H(x-\xi)\left[\mathrm{ch}\lambda(x-\xi) + \cos\lambda(x-\xi)\right],\end{aligned}$$

where H denotes the Heaviside function of the unit jump, i.e.,

$$H(x-c)=1,\ x>c, \qquad H(x-c)=0,\ x<c, \qquad H(x-c)=\frac{1}{2},\ x=c.$$

According to the idea of the method when adapting the solution of the equation (13) to the intervals $\langle 0, x_1) \cup (x_2, l\rangle$ we get

$$\begin{aligned}
X(x) = {} & P\cos\lambda x + Q\sin\lambda x + R\mathrm{ch}\lambda x + S\mathrm{sh}\lambda x \\
& + \frac{4\rho_1 F_1 \eta \Theta}{\lambda^4} X(\frac{x_1+x_2}{2}) H(x - \frac{x_1+x_2}{2}) \\
& \times \left[\mathrm{sh}\lambda(x-\frac{x_1+x_2}{2}) - \sin\lambda(x - \frac{x_1+x_2}{2})\right] \\
& + \frac{3\rho_1 F_1 \eta^2 \Theta}{2\lambda^2}\{H(x-x_2)X(x_2^+)\left[\mathrm{ch}\lambda(x-x_2) - \cos\lambda(x-x_2)\right] \\
& - H(x-x_1)X(x_1^-)\left[\mathrm{ch}\lambda(x-x_1) - \cos\lambda(x-x_1)\right]\} + \\
& + \gamma_1\Theta\{H(x-x_1)(p - X'(x_1^-))\left[\mathrm{sh}\lambda(x-x_1) + \sin\lambda(x-x_1)\right] \\
& + H(x-x_2)(X'(x_2^+) - p)\left[\mathrm{sh}\lambda(x-x_2) + \sin\lambda(x-x_2)\right]\} \\
& + \gamma_2\Theta\{-H(x-x_1)X'''(x_1^-)\left[\mathrm{ch}\lambda(x-x_1) + \cos\lambda(x-x_1)\right] \\
& + H(x-x_2)X'''(x_2^+)\left[\mathrm{ch}\lambda(x-x_2) + \cos\lambda(x-x_2)\right]\}.
\end{aligned}$$

According to the shape of the function

$$W(x,t) = X(x)T(t)$$

and the initial-boundary conditions given by the formulas (8), (9) we obtain the system of thirteen linear equations with thirteen unknown values

$$P,\ Q,\ R,\ S,\ X(\xi),\ X(x_1^-),\ X(x_2^+),\ X'(x_1^-),\ X'(x_2^+),\ X'''(x_1^-),\ X'''(x_2^+),\ p,\ q.$$

The equations are:

1. $X(0) = 0 \iff P + Q = 0,$
2. $X'(0) = 0 \iff R + S = 0,$
3. $X(l) = 0 \iff P\cos\lambda l + Q\sin\lambda l + R\mathrm{ch}\lambda l + S\mathrm{ch}\lambda l = 0,$
4. $X'(l) = 0 \iff -P\sin\lambda l + Q\cos\lambda l + R\mathrm{sh}\lambda l + S\mathrm{ch}\lambda l = 0,$
5. $X''(x_1) = 0 \iff -\lambda^2 P\cos\lambda x_1 - \lambda^2 Q\sin\lambda x_1 + \lambda^2 R\mathrm{ch}\lambda x_1 + \lambda^2 S\mathrm{sh}\lambda x_1 - \frac{1}{3}\rho_1 F_1 \eta^2 \Theta X(x_1^-) = 0,$

6. $$X''(x_2) = 0 \Longleftrightarrow -\lambda^2 P\cos\lambda x_2 - \lambda^2 Q\sin\lambda x_2 + \lambda^2 R\mathrm{ch}\lambda x_2 + \lambda^2 S\mathrm{sh}\lambda x_2 + \frac{4\rho_1 F_1\eta^2\Theta}{\lambda^2} X(\xi)[\mathrm{sh}\lambda\eta + \sin\lambda\eta] + \gamma_1\Theta[p - X'(x_1^-)]\lambda[\mathrm{sh}2\lambda\eta - \sin 2\lambda\eta] - \gamma_2\Theta X'''(x_1^-)\lambda^2[\mathrm{ch}2\lambda\eta - \cos 2\lambda\eta] = 0,$$

7. $$X(\xi) = P\cos\lambda\xi + Q\sin\lambda\xi + R\mathrm{ch}\lambda\xi + S\mathrm{sh}\lambda\xi + \frac{\rho_1 F_1\eta^2\Theta}{3\lambda^2} X(x_2^+)[\mathrm{ch}\lambda\eta - \cos\lambda\eta] + \frac{\gamma_1\Theta}{\lambda}[X'(x_2^+) - p][\mathrm{sh}\lambda\eta + \sin\lambda\eta] - \gamma_2\Theta X'''(x_2^+)[\mathrm{ch}\lambda\eta + \cos\lambda\eta],$$

8. $$X(x_1^-) = P\cos\lambda x_1 + Q\sin\lambda x_1 + R\mathrm{ch}\lambda x_1 + S\mathrm{sh}x_1,$$

9. $$X(x_2^+) = P\cos\lambda x_2 + Q\sin\lambda x_2 + R\mathrm{ch}\lambda x_2 + S\mathrm{sh}x_2 + \frac{4\rho_1 F_1\eta^2\Theta}{\lambda^4} X(\xi)[\mathrm{sh}\lambda\eta - \sin\lambda\eta] - \frac{\rho_1 F_1\eta^2\Theta}{3\lambda^4} X(x_1^-)[\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] + \frac{\gamma_1\Theta}{\lambda}[p - X'(x_1^-)][\mathrm{sh}2\lambda\eta + \sin 2\lambda\eta] - \frac{\gamma_2\Theta}{\lambda} X'''(x_1^-)[\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] + 2\gamma_2\Theta X'''(x_2^+),$$

10. $$X'(x_1^-) = -P\lambda\sin\lambda x_1 + Q\lambda\cos\lambda x_1 + R\lambda\mathrm{sh}\lambda x_1 + S\lambda\mathrm{ch}\lambda x_1,$$

11. $$X'(x_2^+) = -P\lambda\sin\lambda x_2 + Q\lambda\cos\lambda x_2 + R\lambda\mathrm{sh}\lambda x_2 + S\lambda\mathrm{ch}\lambda x_2 + \frac{2\rho_1 F_1\eta\Theta}{\lambda^3} X(\xi)[\mathrm{ch}2\lambda\eta - \cos 2\lambda\eta] + \frac{\gamma_1\Theta}{\lambda}[p - X'(x_1^-)][\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] + \gamma_1\Theta[X'(x_2^+) - q][\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] - \frac{\gamma_2\Theta}{\lambda} X'''(x_1^-)\lambda[\mathrm{sh}2\lambda\eta - \sin 2\lambda\eta],$$

12. $$X'''(x_1^-) = \lambda^3 P\sin\lambda x_1 - \lambda^3 Q\cos\lambda x_1 + \lambda^3 P\mathrm{ch}\lambda x_1 + \lambda^3 S\mathrm{sh}\lambda x_1,$$

13. $$X'''(x_2^+) = \lambda^3 P\sin\lambda x_2 - \lambda^3 Q\cos\lambda x_2 + \lambda^3 P\mathrm{ch}\lambda x_2 + \lambda^3 S\mathrm{sh}\lambda x_2 + \frac{2\rho_1 F_1\eta\Theta}{\lambda} X(\xi)[\mathrm{ch}\lambda\eta + \cos\lambda\eta]$$

$$- \frac{1}{6}\lambda\rho_1 F_1 \eta \Theta X(x_1^-)[\mathrm{sh}2\lambda\eta - \sin 2\lambda\eta]$$
$$+ \lambda^2\gamma_1\Theta[p - X'(x_1^-)][\mathrm{ch}2\lambda\eta - \cos 2\lambda\eta]$$
$$- \lambda^3\gamma_2\Theta X'''(x_1^-)[\mathrm{sh}2\lambda\eta + \sin 2\lambda\eta].$$

The system of the linear equations given above has infinite number of solutions ([2]) and it represents the eigenproblem under consideration. The details of the calculations as well as the explicit solution of the initial boundary problem will be dealt with in a subsequent paper. The problem considered in this paper is also discussed in [5] but in the approach of L. Schwartz [4].

Assuming that the determinant of the matrix of the system 1.-13. of the linear equations is equal to zero we obtain the eigenvalues equation. There is a countable number of such eigenvalues λ_n so we can create an increasing sequence of λ_n . In consequence we put λ_n , T_n , $X_n(x)$, p_n , q_n into the formulas (12) (13) instead of, respectively, λ, T, X, p, q, and form the solution of (10)

$$W(x,t) = \sum_{n=1}^{\infty} \tilde{X}_n(x)(a_n T_{1n}(t) + b_n T_{2n}(t)),$$

where $T_{1n}(t)$ and $T_{2n}(t)$ are the linearly independent particular solutions of (12) while

$$\tilde{X}_n(x) = \begin{cases} X_n(x), & x \in \langle 0, x_1) \cup (x_2, l\rangle \\ p_n x + q_n\,, & x \in (x_1, x_2) \end{cases}.$$

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