DE G DE GRUYTER OPEN

Ann. Univ. Paedagog. Crac. Stud. Math. 17 (2018), 67-87
DOI: 10.2478/aupcsm-2018-0007

FOLIA 233

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XVII (2018)

Anthony To-Ming Lau† *and Yong Zhang*‡

Fixed point properties for semigroups of nonexpansive mappings on convex sets in dual Banach spaces

Communicated by Justyna Szpond

Abstract. It has been a long-standing problem posed by the first author in a conference in Marseille in 1990 to characterize semitopological semigroups which have common fixed point property when acting on a nonempty weak* compact convex subset of a dual Banach space as weak* continuous and norm nonexpansive mappings. Our investigation in the paper centers around this problem. Our main results rely on the well-known Ky Fan's inequality for convex functions.

1. Introduction

A semitopological semigroup is a semigroup with a Hausdorff topology such that the product is separately continuous. Let K be a Hausdorff topological space. We say that $\mathcal{S} = \{T_s : \ s \in S\}$ is a *representation* of the semigroup S on K if for each $s \in S$, T_s is a mapping from K into K and $T_{st}(x) = T_s(T_t x)$ $(s, t \in S, x \in K)$. Sometimes we simply write sx for $T_s(x)$ if there is no confusion in the context. The representation $\mathcal{S}$ is continuous if each $T_s \colon K \to K$ $(s \in S)$ is continuous. We call the representation *separately* (resp. *jointly*) *continuous* if the mapping $(s, x) \mapsto T_s(x)$ from $S \times K$ to K is separately (resp. jointly) continuous. We say that $x \in C$ is a *common fixed point* of (the representation of) S if $T_s(x) = x$ for all $s \in S$.

AMS (2010) Subject Classification: Primary: 43A07; Secondary 43A60, 22D05, 46B20.
Keywords and phrases: amenability, semigroups, non-expansive mappings, weak*-compact convex sets, common fixed point, invariant mean, submean.
† Supported by NSERC Grant ZC912.
‡ Supported by NSERC Grant 1280813.

We are interested in the existence of common fixed points for representations of S on a subset K of a Banach space. We call the representation of S on K *norm nonexpansive* if $\|T_s(x) - T_s(y)\| \leq \|x - y\|$ for all $s \in S$ and all $x, y \in K$. It has been a long-standing open problem to characterize semitopological semigroups which have common fixed points when acting on a nonempty weak* compact convex subset of a dual Banach space as weak* continuous and norm nonexpansive mappings. Our investigation in the paper centers around this problem.

The paper is organised as follows: In section 3 we investigate the notion of invariant submeans. In section 4 we introduce the notion of average Chebyshev centre associated to submeans. We use it to prove some results (Lemmas 4.12 and 4.13) concerning left subinvariant submeans on certain subsets of $\ell^\infty(S)$ and common fixed point property of S on convex subsets of a dual Banach space with normal structure. This is then applied to prove our main results, Theorems 4.14 and 4.16, regarding a left reversible semigroup of norm nonexpansive mappings on a weak* compact convex subset of a dual Banach space with normal structure. The proof depends heavily on a Ky Fan's inequality on convex functions established in [6] and [5]. We refer the readers to [2], [19], [24] and [25] for related works on common fixed point properties of semigroups of nonexpansive mappings.

2. Some preliminaries

Let S be a semigroup. Consider $\ell^\infty(S)$, the Banach space of all real-valued bounded functions on S with the supremum norm. For each $s \in S$ and $f \in \ell^\infty(S)$, denote by $l_s f$ and $r_s f$ the left and right translates of f by s respectively, that is, $l_s f, r_s f \in \ell^\infty(S)$ with $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. Let X be a closed subspace of $\ell^\infty(S)$ containing the constant functions. A linear functional $m \in X^*$ is called a mean if $\|m\| = m(1) = 1$; If in addition X is left (right) translation invariant and m satisfies $m(l_s f) = m(f)$ (resp. $m(r_s f) = m(f)$ for all $s \in S$ and $f \in X$, then the mean m is a *left* (resp. *right*) *invariant mean*, denoted by LIM (resp. RIM).

Let S be a semitopological semigroup. We denote by $C_b(S)$ the space of all bounded continuous real-valued functions on S. Clearly, as a subspace of $\ell^\infty(S)$, $C_b(S)$ is both left and right (translation) invariant. A function $f \in C_b(S)$ is left (right) uniformly continuous if the mapping $s \mapsto l_s(f)$ (resp. $s \mapsto r_s(f)$) from S into $C_b(S)$ is continuous when $C_b(S)$ is equipped with the uniform norm topology. We denote by $LUC(S)$ (resp. $RUC(S)$) the space of all left (resp. right) uniformly continuous functions on S. Both $LUC(S)$ and $RUC(S)$ are left and right invariant subspaces of $C_b(S)$ and they both contain the constant functions. When S is a topological group, then $LUC(S)$ (resp. $RUC(S)$) is indeed the space of bounded right (resp. left) uniformly continuous functions on S as defined in [9, Vol 1]. If S is discrete, all these spaces are equal to $\ell^\infty(S)$. In general they are different. It is well-known that $LUC(S)$ has a LIM if S is a commutative semitopological semigroup or if it is a compact or a solvable group. But for the free group (or free semigroup) $\mathbb{F}_2$ on two generators, $LUC(\mathbb{F}_2) = \ell^\infty(\mathbb{F}_2)$ does not have a left invariant mean. We call a semitopological semigroup S *left amenable* if there is a left invariant mean on $LUC(S)$.

A semitopological semigroup S is *left reversible* if $\overline{sS} \cap \overline{tS} \neq \emptyset$ for all $s, t \in S$. Here and throughout the paper, for a subset A of a topological space, $\overline{A}$ always denotes the closure of A. All groups and all commutative semigroups are left reversible. For a discrete semigroup S, if $LUC(S)$ $(= \ell^\infty(S))$ has a LIM then S is left reversible. However, a general semitopological semigroup S may not be left reversible even when $C_b(S)$ has a LIM (see [10]).

Let S and H be two semitopological semigroups. It is well-known that if S is left amenable and if there is a continuous semigroup homomorphism that maps S onto T, then T is also left amenable. In fact, more is true as asserted in the following proposition.

Proposition 2.1
Let S and H be semitopological semigroups and let $\sigma\colon S \to H$ be a continuous semigroup homomorphism. If S is left amenable and the range $\sigma(S)$ is dense in H, then H is left amenable.

Proof. We give a detailed proof for the sake of completion, although it is standard.

Define $T\colon LUC(H) \to LUC(S)$ by $Tf(s) = f(\sigma(s))$ $(s \in S)$. Then T is norm preserving Banach space homomorphism with $T(1) = 1$. Its conjugate operator is $T^*\colon LUC(S)^* \to LUC(H)^*$. Let m be a LIM on $LUC(S)$. Clearly, $T^*(m)$ is a mean on $LUC(H)$. For each $h = \sigma(t)$ $(t \in S)$ we have

$$\begin{aligned}\langle l_h f, T^*(m)\rangle &= \langle T(l_h f), m\rangle = \langle l_t(Tf), m\rangle \\ &= \langle Tf, m\rangle = \langle f, T^*(m)\rangle.\end{aligned}$$

By density of $\sigma(S)$ and the continuity of $l_h f$ with respect to $h \in H$, the above implies that the identity

$$\langle l_h f, T^*(m)\rangle = \langle f, T^*(m)\rangle$$

holds for all $h \in H$ when $f \in LUC(H)$. Whence $T^*(m)$ is a left invariant mean on $LUC(H)$.

A subset K of a Banach space is said to have *normal structure* if, for each bounded subset W of K that contains more than one point, there is $w \in co(W)$ such that

$$\sup\{\|x - w\| : x \in W\} < \sup\{\|x - y\| : x, y \in W\},$$

where $co(W)$ represents the convex hull of W. It is well-known that a compact set always has normal structure. In a uniformly convex space (e.g. any L^p space with $p > 1$) a bounded convex set always has normal structure. It was shown in [20] that every weak* closed convex subset of ℓ^1 has weak* normal structure (meaning that the above condition holds for each weak* compact convex subset W). However, a weakly compact convex subset of $L^1[0, 1]$ may not have normal structure. Characterizations of normal structure may be seen in [21].

The following fixed point theorems are well-known for even more general space setting.

THEOREM 2.2 ([20])
Let K be a nonempty, weakly compact convex subset of a Banach space, and let S be a left reversible semitopological semigroup acting on K as separately continuous, norm nonexpansive self mappings. If K has normal structure, then K has a common fixed point for S.

THEOREM 2.3 ([11])
Let K be a nonempty, weakly compact convex subset of a Banach space, and let S be a discrete left reversible semigroup acting on K as weakly continuous and norm nonexpansive self mappings. Then K has a common fixed point for S.

3. Subinvariant submeans

The notion of submean was first studied by Mizoguchi and Takahashi in [23]. Further investigations and applications can be seen in [1, 15, 16].

Given a set S, a nonempty subset X of $\ell^\infty(S)$ is called *positively semilinear* if $f, g \in X$ implies $\alpha f + \beta g \in X$ for all $\alpha, \beta \in [0, \infty)$. For any subset X_0 of $\ell^\infty(S)$, the positively semilinear subset generated by X_0 is precisely

$$X = \Big\{ \sum_i^n \alpha_i f_i : \ n \in \mathbb{N}, f_i \in X_0 \text{ and } \alpha_i \in [0, \infty) \text{ for } 1 \le i \le n \Big\}.$$

Let X be a positively semilinear subset of $\ell^\infty(S)$ containing positive constants. A function $\mu\colon X \to \mathbb{R}$ is called a *submean* on X if it satisfies the following conditions.

1° If $f, g_1, g_2 \in X$ and $\alpha, \beta \in [0,1]$ such that $f \le \alpha g_1 + \beta g_2$, then

$$\mu(f) \le \alpha\mu(g_1) + \beta\mu(g_2),$$

2° For every constant $c > 0$, $\mu(c) = c$.

We often write $\mu_t(f(t))$ for the action $\mu(f)$ to emphasize that the variable of the function f is t, in particular when f contains other variables as parameters.

Note that our definition of a submean is slightly different from that given in [15]. But it can be shown easily that both are indeed equivalent. It is also easily seen that a submean is always continuous when X is equipped with the sup norm topology of $\ell^\infty(S)$.

A submean μ is also increasing, i.e. $\mu(f) \ge \mu(g)$ if $f, g \in X$ and $f \ge g$. We call the submean μ *strictly increasing* if for each constant $c > 0$ there is $\delta(c) > 0$ such that

$$\mu(f + c) \ge \mu(f) + \delta(c)$$

for all $f \in X$.

Now suppose further that S is a semigroup. A subset X of $\ell^\infty(S)$ is left invariant if $l_s f \in X$ for all $s \in S$ and $f \in X$. A submean μ on a left invariant, positively semilinear subset X of $\ell^\infty(S)$ containing non-negative constants is called *left subinvariant* if

$$\mu(l_s f) \geq \mu(f) \quad (s \in S,\ f \in X).$$

If the equality $\mu(l_s f) = \mu(f)$ holds for all $s \in S$ and $f \in X$, then we call μ *left invariant.*

Trivially, if X is a left invariant subspace of $\ell^\infty(S)$ containing constants, then any left invariant mean on X is a strictly increasing left invariant submean on X. Some nonlinear examples are given as follows.

Example 1
Let $S = G$ be a group. Then

$$\mu(f) = \sup_{g \in G} f(g) \quad (f \in \ell^\infty(G))$$

is a strictly increasing left invariant submean on $\ell^\infty(G)$.

Example 2
If there is a nonempty $S_0 \subset S$ such that $sS_0 \supset S_0$ for each $s \in S$, then

$$\mu_0(f) = \sup_{s \in S_0} f(s) \quad (f \in \ell^\infty(S))$$

defines a strictly increasing left subinvariant submean on $\ell^\infty(S)$. In particular, if S has a right zero s_0 so that $ss_0 = s_0$ for all $s \in S$, then $\mu_0(f) = f(s_0)$ is a strictly increasing left invariant submean on $\ell^\infty(S)$.

More generally, if S has a left ideal $S_0 = G_0$ which is a group, then μ_0 defined above is a strictly increasing left invariant submean on $\ell^\infty(S)$.

For a left reversible semigroup S, $\ell^\infty(S)$ may have no left invariant mean. But it always has a strictly increasing left subinvariant submean as shown in the following example.

Example 3
Let S be a left reversible semitopological semigroup and let Γ be the collection of all closed right ideals of S. Given any submean ν on a left invariant, positively semilinear subset X of $\ell^\infty(S)$ that contains positive constants, we define

$$\mu(f) = \inf_{J \in \Gamma} \sup_{s \in J} \nu(l_s f) \quad (f \in X).$$

Then μ is a strictly increasing left subinvariant submean on X. Note that in $\ell^\infty(S)$, the semigroup S is regarded as a discrete semigroup. Clearly $X \subset \ell^\infty(S)$ if $X \subset C_b(S)$ for a semitopological semigroup S.

As a special case, we can take the submean ν on $\ell^\infty(S)$ defined by $\nu(f) = \sup_{s \in S} f(s)$. Then $\sup_{s \in J} \nu(l_s f) = \sup_{s \in J} f(s)$, and so

$$\mu(f) = \inf_{J \in \Gamma} \sup_{s \in J} f(s) \quad (f \in \ell^\infty(S))$$

defines a strictly increasing left subinvariant submean on $\ell^\infty(S)$.

We note that left reversibility is crucial in Example 3 to show that μ satisfy the sublinear condition 1º.

Let Y and X be two left invariant, positively semilinear subsets of $\ell^\infty(S)$ containing the positive constant functions such that $Y \subset X$. If μ is a strictly increasing left subinvariant submean on X, then μ, restricting to Y, is also a strictly increasing left subinvariant submean on Y. For the converse, we have the following general observations.

Proposition 3.1
Let Y and X be left invariant, positively semilinear subsets of $\ell^\infty(S)$ containing positive constants, and assume that Y has a left subinvariant submean μ. Suppose that there is a mapping T from X into Y such that $T(l_s f) \geq l_s(Tf)$ for $f \in X$ and $s \in S$, $T(c) = c$ for $c > 0$, and

$$T(f) \leq \sum_i^n \alpha_i T(f_i)$$

if $n \in \mathbb{N}$, $f, f_i \in X$ and $\alpha_i \geq 0$ $(1 \leq i \leq n)$ satisfy $f \leq \sum_i^n \alpha_i f_i$. Then $\mu \circ T$ is a left subinvariant submean on X. Moreover, if T is a projection onto Y, then $\mu \circ T$ extends μ.

Proof. Verification is straightforward.

As an example, we consider $S = G$ to be a locally compact group. Take a $\varphi \in L^1(G)$ with $\|\varphi\|_1 = 1$ and, in $L^\infty(G)$, define $T(f) = \varphi \odot f$, where

$$\varphi \odot f(s) = \int_G \varphi(t) f(st) dt \quad (s \in G).$$

Then T is a linear mapping from $L^\infty(G)$ into $LUC(G)$, satisfying $T(l_s f) = l_s(Tf)$. Through T any (strictly increasing) left subinvariant submean on $LUC(G)$ determines a (strictly increasing) left subinvariant submean on $L^\infty(G)$, and any left invariant mean on $LUC(G)$ determines a left invariant mean on $L^\infty(G)$.

Proposition 3.2
Let S be a left reversible semitopological semigroup and let L be a left invariant subspace of $\ell^\infty(S)$ containing constants. Then any left invariant mean μ on L extends to a strictly increasing left subinvariant submean on $\ell^\infty(S)$.

Proof. First, by the Hahn-Banach theorem, we may extend μ to some $\tilde{\mu} \in \ell^\infty(S)^*$. Regarding the dual space of $\ell^\infty(S)$ as a measure space and using the Jordan decomposition, we may assume that $\tilde{\mu}$ is positive, i.e. $\tilde{\mu}(f) \geq 0$ if $f \geq 0$. Then we define

$$\hat{\mu}(f) := \inf_{J \in \Gamma} \sup_{s \in J} \tilde{\mu}(l_s f) \quad (f \in \ell^\infty(S)),$$

where Γ is the collection of all closed right ideals of S. $\hat{\mu}$ is certainly an extension of μ. As in example 3, one may check that $\hat{\mu}$ is a strictly increasing left subinvariant submean on $\ell^\infty(S)$.

Proposition 3.3
Let S and H be two semigroups. Let X and Y be left invariant, positively semilinear subsets of, respectively, $\ell^\infty(S)$ and $\ell^\infty(H)$. Suppose that X has a (strictly increasing) left subinvariant submean μ. If there is a semigroup epimorphism $\sigma\colon S \to H$ such that $T\colon f \mapsto f \circ \sigma$ maps Y into X, Then $\mu \circ T$ is a (resp. strictly increasing) left subinvariant submean on Y.

Proof. One only needs to notice that $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$, $T(c) = c$ and $T(l_{\sigma(s)} f) = l_s(Tf)$ for $f, g \in Y$, $\alpha, \beta, c \in \mathbb{R}^+$ and $s \in S$. Verification of conditions for $\mu \circ T$ is straightforward.

A submean μ on $\ell^\infty(S)$ is called *supremum admissible* if for any bounded family $\{f_\alpha : \alpha \in \Delta\} \subset \ell^\infty(S)$

$$\mu(\sup_{\alpha\in\Delta} f_\alpha) = \sup_{\alpha\in\Delta} \mu(f_\alpha),$$

where $\sup_{\alpha\in\Delta} f_\alpha \in \ell^\infty(S)$ is defined by $(\sup_{\alpha\in\Delta} f_\alpha)(s) = \sup_{\alpha\in\Delta}(f_\alpha(s))$ $(s \in S)$.

The submeans defined in Examples 1 and 2 are supremum admissible. If S is a finite semigroup, then every submean on $\ell^\infty(S)$ is supremum admissible.

4. The main result

Now let S be a semigroup. For convenience, we call a subset X of $\ell^\infty(S)$ *positively semilinear lattice* if it is positively semilinear, contains positive constant functions and $\max\{f, g\} \in X$ whenever $f, g \in X$, where

$$\max\{f, g\}(s) = \max\{f(s), g(s)\} \quad (s \in S).$$

For example, if S is a semitopological semigroup, then it is readily seen that $C_b(S)$, $LUC(S)$, and $RUC(S)$ are all (positively semilinear, left invariant) lattice subspaces of $\ell^\infty(S)$. So are $AP(S)$ and $WAP(S)$ since $AP(S) = C(S^a)$ and $WAP(S) = C(S^w)$, where S^a is the spectrum of $AP(S)$ and S^w is the spectrum of $WAP(S)$. Here we recall that $AP(S)$ (resp. $WAP(S)$), the space of almost periodic functions (resp. weakly almost periodic functions) on S, consists of all functions $f \in C_b(S)$ such that the left orbit $\{l_s f :\ s \in S\}$ of f is precompact in the norm topology (resp. weak topology) of $C_b(S)$. It is well known that both S^a and S^w are compact semitopological semigroups [3].

If X is a positively semilinear lattice subset, it is readily seen that for any finite set $\Lambda \subset X$ we have $\max\{f :\ f \in \Lambda\} \in X$, where

$$\max\{f :\ f \in \Lambda\}(s) = \max\{f(s) :\ f \in \Lambda\}.$$

Suppose that $\mathcal{S} = \{T_s :\ s \in S\}$ is a representation of S on a subset K of a Banach space E. Denote the unit ball of the dual space E^* by $(E^*)_1$. Let Δ be a weak* dense subset of $(E^*)_1$. Denote the collection of all finite subsets of Δ by Γ. For $x, y \in K$ and $\phi \in \Delta$ we consider the function

$$\varphi_{(x,\phi,y)}(s) = |\langle \phi, T_s x - y\rangle| \quad (s \in S).$$

Let X be a positively semilinear lattice subset of $\ell^\infty(S)$ with a submean μ. Suppose that $\varphi_{(x,\phi,y)} \in X$ for all $\phi \in \Delta$. We then can define $\rho_x(y)$ by

$$\rho_x(y) = \sup_{\Lambda\in\Gamma} \mu(\max_{\phi\in\Lambda} \varphi_{(x,\phi,y)}) = \sup_{\Lambda\in\Gamma} \mu_t(\max_{\phi\in\Lambda} |\langle \phi, T_t x - y\rangle|).$$

If $\varphi_{(x,\phi,y)} \in X$ for all $\phi \in \Delta$ and all $y \in K$, we define the *μ-average Chebyshev radius* of K at x with respect to Δ to be

$$\rho_x = \inf_{y\in K} \rho_x(y).$$

We call

$$K_x = \{y \in K : \ \rho_x(y) \le \rho_x\}$$

the *μ-average Chebyshev center* of K at x with respect to Δ.

For example, if S is a semitopological semigroup and if its representation $\mathcal{S}$ on the set K is separately continuous and equicontinuous when K is equipped with the $\sigma(E,\Delta)$-topology, then we may consider the subspace $X = RUC(S)$ of $\ell^\infty(S)$. Let $x \in K$ be such that Sx is bounded. Then each $\varphi_{(x,\phi,y)}$ automatically belongs to X; If K is $\sigma(E,\Delta)$-compact and the representation is jointly continuous when K is equipped with the $\sigma(E,\Delta)$-topology, then $\varphi_{(x,\phi,y)} \in LUC(S)$ for all $x, y \in K$ and $\phi \in \Delta$. So, we may consider $X = LUC(S)$. If μ is a mean on X, then μ-average Chebyshev radius ρ_x is well defined.

Remark 4.1
If $X = \ell^\infty(S)$ and μ is supremum admissible, then one sees easily that $\rho_x(y) = \mu_t(\|T_t x - y\|)$ for $x, y \in K$. But in general this is not true.

Let T be a self mapping on a subset K of a Banach space E, and Let Δ be a weak* dense subset of $(E^*)_1$. We call T *pseudo Δ-nonexpansive* if, for each $\phi \in \Delta$ and each $\varepsilon > 0$, there exists a finite set $\Lambda \subset \Delta$ such that

$$|\langle \phi, Tx - Ty\rangle| \le \max_{\phi'\in\Lambda} |\langle \phi', x - y\rangle| + \varepsilon$$

for all $x, y \in K$. In particular, a pseudo $(E^*)_1$ nonexpansive mapping is called a *pseudo weakly nonexpansive* mapping. If K is a subset of a dual Banach space E and a predual space of E is E_*, then a pseudo $(E_*)_1$ nonexpansive mapping is called a *pseudo weak* nonexpansive* mapping. For example, if K is a left translation invariant subset of $E = \ell^\infty(S)$, where S is a semigroup, then for each $s \in S$ the translation operator l_s on K is pseudo weakly nonexpansive, since the dual operator l_s^* maps $(E^*)_1$ into itself. If K is a subset of the dual space E of a left invariant subspace E_* of $\ell^\infty(S)$ such that $l_s^*(K) \subset K$ for all $s \in S$, then each l_s^* is a pseudo weak* nonexpansive self mapping on K because $l_s\left((E_*)_1\right) \subset (E_*)_1$.

We call a semigroup S acting on a subset K of a Banach space pseudo weakly nonexpansive if each T_s $(s \in S)$ is pseudo weakly nonexpansive on K. The notion of a pseudo weak* nonexpancive S-action is defined similarly.

Proposition 4.2
Suppose that Δ_1 and Δ_2 be two weak dense subsets of $(E^*)_1$ and $\Delta_1 \subset \Delta_2$. Let $K \ne \emptyset$ be a $\sigma(E,\Delta_2)$ compact subset of E. Then $T\colon K \to K$ is pseudo Δ_2-nonexpansive if and only if it is pseudo Δ_1-nonexpansive.*

Proof. First we note that for each $\psi \in \Delta_2$ and each $\varepsilon > 0$ there is a finite set $\Lambda_\psi \subset \Delta_1$ such that

$$|\langle \psi, x-y\rangle| \leq \max_{\phi\in\Lambda_\psi} |\langle \phi, x-y\rangle| + \varepsilon/2 \tag{1}$$

for all $x, y \in K$. In fact, since Δ_1 is weak* dense in Δ_2, for each pair $a, b \in K$ there is $\phi \in \Delta_1$ such that

$$|\langle \psi, a-b\rangle| < |\langle \phi, a-b\rangle| + \varepsilon/2.$$

The inequality holds also for (x, y) in a $\sigma(E, \Delta_2)$ neighbourhood of (a, b) in $K\times K$. Using finite covering argument we derive the wanted finite set $\Lambda_\psi \subset \Delta_1$.

Now assume that T is pseudo Δ_1-nonexpansive. Given $\psi \in \Delta_2$ and $\varepsilon > 0$, Let $\Lambda_\psi \subset \Delta_1$ be the finite set obtained above. For each $\phi \in \Lambda_\psi$ there is a finite set Λ_ϕ such that

$$|\langle \phi, Tx-Ty\rangle| \leq \max_{\phi'\in\Lambda_\phi} |\langle \phi', x-y\rangle| + \varepsilon/2$$

for all $x, y \in K$. Denote $\Lambda = \bigcup_{\phi\in\Lambda_\psi} \Lambda_\phi$. Then

$$|\langle \psi, Tx-Ty\rangle| \leq \max_{\phi\in\Lambda_\psi} |\langle \phi, Tx-Ty\rangle| + \varepsilon/2 \leq \max_{\phi'\in\Lambda} |\langle \phi', x-y\rangle| + \varepsilon$$

for all $x, y \in K$. So T is pseudo Δ_2-nonexpansive.

Conversely, assume T is pseudo Δ_2-nonexpansive. Then for $\phi \in \Delta_1$ and $\varepsilon > 0$ there is a finite set $\Lambda_\phi \subset \Delta_2$ such that

$$|\langle \phi, Tx-Ty\rangle| \leq \max_{\psi\in\Lambda_\phi} |\langle \psi, x-y\rangle| + \varepsilon/2$$

for all $x, y \in K$. For each $\psi \in \Lambda_\phi$ let $\Lambda_\psi \subset \Delta_1$ be the set such that (1) holds for all $x, y \in K$. Let $\Lambda = \bigcup_{\psi\in\Lambda_\phi} \Lambda_\psi$. We then have

$$|\langle \phi, Tx-Ty\rangle| \leq \max_{\phi'\in\Lambda} |\langle \phi', x-y\rangle| + \varepsilon$$

for all $x, y \in K$. So, by definition, T is pseudo Δ_1-nonexpansive.

If T is a pseudo Δ-nonexpansive mapping from K_1 to K_2, from definition it is clear that, for any finite set $\Lambda \subset \Delta$ and $\varepsilon > 0$, there is a finite set $\Lambda' \subset \Delta$ such that

$$\max_{\phi\in\Lambda} |\langle \phi, Tx-Ty\rangle| \leq \max_{\phi'\in\Lambda'} |\langle \phi', x-y\rangle| + \varepsilon$$

for all $x, y \in K_1$. So a pseudo Δ-nonexpansive mapping from K_1 to K_2 is Δ-uniformly continuous. It is also easily seen that a pseudo Δ-nonexpansive mapping must be norm nonexpansive. The converse is not true. However, the converse is true if K is compact in the $\sigma(E, \Delta)$ topology and the mapping T is continuous in this topology. Notice that the notions of Δ-nonexpansiveness and norm nonexpansiveness are still valid for a mapping $T\colon K_1 \to K_2$, where K_1 and K_2 are any two subsets of the Banach space E. We state and prove a more general result as follows.

Proposition 4.3
Let K_1 and K_2 be subsets of a Banach space E and Δ be a weak dense subset of $(E^*)_1$. Suppose that K_1 is $\sigma(E,\Delta)$ compact and that $T\colon K_1 \to K_2$ is continuous when both K_1 and K_2 are equipped with the $\sigma(E,\Delta)$ topology. Then T is pseudo Δ-nonexpansive if and only if it is norm nonexpansive.*

Proof. The necessity is trivial. So we only prove the sufficiency. Suppose that T is norm nonexpansive. Given $\phi \in \Delta$ and $\varepsilon > 0$, for each pair $a, b \in K_1$ there is $\phi' \in \Delta$ such that

$$|\langle \phi, Ta - Tb\rangle| \le \|Ta - Tb\| \le \|a - b\| < |\langle \phi', a - b\rangle| + \varepsilon.$$

Since T is $\sigma(E,\Delta)$ continuous, there is a neighbourhood $N_{(a,b)}$ of the point (a,b) in $K_1 \times K_1$ such that the inequality

$$|\langle \phi, Tx - Ty\rangle| < |\langle \phi', x - y\rangle| + \varepsilon$$

holds for all $x, y \in N_{(a,b)}$, where K_1 is equipped with the $\sigma(E,\Delta)$ topology. The product space $K_1 \times K_1$ is compact. Using the finite subcovering property, we obtain finite set $\Lambda \subset \Delta$ such that

$$|\langle \phi, Tx - Ty\rangle| \le \max_{\phi' \in \Lambda} |\langle \phi', x - y\rangle| + \varepsilon$$

for all $x, y \in K_1$. Therefore T is pseudo Δ-nonexpansive.

Remark 4.4
Let K_1, K_2 be sets as described in Proposition 4.3. Let Σ be a collection of norm nonexpansive $\sigma(E,\Delta)$-continuous mappings from K_1 to K_2. We wonder whether the mappings in Σ are $\sigma(E,\Delta)$ equicontinuous. If the answer is affirmative, then the weak* equicontinuity condition may be removed from Corollaries 4.18 and 4.21.

Proposition 4.3 allows us to use pseudo Δ-nonexpansiveness techniques to deal with norm nonexpansive mappings.

Lemma 4.5
Let S be a semigroup acting on a subset K of a Banach space E as self mappings. Let Δ be a weak dense subset of $(E^*)_1$ and let X be a positively semilinear lattice subset of $\ell^\infty(S)$ with a submean μ. Suppose that for some $b \in K$, $\varphi_{(b,\phi,y)} \in X$ for all $\phi \in \Delta$ and all $y \in K$. Then the μ-average Chebyshev radius function $\rho_b(y)\colon K \to \mathbb{R}^+$ with respect to Δ is lower semicontinuous when K is equipped with the $\sigma(E,\Delta)$-topology.*

Proof. For each $y \in K$ and $(y_\alpha) \subset K$ such that $y_\alpha \to y$ in $\sigma(E,\Delta)$-topology, we show

$$\liminf_{\alpha} \rho_b(y_\alpha) \ge \rho_b(y).$$

For each $\Lambda \in \Gamma$, from definition we have

$$\rho_b(y_\alpha) \ge \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t(b) - y_\alpha\rangle|).$$

Since $\langle \phi, y_\alpha \rangle \to \langle \phi, y \rangle$ for each $\phi \in \Delta$, Λ is finite, and μ is continuous, we obtain

$$\liminf_{\alpha} \rho_b(y_\alpha) \geq \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t(b) - y \rangle|)$$

for each $\Lambda \in \Gamma$. So

$$\liminf_{\alpha} \rho_b(y_\alpha) \geq \sup_{\Lambda \in \Gamma} \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t(b) - y \rangle|) = \rho_b(y).$$

Therefore, $\rho_b(y)$ is lower semicontinuous.

Assuming that the function $s \mapsto \|T_s x - y\|$ belongs to X for all $y \in K$, one may consider $\varrho_b(y) = \mu_t(\|T_t b - y\|)$ for all $y \in K$. However, $\varrho_b(y)$ may not be lower semicontinuous in the $\sigma(E, \Delta)$-topology. So the above lemma is no longer valid if $\rho_b(y)$ is replaced by $\varrho_b(y)$.

Lemma 4.6
Let E be a Banach space and Δ be a weak dense subset of $(E^*)_1$. Suppose that K is a convex $\sigma(E, \Delta)$ compact subset of E. Let S be a semigroup acting on K as pseudo Δ nonexpansive self mappings. Let X be a left invariant positively semilinear lattice subset of $\ell^\infty(S)$ that has a strictly increasing left subinvariant submean μ. Suppose that $b \in K$ such that the function $\varphi_{(b,\phi,y)}(s) = |\langle \phi, T_s b - y \rangle|$ belongs to X for all $\phi \in \Delta$ and all $y \in K$. Then the μ-average Chebyshev center K_b of K at b with respect to Δ is a nonempty $\sigma(E, \Delta)$ compact convex S-invariant subset of K.*

Proof. Using the uniform boundedness principle, we have that K is norm bounded. So the μ-average Chebyshev radius of K at b, ρ_b, is finite. For each $r > \rho_b$, by definition the set

$$K_r = \{y \in K : \ \rho_b(y) \leq r\}$$

is nonempty. For $y_1, y_2 \in K_r$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, we have

$$\begin{aligned}
\rho_b(\alpha y_1 + \beta y_2) &= \sup_{\Lambda \in \Gamma} \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t(b) - (\alpha y_1 + \beta y_2 \rangle|) \\
&\leq \alpha \sup_{\Lambda \in \Gamma} \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t(b) - y_1 \rangle|) + \beta \sup_{\Lambda \in \Gamma} \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t(b) - y_2 \rangle|) \\
&= \alpha \rho_b(y_1) + \beta \rho_b(y_2) \leq r.
\end{aligned}$$

So $\alpha y_1 + \beta y_2 \in K_r$, showing that K_r is a convex subset of K. We show further that K_r is indeed $\sigma(E, \Delta)$ closed which then implies that it is $\sigma(E, \Delta)$ compact. In fact, by Lemma 4.5, $\rho_b(y)$ is $\sigma(E, \Delta)$ lower semicontinuous. If $(y_\alpha) \subset K_r$ and $y_\alpha \to y$ in the $\sigma(E, \Delta)$ topology, we have $y \in K$ and

$$\rho_b(y) \leq \liminf_{\alpha} \rho_b(y_\alpha) \leq r.$$

So $y \in K_r$, and hence K_r is $\sigma(E, \Delta)$ closed.

K_r is also S-invariant. For $y \in K_r$ $s \in S$ and any finite set $\Lambda \subset \Delta$, since μ is left subinvariant we have

$$\mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_t b - T_s y \rangle|) \leq \mu_t(\max_{\phi \in \Lambda} |\langle \phi, T_{st} b - T_s y \rangle|).$$

From hypothesis, T_s is pseudo Δ-nonexpansive. So for each $\varepsilon > 0$ there is another finite set $\Lambda' \subset \Delta$ such that

$$\max_{\phi\in\Lambda} |\langle \phi, T_{st}b - T_s y\rangle| \le \max_{\phi'\in\Lambda'} |\langle \phi', T_t b - y\rangle| + \varepsilon$$

for all $t \in S$. This leads to

$$\rho_b(T_s y) \le \rho_b(y) + \varepsilon$$

for all $\varepsilon > 0$. Thus $\rho_b(T_s y) \le \rho_b(y) \le r$. Therefore $T_s y \in K_r$ for each $s \in S$, showing that K_r is S-invariant. By the finite intersection property, $K_b = \bigcap_{r>\rho_b} K_r$ is nonempty $\sigma(E,\Delta)$ compact, convex and S-invariant.

If K is a subset of a dual Banach space $E = (E_*)^*$ and $\Delta = (E_*)_1$, then the $\sigma(E,\Delta)$ (i.e. weak*) compactness assumption on K may be weakened in the above result. Precisely, we have the following.

Lemma 4.7
Suppose that K is a weak closed convex subset of a dual Banach space $E = (E_*)^*$. Let S be a semigroup acting on K as pseudo weak* nonexpansive self mappings. Let X be a left invariant positively semilinear lattice subset of $\ell^\infty(S)$ that has a strictly increasing left subinvariant submean μ. Suppose that $b \in K$ such that Sb is bounded and such that the function $\varphi_{(b,\phi,y)}(s) = |\langle \phi, T_s b - y\rangle|$ belongs to X for all $\phi \in (E_*)_1$ and $y \in K$. Then the μ-average Chebyshev center K_b of K at b with respect to $(E_*)_1$ is a nonempty weak* compact convex S-invariant subset of K.*

Proof. Let $\Delta = (E_*)_1$. Following the proof of Lemma 4.6, we have $\rho_b < \infty$ and for each $r > \rho_b$ the set $K_r = \{y \in K : \ \rho_b(y) \le r\}$ is a nonempty, bounded, and weak* closed subset of K. So K_r is weak* compact according to Alaoglu's Theorem. Also, as shown in the proof of Lemma 4.6, K_r is convex and S-invariant. So $K_b = \bigcap_{r>\rho_b} K_r$ is nonempty weak* compact convex and S-invariant.

Remark 4.8
The pseudo Δ-nonexpansive assumption in the above two lemmas is only used in showing that $\rho_b(T_s y) \le \rho_b(y)$ for the S-invariance of K_r. If $X = \ell^\infty(S)$ and μ is supremum admissible, then by Remark 4.1 this inequality holds if the representation $\mathcal{S}$ is norm nonexpansive. So for this case Lemmas 4.6 and 4.7 remain true if the condition of pseudo Δ-(or pseudo weak*) nonexpansiveness on $\mathcal{S}$ is replaced by norm nonexpansiveness. This fact will be used later to establish Theorem 4.14.

Let us return to the general setting that $K \subset E$ and Δ is a weak* dense subset of $(E^*)_1$. For $x \in K$ we denote

$$r_x = \sup_{k\in K} \|x - k\|,$$

and let $r_K = \inf\{r_x : \ x \in K\}$.

Since Δ is weak* dense in $(E^*)_1$, we have

$$r_K = \inf_{x\in K} \sup\{|\langle \phi, x - k\rangle| : \ \phi \in \Delta, \ k \in K\}.$$

Assume that S acts on K and $b \in K$ so that the conditions of Lemma 4.6 are satisfied. From the definition one sees clearly that the relation

$$\rho_b \le r_K$$

holds. We show further the following.

LEMMA 4.9
Under the condition of Lemma 4.6, if $K_b = K$ then $\rho_b = r_K$.

Lemma 4.9 is crucial for us to prove our main theorems. Its proof relies on the well-known Ky Fan's inequality on convex functions as stated below.

LEMMA 4.10 (Ky Fan [6])
Let K be a compact convex subset of a topological vector space. Let $\{f_\nu\}_{\nu \in I}$ be a family of lower semicontinuous convex functions defined on K. If for each finite set of indices $\nu_1, \nu_2, \dots, \nu_n \in I$ and any numbers $\lambda_1 \ge 0, \lambda_2 \ge 0, \dots, \lambda_n \ge 0$ such that $\sum_{i=1}^n \lambda_i = 1$ the inequality

$$\min_{x \in K} \sum_{i=1}^{n} \lambda_i f_{\nu_i}(x) \le c$$

holds, then there is $x_0 \in K$ such that $\sup_{\nu \in I} f_\nu(x_0) \le c$.

Proof of Lemma 4.9. It suffices to show $r_K \le \rho_b$. If $K_b = K$ then

$$\mu_t(|\langle \phi, T_t(b) - k \rangle|) \le \rho_b$$

for all $\phi \in \Delta$ and all $k \in K$. Let (ϕ_i, k_i), $i = 1, 2, \dots, n$, be any finite set of $\Delta \times K$, and let $\lambda_1 \ge 0, \lambda_2 \ge 0, \dots, \lambda_n \ge 0$ be finite numbers such that $\sum_{i=1}^n \lambda_i = 1$. Then, as μ is a submean,

$$\mu_t\Big(\sum_{i=1}^{n} \lambda_i |\langle \phi_i, T_t b - k_i \rangle|\Big) \le \sum_{i=1}^{n} \lambda_i \mu_t(|\langle \phi_i, T_t b - k_i \rangle|) \le \rho_b.$$

By the monotone property of μ, for any $\varepsilon > 0$, there must exists $t_\varepsilon \in S$ such that

$$\sum_{i=1}^{n} \lambda_i |\langle \phi_i, T_{t_\varepsilon} b - k_i \rangle| \le \rho_b + \varepsilon.$$

This shows that

$$\min_{x \in K} \sum_{i=1}^{n} \lambda_i |\langle \phi_i, x - k_i \rangle| \le \rho_b.$$

Now the function $f_{(\phi,k)}(x) = |\langle \phi, x - k \rangle|$ is $\sigma(E, \Delta)$ continuous convex function on K for each $(\phi, k) \in \Delta \times K$, and K is $\sigma(E, \Delta)$ compact convex set. By Lemma 4.10, there is $x_0 \in K$ such that

$$\sup_{(\phi,k) \in \Delta \times K} |\langle \phi, x_0 - k \rangle| \le \rho_b.$$

By definition, we then have $r_K \le \rho_b$.

Lemma 4.11

Let E be a Banach space and Δ be a weak dense subset of $(E^*)_1$. Suppose that K is a $\sigma(E,\Delta)$ compact convex subset of E containing more than one point. Let S be a semigroup acting on K as pseudo Δ-nonexpansive self mappings. Let X be a left invariant positively semilinear lattice subset of $\ell^\infty(S)$ that has a strictly increasing left subinvariant submean μ. Suppose that the functions $\varphi_{(x,\phi,y)}(s) = |\langle \phi, T_s x - y\rangle|$, $s \in S$, belong to X for all $\phi \in \Delta$ and all $x, y \in K$. If K has normal structure, then there is $x \in K$ such that $K_x \subsetneq K$.*

Proof. Since K is norm bounded and has more than one point, $0 < r_K < \infty$. Assume to the contrary that $K_x = K$ for all $x \in K$. We aim to construct a sequence $(x_n) \subset K$ such that

$$\|x_n - x_m\| \le r_K \quad \text{and} \quad \|x_{n+1} - \bar{x}_n\| \ge r_K - \frac{1}{n^2}$$

for all $n, m \in \mathbb{N}$, where $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i$. Then, by Lim's characterization of normal structure [22, Lemma 1], K could not have normal structure. This would be a contradiction to the hypothesis.

First, since K is $\sigma(E,\Delta)$ compact, by the standard finite intersection argument one sees that the Chebyshev center $C_K = \{k \in K : \ \sup_{x\in K} \|x - k\| \le r_K\}$ is nonempty. Take $k_0 \in C_K$. We clearly have

$$\|T_s x - k_0\| \le r_K \quad (s \in S,\ x \in K). \tag{2}$$

Let $x_1 = k_0$. Since $\bar{x}_1 = x_1 \in K = K_{k_0}$ by assumption, we have

$$\sup_{\Lambda\in\Gamma} \mu_t(\max_{\phi\in\Lambda} |\langle \phi, T_t k_0 - \bar{x}_1\rangle|) = \rho_{k_0}(\bar{x}_1) = \rho_{k_0} = r_K$$

due to Lemma 4.9, where Γ denotes the collection of all finite subsets of Δ. The identity implies that there exist $\phi \in \Delta$ and $t_1 \in S$ such that

$$|\langle \phi, T_{t_1} k_0 - \bar{x}_1\rangle| \ge r_K - 1.$$

This then ensures that $\|T_{t_1} k_0 - \bar{x}_1\| \ge r_K - 1$. Let $x_2 = T_{t_1} k_0$. Then

$$\|x_2 - \bar{x}_1\| \ge r_K - 1.$$

On the other hand, by (2) we also have

$$\|x_2 - x_1\| = \|T_{t_1} k_0 - k_0\| \le r_K.$$

In general, let x_p $(1 \le p \le n)$ have been chosen, with the forms $x_1 = k_0$ and $x_p = T_{t_1 t_2 \cdots t_{p-1}} k_0$ for $1 < p \le n$, so that

$$\|x_p - x_q\| \le r_K \quad \text{and} \quad \|x_{p+1} - \bar{x}_p\| \ge r_K - \frac{1}{p^2}$$

for $1 \le p \le n-1$ and $1 \le q \le n$. Since $\bar{x}_n \in K = K_{k_0}$ by assumption, we have again

$$\sup_{\Lambda\in\Gamma} \mu_t(\max_{\phi\in\Lambda} |\langle \phi, T_t k_0 - \bar{x}_n\rangle|) = \rho_{k_0}(\bar{x}_n) = \rho_{k_0} = r_K.$$

From subinvariance of μ we have

$$\sup_{\Lambda\in\Gamma}\mu_t(\max_{\phi\in\Lambda}|\langle\phi,T_{t_1t_2\cdots t_{n-1}t}k_0-\bar{x}_n\rangle|)\geq r_K.$$

So there is $\phi\in\Delta$ and $t_n\in S$ such that

$$|\langle\phi,T_{t_1\cdots t_{n-1}t_n}k_0-\bar{x}_n\rangle|\geq r_K-\frac{1}{n^2}.$$

This ensures that $\|T_{t_1\cdots t_{n-1}t_n}k_0-\bar{x}_n\|\geq r_K-\frac{1}{n^2}$. Let $x_{n+1}=T_{t_1\cdots t_{n-1}t_n}k_0$. Then

$$\|x_{n+1}-\bar{x}_n\|\geq r_K-\frac{1}{n^2}.$$

On the other hand, by (2)

$$\|x_{n+1}-x_1\|=\|T_{t_1\cdots t_{n-1}t_n}k_0-k_0\|\leq r_K$$

and, since the pseudo Δ-nonexpansive mapping T_s is norm nonexpansive, for each $s\in S$ we also have

$$\|x_{n+1}-x_p\|=\|T_{t_1\cdots t_{p-1}\cdots t_n}k_0-T_{t_1\cdots t_{p-1}}k_0\|\leq\|T_{t_p\cdots t_n}k_0-k_0\|\leq r_K$$

for each $1<p\leq n$.

By induction, the sequence (x_n) that we were seeking does exist. The proof is completed.

Lemma 4.12
Let E be a Banach space and Δ a weak dense subset of $(E^*)_1$. Let S be a semigroup acting on a $\sigma(E,\Delta)$ compact convex subset K of E as pseudo Δ-nonexpansive self mappings. Suppose that X is a left invariant positively semilinear lattice subset of $\ell^\infty(S)$ that has a strictly increasing left subinvariant submean μ and contains the functions $\varphi_{(x,\phi,y)}(s)=|\langle\phi,T_sx-y\rangle|$ ($s\in S$) for all $\phi\in\Delta$ and all $x,y\in K$. If K has normal structure then K has a common fixed point for S.*

Proof. By Zorn's Lemma, there is a minimal nonempty $\sigma(E,\Delta)$ compact S-invariant convex subset $K_0\neq\emptyset$ of K. By the hypothesis, if K_0 is not a singleton, then K_0 has normal structure. By Lemma 4.11 there is $x\in K_0$ such that $K_x\subsetneq K_0$, where K_x is the μ-average Chebyshev center of K_0 at x with respect to Δ. However, K_x is a nonempty $\sigma(E,\Delta)$ compact convex S-invariant subset of K_0 due to Lemma 4.6. This contradicts the minimum assumption of K_0. So $K_0=\{k\}$ is a singleton. Then k is a common fixed point for S in K.

When $\Delta=(E^*)_1$ we can even allow K to be unbounded.

Lemma 4.13
Let S be a semigroup that acts on a weak closed convex set $K\neq\emptyset$ of a dual Banach space $E=(E_*)^*$ as pseudo weak* nonexpansive self mappings. Suppose that X is a left invariant positively semilinear lattice subset of $\ell^\infty(S)$ that has a strictly increasing left subinvariant submean μ and contains the function $\varphi_{(x,\phi,y)}(s)=|\langle\phi,T_sx-y\rangle|$ ($s\in S$) for all $\phi\in(E_*)_1$ and all $x,y\in K$ such that Sx is bounded. If K has normal structure and there is $b\in K$ such that Sb is bounded, then K has a common fixed point for S.*

Proof. From Lemma 4.7, K_b is a nonempty weak* compact convex S-invariant subset of K. Then replace K by K_b. The result then follows from Lemma 4.12 for the case $\Delta = (E_*)_1$.

In light of Remark 4.8 we derive our first main theorem concerning norm nonexpansive semigroup actions on weak* closed convex sets.

Theorem 4.14
Let S be a semigroup that acts on a weak closed convex set $K \neq \emptyset$ of a dual Banach space $E = (E_*)^*$ as norm nonexpansive self mappings. Suppose that $\ell^\infty(S)$ has a strictly increasing supremum admissible left subinvariant submean. If K has normal structure and there is $b \in K$ such that Sb is bounded, then K has a common fixed point for S.*

Proof. The function $\varphi_{(x,\phi,y)}(s) = |\langle \phi, T_s x - y\rangle|$, $s \in S$, belongs to $X = \ell^\infty(S)$ for all $\phi \in (E_*)_1$ and all $x, y \in K$ such that Sx is bounded. Note that the boundedness of Sb and nonexpansiveness of the S-action imply Sx is bounded for all $x \in K$. Lemma 4.13 and Remark 4.8 then lead to the result.

A special case is when S is a group.

Corollary 4.15
Let G be a group that acts on a weak closed convex set $K \neq \emptyset$ of a dual Banach space $E = (E_*)^*$ as norm nonexpansive self mappings. If K has normal structure and there is $b \in K$ such that Gb is bounded, then K has a common fixed point for G.*

Proof. $\ell^\infty(G)$ has a strictly increasing left invariant submean (Example 1) that is supremum admissible.

We remark that since nonexpansive group actions are isometries, Corollary 4.15 also follows from [4, Theorem 3].

From Proposition 4.3, Lemma 4.12 immediately yields the following.

Theorem 4.16
Let E be a Banach space and Δ be a weak dense subset of $(E^*)_1$. Let S be a semigroup acting on a $\sigma(E, \Delta)$ compact convex subset K of E as $\sigma(E, \Delta)$ continuous and norm nonexpansive self mappings. Suppose that X is a left invariant positively semilinear lattice subset of $\ell^\infty(S)$ that has a strictly increasing left subinvariant submean μ and contains the functions $\varphi_{(x,\phi,y)}(s) = |\langle \phi, T_s x - y\rangle|$ $(s \in S)$ for all $\phi \in \Delta$ and all $x, y \in K$. If K has normal structure then K has a common fixed point for S.*

We now consider special types of semigroups S that ensure certain subspaces X of $\ell^\infty(S)$ that fulfill the requirements of our general results above.

Corollary 4.17
Let S be a left reversible semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a norm nonexpansive representation of S on a nonempty weak compact convex subset K of a dual Banach space $E = (E_*)^*$. If K has normal structure and the representation is weak* continuous, then K contains a common fixed point for S.*

Proof. We choose $X = \ell^\infty(S)$. From Example 3, there is a strictly increasing left subinvariant submean on X. Trivially, $\varphi_{(x,\phi,y)} \in X$ for all $\phi \in (E_*)_1$ and all $x, y \in K$. The conclusion follows from Theorem 4.16 for the case $\Delta = (E_*)_1$.

We wonder whether the weak* continuity assumption on the representation is removable in the above corollary.

Recall that a function $f \in C_b(S)$ is almost periodic if the orbit $\{l_s f : s \in S\}$ of f is norm precompact in $C_b(S)$. The set of almost periodic functions on S is denoted by $AP(S)$. This is a translation invariant subspace of $C_b(S)$, containing the constant functions.

Corollary 4.18
Let S be a semitopological semigroup such that $AP(S)$ has a LIM. Let $\mathcal{S} = \{T_s : s \in S\}$ be a norm nonexpansive representation of S on a nonempty weak compact convex subset K of a dual Banach space $E = (E_*)^*$. If K has normal structure and the representation is separately continuous and equicontinuous when K is equipped with the weak* topology of E, then K contains a common fixed point for S.*

Proof. We consider $X = AP(S)$. A LIM on $AP(S)$ may be regarded as a strictly increasing left subinvariant submean on X. Proposition 4.3 ensures that the representation is pseudo weak* nonexpansive. Since the representation is weak* equicontinuous, we have $\varphi_{(x,\phi,y)} \in AP(S) = X$ for all $\phi \in (E_*)_1$ and all $x, y \in K$ (see [12, Lemma 3.1]). The result then follows from Theorem 4.16 for the case $\Delta = (E_*)_1$.

Corollary 4.19
Let S be a semitopological semigroup such that $LUC(S)$ has a left invariant mean. Let $\mathcal{S} = \{T_s : s \in S\}$ be a norm nonexpansive representation of S on a nonempty weak compact convex subset K of a dual Banach space $E = (E_*)^*$ and the mapping $(s, x) \mapsto T_s x$ from $S \times K$ into K is jointly continuous when K is equipped with the weak* topology of E. If K has normal structure, then it contains a common fixed point for S.*

Proof. We consider $X = LUC(S)$. From the hypothesis, X has a left invariant mean which is certainly a strictly increasing left subinvariant submean. Since the representation of S on K is weak* jointly continuous and K is weak* compact, the function $\varphi_{(x,\phi,y)}(s) = |\langle \phi, T_s x - y\rangle|$ is left uniformly continuous, i.e. $\varphi_{(x,\phi,y)} \in X$ for all $\phi \in (E_*)_1$ and all $x, y \in K$. So again the result follows from Theorem 4.16 for the case $\Delta = (E_*)_1$.

Remark 4.20
Corollary 4.19 is indeed [17, Proposition 6.1], which partially answers the open question raised in [14] (see also page 2962 of [17]). One may relax the normal structure assumption to weak* normal structure on K. We wonder whether the normal structure assumption on K is removable.

Corollary 4.21
Let S be a semitopological semigroup such that $RUC(S)$ has a left invariant mean. Let $\mathcal{S} = \{T_s : s \in S\}$ be a norm nonexpansive representation of S on a nonempty weak compact convex subset K of a dual Banach space $E = (E_*)^*$. If K has normal structure and if the representation is separately continuous and equicontinuous when K is equipped with the weak* topology of E, then K contains a common fixed point for S.*

Proof. Take $X = RUC(S)$. By assumption, it has a left invariant mean. On the other hand, the representation is pseudo weak* nonexpansive due to Proposition 4.3. Since the representation is weak* equicontinuous, for each $\phi \in (E_*)_1$ and all $x, y \in K$ we have that the function $\varphi_{(x,\phi,y)}(s) = |\langle \phi, T_s x - y\rangle|$ is right uniformly continuous, i.e. $\varphi_{(x,\phi,y)} \in X$. The result follows from Theorem 4.16.

5. Some open questions and remarks

Let S be a semitopological semigroup. Consider the following fixed point property for S.

(F_j) : Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a norm nonexpansive representation of S on a nonempty weak* compact convex subset K of a dual Banach space $E = (E_*)^*$ such that the mapping $(s, x) \mapsto T_s x$ is jointly continuous from $S \times K$ into K when K has the weak*-topology of E, K contains a common fixed point for S.

Problem 1
If $LUC(S)$ has a left invariant mean, does S have the fixed point property (F_j)?

This problem was posed in a conference in Marseille in 1990 by the first author (see [14]). Corollary 4.19 partially answers this open problem. Note that if a semitopological semigroup S has the fixed point property (F_j), then $LUC(S)$ must have a left invariant mean. In fact, let E be $LUC(S)^*$, K be the set of means on $LUC(S)$ and $\mathcal{S} = \{\ell_s^* : s \in S\}$, then K and $\mathcal{S}$ satisfy the conditions of (F_j). A common fixed point in this K for this representation of S is indeed a left invariant mean on $LUC(S)$.

Problem 2
If $LUC(S)$ has a left invariant mean, when does the linear span of the set of left invariant means on $LUC(S)$ (i.e. the fixed point set of the adjoint operators of left translations on the set of means) form a finite dimensional space?

For discrete S this question was answered affirmatively by E. E. Graniner [7, 8].

Problem 3
Is the condition of $\sigma(E, \Delta)$ continuity on T removable in Proposition 4.3?

Any partial affirmative answer to this problem can notably improve Theorem 4.16.

For a semitopological semigroup S, it is known that $AP(S)$ has a left invariant mean if S is left reversible [10]. The converse is not true.

Problem 4
If S is a semitopological semigroup such that $AP(S)$ has a LIM, does the conclusion of Corollary 4.17 hold?

Note that Corollary 4.18 answers the question affirmatively under the strong condition that the representation of S is separately weak* continuous and weak* equicontinuous.

An F-algebra is a Banach algebra $\mathfrak{A}$ which is a predual of a von Neumann algebra $\mathfrak{M}$ such that the identity 1 of $\mathfrak{M}$ is a multiplicative linear functional on $\mathfrak{A}$ [13]. The F-algebra $\mathfrak{A}$ is left amenable if there is a topological left invariant mean m on $\mathfrak{A}^* = \mathfrak{M}$, i.e. if there is $m \in \mathfrak{M}^*$ such that $\|m\| = 1$ and $\langle m, \varphi \cdot f\rangle = \langle m, f\rangle$ for all $f \in \mathfrak{M}$ and all $\varphi \in \mathfrak{A}$ such that $\|\varphi\| = \langle 1, \varphi\rangle = 1$, where $\langle \varphi \cdot f, \psi\rangle = \langle f, \psi\varphi\rangle$ for $\psi \in \mathfrak{A}$. In a recent paper [18] the authors showed that $\mathfrak{A}$ is left amenable if and only if the metric semigroup $S = P_1(\mathfrak{A}) = \{\varphi \in \mathfrak{A} : \ \varphi \geq 0, \ \|\varphi\| = 1\}$ with the product and topology inherited from $\mathfrak{A}$ has the following fixed point property:

(F_U) : Whenever S acts on a compact subset K of a locally convex space such that the mapping $(s, y) \mapsto T_s y \colon S \times K \to K$ is separately continuous and is uniformly continuous in s for each $y \in K$, then K has a common fixed point for S.

Related to Problem 2 we pose the following problem.

Problem 5
Suppose that the F-algebra $\mathfrak{A}$ is left amenable. When is the space spanned by the set of topological left invariant means on $\mathfrak{A}$ finite dimensional?

The authors are grateful to the referee for careful reading of the paper and valuable suggestions.

References

[1] Atsushiba, Sachiko and Wataru Takahashi. "Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces." *J. Nonlinear Convex Anal.* 14, no. 2 (2013): 209-219. Cited on 70.

[2] Belluce, Lawrence P., and William A. Kirk. "Nonexpansive mappings and fixed-points in Banach spaces." *Illinois J. Math.* 11 (1967): 474-479. Cited on 68.

[3] Berglund, John F., and Hugo D. Junghenn, and Paul Milnes. *Analysis on semigroups.* New York: John Wiley & Sons, Inc., 1989. Cited on 73.

[4] Brodskiĭ, M. S. and D. P. Mil'man. "On the center of a convex set." *Doklady Akad. Nauk SSSR (N.S.)* 59 (1948): 837-840. Cited on 82.

[5] Fan, Ky. "Minimax theorems." *Proc. Nat. Acad. Sci. U. S. A.* 39, (1953): 42-47. Cited on 68.

[6] Fan, Ky. "Existence theorems and extreme solutions for inequalities concerning convex functions or linear transformations." *Math. Z.* 68 (1957): 205-216. Cited on 68 and 79.

[7] Granirer, Edmond E. "On amenable semigroups with a finite-dimensional set of invariant means. I." *Illinois J. Math.* 7 (1963): 32-48. Cited on 84.

[8] Granirer, Edmond E. "On amenable semigroups with a finite-dimensional set of invariant means. II." *Illinois J. Math.* 7 (1963): 49-58. Cited on 84.

[9] Hewitt, Edwin and Kenneth A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations.* Vol 115 of *Die Grundlehren der mathematischen Wissenschaften*, New York: Academic Press, Inc., 1963. Cited on 68.

[10] Holmes, R. D., and Anthony To-Ming Lau. "Non-expansive actions of topological semigroups and fixed points." *J. London Math. Soc.* 5, no. 2 (1972): 330-336. Cited on 69 and 84.

[11] Hsu, R. *Topics on weakly almost periodic functions.* Ph.D. diss., State University of New York at Buffalo, 1985. Cited on 70.

[12] Lau, Anthomy To-Ming. "Invariant means on almost periodic functions and fixed point properties." *Rocky Mountain J. Math.* 3 (1973): 69-76. Cited on 83.

[13] Lau, Anthony To-Ming. "Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups." *Fund. Math.* 118, no. 3 (1983): 161-175. Cited on 85.

[14] Lau, Anthony To-Ming. "Amenability and fixed point property for semigroup of nonexpansive mappings." In *Fixed point theory and applications (Marseille, 1989)*, 303–313. Vol. 252 of *Pitman Res. Notes Math. Ser.* Harlow: Longman Sci. Tech., 1991. Cited on 83 and 84.

[15] Lau, Anthony To-Ming, and Wataru Takahashi. "Invariant submeans and semigroups of nonexpansive mappings on Banach spaces with normal structure." *J. Funct. Anal.* 142, no. 1 (1996): 79-88. Cited on 70.

[16] Lau, Anthony To-Ming, and Wataru Takahashi. "Nonlinear submeans on semigroups." *Topol. Methods Nonlinear Anal.* 22, no. 2 (2003): 345-353. Cited on 70.

[17] Lau, Anthony To-Ming, and Yong Zhang. "Fixed point properties of semigroups of non-expansive mappings." *J. Funct. Anal.* 254, no. 10 (2008): 2534-2554. Cited on 83.

[18] Lau, Anthony To-Ming, and Yong Zhang. "Finite-dimensional invariant subspace property and amenability for a class of Banach algebras." *Trans. Amer. Math. Soc.* 368, no. 6 (2016): 3755-3775. Cited on 85.

[19] Lennard, Chris. $\mathscr{C}_1$ is uniformly Kadec-Klee." *Proc. Amer. Math. Soc.* 109, no. 1 (1990): 71-77. Cited on 68.

[20] Lim, Teck Cheong. "Asymptotic centers and nonexpansive mappings in conjugate Banach spaces." *Pacific J. Math.* 90, no. 1 (1980) 135-143. Cited on 69 and 70.

[21] Lim, Teck Cheong. "Characterizations of normal structure." *Proc. Amer. Math. Soc.* 43 (1974): 313-319. Cited on 69.

[22] Lim, Teck Cheong. "A fixed point theorem for families on nonexpansive mappings." *Pacific J. Math.* 53: (1974) 487-493. Cited on 80.

[23] Mizoguchi, Noriko, and Wataru Takahashi. "On the existence of fixed points and ergodic retractions for Lipschitzian semigroups in Hilbert spaces." *Nonlinear Anal.* 14, no. 1 (1990): 69-80. Cited on 70.

[24] Randrianantoanina, Narcisse. "Fixed point properties of semigroups of nonexpansive mappings." *J. Funct. Anal.* 258, no. 11 (2010): 3801-3817. Cited on 68.

[25] Wiśnicki, Andrzej. "Amenable semigroups of nonexpansive mappings on weakly compact convex sets." *J. Nonlinear Convex Anal.* 17, no. 10 (2016): 2119-2127. Cited on 68.

Anthony To-Ming Lau
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta
T6G 2G1 Canada
E-mail: anthonyt@ualberta.ca

Yong Zhang
Department of Mathematics
University of Manitoba
Winnipeg, Manitoba
R3T 2N2 Canada
E-mail: yong.zhang@umanitoba.ca

Received: May 5, 2018; final version: August 28, 2018;
available online: October 22, 2018.