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Pseudo-amenability and pseudo-contractibility of restricted semigroup algebra

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Abstract. In this article the pseudo-amenability and pseudo-contractibility of restricted semigroup algebra $l^1_r(S)$ and semigroup algebra, $l^1(S_r)$ on restricted semigroup, S_r are investigated for different classes of inverse semigroups such as Brandt semigroup, and Clifford semigroup. We particularly show the equivalence between pseudo-amenability and character amenability of restricted semigroup algebra on a Clifford semigroup and semigroup algebra on a restricted semigroup. Moreover, we show that when $S = M^0(G, I)$ is a Brandt semigroup, pseudo-amenability of $l^1(S_r)$ is equivalent to its pseudo-contractibility.

1. Introduction

The notions of pseudo-amenability and pseudo-contractibility in Banach algebra which were introduced in [9], have been studied for different classes of semigroups. The notable ones among these are the research work in [7], [8] and [19]. The authors in [7] particularly showed that for a Brandt semigroup $S = M^0(G, I)$, the semigroup algebra $l^1(S)$ is pseudo-contractible if and only G and I are finite.

Recently, the notions of module pseudo-amenability and module pseudo-contractibility in Banach algebras were introduced in [2], where necessary and sufficient conditions were particularly obtained for the semigroup algebra $l^1(S)$ and its dual to be $l^1(E)$-module pseudo-amenable for every inverse semigroup S with subsemigroup E of idempotent.

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The concept of restricted representation for an inverse semigroup S was introduced in [11] and the restricted forms of some important Banach algebras on S were studied by the same author.

In [14], the amenability of restricted semigroup algebras was studied where it was shown that for an inverse semigroup S, $l^1_r(S)$ is amenable if and only if $l^1(S)$ is amenable. The authors in [13] continued further study on restricted semigroups by investigating character amenability of restricted semigroup algebras and show that for an inverse semigroup S, the restricted semigroup algebra $l^1_r(S)$ is character amenable if and only if $l^1(S_r)$ is character amenable and that for the same inverse semigroup, the semigroup algebra $l^1(S_r)$ on restricted semigroup S_r is character amenable if and only if $l^1(S)$ is character amenable.

In this paper, S is a discrete semigroup and $l^1(S)$ is a discrete semigroup algebra. We show the pseudo-amenability and pseudo-contractibility of restricted semigroup algebra $l^1_r(S)$ and semigroup algebra $l^1(S_r)$ on restricted semigroup S_r.

2. Preliminaries and definitions

In this section, we recall some standard notations and define some basic concepts that are relevant to this study.

Let A be a Banach algebra. A derivation $D\colon A \to X$ is *approximately inner* if there is a net $(x_\alpha) \subset X$ such that

$$D(a) = \lim_\alpha (a.x_\alpha - x_\alpha.a) \qquad \text{for all } a \in A.$$

The limit is being taken in $(X, \|\cdot\|)$, i.e. $D(a) = \lim_\alpha \delta_\alpha(a)$, where (δ_{x_α}) is a net of inner derivations.

The Banach algebra A is *approximately amenable* if for each Banach A-bimodule X, every continuous derivation $D\colon A \to X$ is approximately inner.

Let A be a Banach algebra, a *character on* A is a homomorphism $\varphi\colon A \to \mathbb{C}$. A character φ is a non-zero linear functional on A such that

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad \text{for all } a, b \in A.$$

By Φ_A we denote the set of all characters on A, called the *character space of* A.

Let A be a Banach algebra and let $\varphi \in \Phi_A$. A is left *φ-amenable* if every continuous derivation $D\colon A \to X'$ is inner for every $X \in M^A_{\varphi_r}$, where $M^A_{\varphi_r}$ denotes the class of Banach A-bimodule X for which the right module action of A on X is given by

$$x.a = \varphi(a)x \qquad \text{for all } a \in A,\ x \in X,\ \varphi \in \Phi_A.$$

A *right φ-amenable* Banach algebra is similarly defined. Algebra A is *left (right) character amenable* if it is left (right) φ-amenable for every $\varphi \in \Phi_A$. Finally we say that A is *character amenable* if it is both left and right amenable.

A Banach algebra A is said to be *pseudo-amenable* if there is a net $(m_\alpha)_{\alpha\in I} \subseteq A\widehat{\otimes}A$, (not necessarily bounded) called an approximate diagonal for A, such that for each $a \in A$,

$$a.m_\alpha - m_\alpha.a \to 0 \quad \text{and} \quad \pi(m_\alpha)a \to a.$$

Moreover, A is *pseudo-contractible* if there is an approximate diagonal $(m_\alpha)_{\alpha\in I}$ for A which is central, that is $a.m_\alpha = m_\alpha.a$ for each $a \in A$ and $\alpha \in I$.

Suppose that A and B are Banach algebras. We denote the projective tensor product of A and B by $A\widehat{\otimes}B$. The Banach algebra $A\widehat{\otimes}A$ is a Banach A-bimodule with the following actions

$$a.(b\otimes c) = ab\otimes c, \quad (b\otimes c).a = b\otimes ca \qquad \text{for all } a,b,c\in A.$$

Let A be a Banach algebra and let I be a non-empty set. We denote by $\mathbb{M}_I(A)$, the set of $I\times I$ matrices (a_{ij}) with entries in A such that

$$\|(a_{ij})\| = \sum_{i,j\in I}\|a_{ij}\| < \infty,$$

see [16]. Then $\mathbb{M}_I(A)$ with the usual matrix multiplication is a Banach algebra that belongs to the class of l^1-Munn algebras ([15]). It is an easy verification that the map $\theta\colon \mathbb{M}_I(A)\to\mathbb{M}_I(\mathbb{C})\widehat{\otimes}A$ defined by

$$\theta((a_{ij})) = \sum_{i,j\in I} E_{ij}\otimes a_{ij}, \qquad (a_{ij})\in\mathbb{M}_I(A),$$

is an isometric isomorphism of Banach algebras, where (E_{ij}) are the matrix units in $\mathbb{M}_I(\mathbb{C})$.

Let $\{A_\alpha : \alpha\in I\}$ be a collection of Banach algebras. Then the l^1-direct sum of A_α is denoted by $l^1 - \oplus\{A_\alpha : \alpha\in I\}$, which is a Banach algebra with componentwise operations.

A non empty set S with an associative binary operation denoted by

$$S\times S\to S, \qquad (s,t)\mapsto st$$

is called a *semigroup*. For example, $(\mathbb{N},+)$, $(\mathbb{Z},+)$ and $\mathbb{Z}^2=\mathbb{Z}\times\mathbb{Z}$ with binary operation

$$(m_1,n_1).(m_2,n_2) = (m_1+m_2,n_2)$$

are semigroups.

The following definitions are recalled from [12]. Let S be a semigroup.

- (i) Let $s\in S$. An element $s^*\in S$ is called an *inverse* of s if $ss^*s=s$ and $s^*ss^*=s^*$.
- (ii) An element $s\in S$ is called *regular* if there exists $t\in S$ with $sts=s$.
- (iii) An element $s\in S$ is called *completely regular* if there exists $t\in S$ with $sts=s$ and $ts=st$.
- (iv) S is called *regular* if each $s\in S$ is a regular element.
- (v) S is called *completely regular* if each $s\in S$ is a completely regular element.
- (vi) S is called an *inverse semigroup* if every element in S has a unique inverse.
- (vii) An element $p\in S$ is called an *idempotent* if $p^2=p$; the set of idempotents of S is denoted by $E(S)$.
- $(viii)$ S is called a *semilattice* if it commutes and $E(S)=S$.

(ix) S is called a *band semigroup* if it is a semilattice.

(x) S is called a *rectangular band semigroup* if it is a band semigroup and if S is regular.

An inverse semigroup S is called a *Clifford semigroup* if $ss^{-1} = s^{-1}s$ for each $s \in S$.

Let S be a Clifford semigroup and let $s \in S$. Then $s \in G_{ss^{-1}}$ and hence S is a disjoint union of the groups G_p, $p \in E(S)$, that is $S = \bigcup_{p \in E(S)} G_p$ where G_p's are the maximal subgroups of S.

Let S be a non-empty set. Then

$$l^1(S) = \Big\{ f \in \mathbb{C}^S : \sum_{s \in S} |f(s)| < \infty \Big\},$$

with the norm $\|\cdot\|_1$ given by $\|f\|_1 = \sum_{s \in S} |f(s)|$ for $f \in l^1(S)$. We write δ_s for the characteristic function of $\{s\}$ when $s \in S$.

Now suppose that S is a semigroup. For $f, g \in l^1(S)$ we set

$$(f * g)(t) = \Big\{ \sum f(r)g(s) : \ r, s \in S, \ rs = t \Big\}, \qquad t \in S,$$

so that $f * g \in l^1(S)$. It is standard that $(l^1(S), *)$ is a Banach algebra, called the *semigroup algebra on S*. Elements of $l^1(S)$ are of the form $f = \sum_{s \in S} \alpha_s \delta_s$ and the dual space of $l^1(S)$ with the duality

$$\langle f, \lambda \rangle = \sum_{s \in S} f(s)\lambda(s), \qquad f \in l^1(S), \ \lambda \in l^\infty(S).$$

Notice that $l^1(S)$ is commutative if and only if S is abelian and $l^1(S^\#) = l^1(S)^\#$. If $f \in l^1(S)$, then $f = 0$ on S except at most on a countable subset of S. In other words, the set $D = \{s \in S : \ f(s) \neq 0\}$ is at most countable since if $D_n = \{s \in S : \ |f(s)| \geq \frac{1}{n}\}$, $D = \bigcup_{n \in \mathbb{N}} D_n$. There is always one character in the Banach algebra $l^1(S)$, this is the *augmentation character*.

Let T be a subsemigroup of a semigroup S. Then

$$\Phi_{l^1(T)} = \{\varphi_s|_{l^1(T)} : \ \varphi_s \in \Phi_{l^1(S)}\}.$$

See [4, Chapter 4] for details about this algebra.

3. General results

In this section, we prove some general results which are useful in establishing our main results on restricted semigroup algebras.

For a semigroup S, $l^1(S)\widehat{\otimes} l^1(S)$ is isometrically isomorphic to $l^1(S \times S)$, and so, we identify $(l^1(S)\widehat{\otimes} l^1(S))''$ with $l^1(S \times S)''$. We define the bimodule operations under this identification as follows. Let $M \in (l^\infty(S \times S))'$ and $s \in S$, then for all $f \in l^\infty(S \times S)$,

$$Ms(f) = M(sf), \qquad sM(f) = M(fs),$$

where

$$fs(u,v) = f(su,v), \qquad sf(u,v) = f(u,vs).$$

Clearly, $l^1(S)$ is a Banach $l^1(S)$-bimodule. In the case where S is a semilattice, $l^1(S)$ is a commutative $l^1(S)$-module. The dual module action of $s \in S$ on the dual space $l^1(S)' = l^\infty(S)$ is given by

$$\langle t, s.\lambda\rangle = \langle ts, \lambda\rangle, \qquad \langle t, \lambda.s\rangle = \langle st, \lambda\rangle, \qquad t \in S.$$

With this module action, it follows that a continuous linear map $D\colon l^1(S) \to l^\infty(S)$ is a derivation if and only if

$$\langle r, D(st)\rangle = \langle tr, D(s)\rangle + \langle rs, D(t)\rangle, \qquad r,s,t \in S$$

and D is inner if and only if there exists $\lambda \in l^\infty(S)$ such that

$$\langle t, D(s)\rangle = \langle ts - st, \lambda\rangle, \qquad s,t \in l^1(S).$$

For a Banach algebra A, we recall from [18] that a Banach A-bimodule X is *pseudo-unital* if $X = \{a.x.b : \ a,b \in A,\ x \in X\}$, and X is *essential* if the linear hull of $\{a.x.b : \ a,b \in A,\ x \in X\}$ is dense in X. If A has a bounded approximate identity and X is essential, then X is *pseudo-unital*.

By following similar argument as in [18, Proposition 2.1.5], we have the following results.

Proposition 3.1
Let A be a Banach algebra with a bounded approximate identity. Then A is pseudo-amenable if and only if every continuous derivation $D\colon A \to X'$ is approximately inner for each pseudo-unital Banach A-bimodule X.

Theorem 3.2
Let S be a semilattice and let $l^1(S)$ have a bounded approximate identity. Then $l^1(S)$ is pseudo-amenable if and only if every continuous derivation $D\colon l^1(S) \to X'$ is approximately inner for each pseudo-unital Banach A-bimodule X.

Proof. Suppose $l^1(S)$ is pseudo-amenable, then there exists an approximate diagonal $m_\alpha \in l^1(S \times S)$ for $l^1(S)$ such that $m_\alpha \delta_s - \delta_s m_\alpha \to 0$ and $\pi m_\alpha \delta_s \to \delta_s$. Let $D\colon l^1(S) \to X'$ be a bounded derivation and suppose X is a pseudo-unital $l^1(S)$-bimodule. Then for each $x \in X$ there exist $f,g \in l^1(S)$ and there is $y \in X$ such that $y = f.x.g$. Since D is bounded there exists $M > 0$ such that $\|D\| \le M$. Let $D \circ \pi = \Phi\colon l^1(S)\widehat{\otimes} l^1(S) \to X'$ be defined by $\Phi(\delta_s \otimes e_\alpha) = D(\delta_s).e_\alpha$, where $\pi\colon l^1(S)\widehat{\otimes} l^1(S) \to l^1(S)$ is an induced product map and e_α is the bounded approximate identity in $l^1(S)$. Clearly, Φ is a bounded Banach $l^1(S)$ morphism and $\|\Phi\| = \|D \circ \pi\| \le \|D\|$ implies that $\|\pi\| \le 1$. For every $\delta_s \in l^1(S)$, we have

$$\Phi(m_\alpha.\delta_s - \delta_s.m_\alpha) = \Phi(m_\alpha)\delta_s - \delta_s.\Phi(m_\alpha) \to D(\delta_s)e_\alpha.$$

Let $\Phi(m_\alpha) = -\Psi_\alpha$. Then we have $D(\delta_s).e_\alpha = \delta_s.\Psi_\alpha - \Psi_\alpha\delta_s$. Now

$$\langle f.x.g, \Phi(\delta_s \otimes e_\alpha)\rangle = \langle y, D(\delta_s).e_\alpha\rangle$$

implies that $\langle y, D(\delta_s)\rangle = \lim_\alpha \langle y, D(\delta_s)e_\alpha\rangle$. Since X is a pseudo-unital Banach $l^1(S)$-module, then $D(\delta_s).e_\alpha \to D(\delta_s)$ in the weak* topology of X'. Hence $D(\delta_s) = \lim_\alpha D(\delta_s)e_\alpha = \lim_\alpha(\delta_s\Psi_\alpha - \Psi_\alpha\delta_s)$. This clearly shows Ψ_α is a net in X' and hence every continuous derivation D is approximately inner for each pseudo-unital Banach A-bimodule X.

Conversely, suppose every continuous derivation $D\colon A \to X'$ is approximately inner, then $l^1(S)$ is approximately amenable. Since $l^1(S)$ has a bounded approximate identity, hence it follows from [9, Proposition 3.2] that $l^1(S)$ is pseudo-amenable.

Proposition 3.3

Let A be a Banach algebra and let $M_J(A)$ be a unital Banach algebra where J is a non empty set. Then A is pseudo-amenable if and only if $M_J(A)$ is pseudo-amenable.

Proof. It is a well-known result that

$$M_J(A) \cong M_J(\mathbb{C})\widehat{\otimes}A.$$

If $M_J(A)$ is pseudo-amenable, then by [9, Proposition 2.2], $M_J(\mathbb{C})\widehat{\otimes}A$ is pseudo-amenable. Hence, it suffices to say that A is pseudo-amenable.

Conversely, if A is pseudo-amenable, then $M_J(A)$ is clearly pseudo-amenable by the same result as in [9].

Proposition 3.4

Let S be an inverse semigroup with $E(S)$ finite. Then $l^1(S)$ is pseudo-amenable if and only if $l^1(E(S))$ is pseudo-amenable.

Proof. Let $T\colon S \to E(S)$ be defined by $Ts = ss^*$. Then T extends to a norm decreasing linear map $T\colon l^1(S) \to l^1(E(S))$ defined by $T\delta_s = \delta_{ss^*} = \delta_e$. Suppose $l^1(S)$ is pseudo-amenable, then there exists an approximate diagonal $m_\alpha \in l^1(S)\widehat{\otimes}l^1(S)$ such that $m_\alpha\delta_s - \delta_s m_\alpha \to 0$ and $\pi m_\alpha\delta_s - \delta_s \to 0$. Let

$$\|T\delta_s - \delta_e\| < \epsilon/2. \tag{1}$$

For each $m_\alpha \in l^1(S \times S)$ we have

$$\|Tm_\alpha\delta_s - \delta_e + \delta_e - T\delta_s m_\alpha\| < \epsilon/2 + \epsilon/2$$

and

$$\|Tm_\alpha\delta_s - T\delta_s m_\alpha\| \le \|T\|\|m_\alpha\delta_s - \delta_s m_\alpha\| < \epsilon. \tag{2}$$

Suppose $\|T\| \le 1$, then $m_\alpha\delta_s - \delta_s m_\alpha \to 0$. Putting $T\delta_s = \delta_e$ in (2) we have $\|m_\alpha\delta_e - \delta_e m_\alpha\| < \epsilon$, which implies that $m_\alpha\delta_e - \delta_e m_\alpha \to 0$. This shows that m_α is an approximate diagonal for $l^1(E(S))$. Now let $\pi\colon l^1(S\times S) \to l^1(S)$ be an induced product map. We consider the composition map $T\circ\pi\colon l^1(S\times S) \to l^1(E(S))$. Then for every $\delta_s \in l^1(S)$, we have

$$\|T\pi m_\alpha\delta_s - T\delta_s\| < \epsilon/2. \tag{3}$$

Combining (1) and (3) gives

$$\|\pi m_\alpha \delta_e - \delta_e\| = \|\pi m_\alpha T\delta_s - \delta_e\| = \|T\pi m_\alpha \delta_s - T\delta_s + T\delta_s - \delta_e\| < \epsilon,$$

so that $\pi m_\alpha \delta_e - \delta_e \to 0$. This clearly shows that $l^1(E(S))$ is pseudo-amenable.

The converse is clear.

Proposition 3.5

Let S be an inverse semigroup. Then $l^1(S)$ is pseudo-amenable if and only if S is finite.

Proof. Suppose that S is finite, then it is amenable [5]. Let G be a maximal group homomorphic image of S, then by [5, Theorem 1], G is amenable. It then follows from [19, Theorem 3], that $l^1(S)$ is pseudo-amenable.

Conversely, suppose $l^1(S)$ is pseudo-amenable, then G is an amenable group [8, Corollary 3.8]. By Theorem [5, Theorem 1], S is amenable, thus this implies that S is finite.

We recall the definition of a biflat Banach algebra. A Banach algebra A is *biflat* if the dual of the diagonal map $\triangle^*\colon A^* \to (A\widehat{\otimes}A)^*$ has a bounded left inverse which is an A-bimodule homomorphism [18, Definition 4.3.21]. Equivalently, we define a biflat Banach algebra as follows. Let A be a Banach algebra and let $\rho\colon A \to (A\widehat{\otimes}A)''$ be an A-bimodule. A is said to be biflat if there is a canonical embedding $\triangle^{**} \circ\rho$ of A into A'' [18, Lemma 4.3.22].

Theorem 3.6

Let S be a finite semilattice. If $l^1(S)$ is biflat then

(*i*) *there is an isometric isomorphism between $l^\infty(S)$ and $l^\infty(S\times S)$,*

(*ii*) *it is pseudo-contractible.*

Proof. Let $\rho\colon l^1(S) \to l^\infty(S\times S)'$ be an algebra homomorphism. Since $l^1(S)$ is biflat, then there exists a canonical embedding map $k_{l^1(S)}\colon l^1(S) \to l^\infty(S)'$. Let $\pi'\colon l^\infty(S) \to l^\infty(S\times S)$ be defined by $\pi'(\Phi) = \Psi$ for $\Phi \in l^\infty(S)$, $\Psi \in l^\infty(S\times S)$, where $\pi\colon l^1(S\times S) \to l^1(S)$ is a diagonal map. We note that $\rho(\delta_s) \in l^\infty(S\times S)'$ for $\delta_s \in l^1(S)$. Hence

$$\rho(\delta_s)\Psi = \langle \Psi, \rho(\delta_s)\rangle. \tag{4}$$

If $k_{l^1(S)}(\delta_s) \in l^\infty(S)'$, then

$$k_{l^1(S)}\delta_s(\Phi) = \langle \Phi, k_{l^1(S)}(\delta_s)\rangle = \langle \Phi, \pi''\rho(\delta_s)\rangle = \langle \pi'(\Phi), \rho(\delta_s)\rangle = \langle \Psi, \rho(\delta_s)\rangle. \tag{5}$$

From $\|\pi''\circ\rho\| \le \|\rho\| = \|\pi''\| \le 1$. Let $\pi''|_{l^1(S)\widehat{\otimes}l^1(S)} = \pi$, then $\pi \subseteq \pi''$ and $\|\pi\| \le \|\pi''\| \le 1$. By considering (4) and (5),

$$\rho(\delta_s)\Psi = k_{l^1(S)}(\delta_s)\Phi,$$

then we can conclude that $l^\infty(S) \cong l^\infty(S\times S)$.

Now suppose $M \in l^1(S\times S)$ is a diagonal element for $l^1(S)$, then $\pi M \in l^1(S)$ and hence $\rho\pi M = \rho(\delta_s) = \pi M = (\delta_s)$. Now for each $\delta_s \in l^1(S)$ we have $\pi M\delta_s = \delta_s$. We can therefore conclude that M is a central approximate diagonal for $l^1(S)$ as $M\delta_s = \delta_s M$.

4. Results on restricted semigroup algebras

In this section we shall consider the pseudo-amenability properties of the restricted semigroup $l^1_r(S)$ and that of the semigroup algebra on restricted semigroup S_r. For details on restricted semigroups and restricted semigroup algebra, see [11] and [14].

For any inverse semigroup S, the restricted product of elements s and t of S is st if $s^*s = tt^*$ and undefined otherwise. The set S with this restricted product $\bullet$ forms a discrete groupoid [10, 3.1.4]. Adjoining a zero element 0, to this groupoid and putting $0^* = 0$ gives an inverse semigroup S_r [10, 3.3.3] with the multiplication rule

$$s \bullet t = \begin{cases} st, & \text{if } s^*s = tt^*, \\ 0, & \text{otherwise}, \end{cases}$$

for $s, t \in S \cup \{0\}$, which is called in [11], the *restricted semigroup* of S.

It is clear that $E(S_r) = E(S) \cup \{0\}$. Suppose S is a $*$-semigroup, given a Banach space $l^1(S)$ with the usual l^1-norm, we set $\tilde{f}(x) = \overline{f(x)}$ and define the following multiplication on $l^1(S)$,

$$(f \bullet g)(s) = \sum_{s^*s = tt^*} f(st)g(t^*), \qquad s \in S.$$

Then $(l^1(S), \bullet)$, with the l^1-norm is a Banach $*$-algebra denoted by $l^1_r(S)$, called the *restricted semigroup algebra* of S. For a restricted semigroup S_r of an inverse semigroup S the set $l^1(S_r)$ is called the *semigroup algebra on the restricted semigroup S_r*.

4.1. Pseudo-amenability of restricted semigroup algebras

In this section, we give some results about pseudo-amenable restricted semigroup algebras and a pseudo-amenable semigroup algebra on a restricted semigroup S_r.

THEOREM 4.1
Let S be an inverse semigroup with $E(S)$ finite. Then $l^1(S_r)$ is pseudo-amenable if and only if each G_i is an amenable group while G_i is the corresponding group in the Brandt semigroup S_i.

Proof. Suppose each G_i is an amenable group, then each S_i is amenable for $S_i = \bigcup_{i=1}^{n} G_i$. Now from the fact that $S_r = \bigcup_{i \in I} S_i$ for Brandt semigroups S_i and by using [19, Theorem 3], we get that $l^1(S_r)$ is pseudo-amenable.

Conversely, suppose $l^1(S_r)$ is pseudo-amenable and since $l^1(S_r) = \bigcup l^1(S_i)$, then S_i is an amenable semigroup for each i [7, Theorem 3.1]. Thus, each G_i is an amenable group.

THEOREM 4.2
Let $S = M^0(G, I)$ be a Brandt semigroup. Then $l^1(S_r)$ is pseudo-amenable if and only if S_r has finitely many idempotents.

Proof. Suppose S_r has finitely many idempotents, then $l^1(S_r)$ has a bounded approximate identity, [14, Theorem 3.6]. Let $S = S_r$ be as in [14, Example 1.2] and suppose $l^1(S_r)$ is approximately amenable, then by Proposition 3.1, it is pseudo-amenable.

Conversely, if $l^1(S_r)$ is approximately amenable and has a bounded approximate identity, then by [9, Proposition 3.2] it is pseudo-amenable and has a bounded approximate identity. It then follows from [14, Theorem 3.6] that S_r has finitely many idempotents.

Proposition 4.3
Let S be an inverse semigroup. The restricted semigroup algebra $l^1_r(S)$ is pseudo-amenable if and only if $l^1(S_r)$ is pseudo-amenable.

Proof. Suppose $l^1(S_r)$ is pseudo-amenable and $\mathbb{C}\delta_0$ is a closed ideal of $l^1(S_r)$, if $\mathbb{C}\delta_0$ has a bounded approximate identity, then $\mathbb{C}\delta_0$ is pseudo-amenable [9, Corollary 2.7]. Let there exist an epimorphism $\theta\colon l^1(S_r) \to l^1_r(S)$ which kernel is $\mathbb{C}\delta_0$. By [11, Theorem 3.7], $l^1(S_r)/\mathbb{C}\delta_0$ is isometrically isomorphic to $l^1_r(S)$. Hence $l^1_r(S)$ is pseudo-amenable by [9, Theorem 2.2].

Proposition 4.4
Let S be a left cancellative semigroup. $l^1(S_r)$ is pseudo-amenable if and only if $l^1(S_r)$ has a bounded approximate identity.

Proof. Suppose $l^1(S_r)$ has a bounded approximate identity, then S has finitely many idempotents [14, Theorem 3.6]. Then, by [14, Corollary 3.7], $l^1(S_r)$ is amenable. It then follows from [7, Theorem 3.6] and Proposition 4.3 that $l^1(S_r)$ is pseudo-amenable.

Conversely, suppose $l^1(S_r)$ is pseudo-amenable and S_r is an amenable group [7, Theorem 3.6], then S is equally an amenable group. This implies that S is finite and hence has finitely many idempotents. Then by [14, Theorem 3.6], $l^1(S_r)$ has a bounded approximate identity.

Corollary 4.5
If an inverse semigroup S is infinite and $l^1(S)$ has no bounded approximate identity, then $l^1(S)$, $l^1(S_r)$ and $l^1_r(S)$ are not pseudo-amenable.

Proposition 4.6
Let $S = \bigcup_{i=1} G_i$ be a Clifford semigroup with $E(S)$ finite. Then the following are equivalent

- (i) *$l^1_r(S)$ is pseudo-amenable,*
- (ii) *$l^1(S_r)$ is pseudo-amenable,*
- (iii) *G_i is an amenable group for each i.*

Proof. Equivalence $(i)\Leftrightarrow(ii)$ follows from Proposition 4.3.

To show $(ii)\Leftrightarrow(iii)$ observe that $S_r = S \cup \{0\}$. Let $S_i = G_i \cup \{0\}$ for $i = 1, 2, \ldots, n$, then each S_i is a Brandt semigroup with the group G_i. Thus $S_r = \bigcup_{i=1} S_i$ with $S_i \cap S_j = S_iS_j = \{0\}$ and so the result follows by applying Theorem 4.1.

To prove $(iii)\Leftrightarrow(i)$ suppose that each G_i is an amenable group, then by Theorem 4.1 and Proposition 4.3, $l^1_r(S)$ is pseudo-amenable.

Proposition 4.7
Let $S = M^0(G, I, n)$ be a Brandt semigroup and let $l^1(S)$ be a unital Banach algebra. Then $l^1_r(S)$ is pseudo-amenable if and only if $M_n(l^1(G))$ is pseudo-amenable.

Proof. Suppose G is discrete then $L^1(G) = l^1(G)$. If G is amenable then $l^1(G)$ is pseudo-amenable, see [9, Proposition 4.1]. Let $S = S_r$ therefore $l^1(S) = l^1(S_r)$ [14, Example 1.2]. Let $l^1_r(S) = l^1(S)/\mathbb{C}\delta_0 \cong M^0(l^1(G), I, n)$, where $\mathbb{C}\delta_0$ is a closed ideal of $l^1(S_r)$. Now put $\tilde{A} = l^1_r(S)$ and $\mathrm{A} = l^1(G)$, then the result follows from Proposition 3.3.

We recall that $S_r = S \cup \{0\}$. We have $l^1(E_r) = (l^1(E \cup [0]), \bullet)$ as a subalgebra of $l^1(S_r)$. Hence $l^1(E_r) \subseteq l^1(S_r)$. Now suppose S_r is a finite semilattice. Let $A = l^1(S_r)$ and let $A_{s_r} = l^1(E_r)$. Hence $A = l^1 \oplus A_{s_r} :\ A_{s_r}A_{t_r} \subset A_{st_r}$, $s_r, t_r \in S_r$. Clearly, each A_{s_r} is a closed subalgebra of A.

Proposition 4.8
Let S_r be a finite semilattice and let A be a Banach algebra graded over S_r. Then A is pseudo-amenable if and only if each A_{s_r} is pseudo-amenable.

The following is a modified Example of [14, Example 3.9].

Example 4.9
Let $S_r = \mathbb{N}_\wedge$, where $m \wedge n = \max(m, n)$ and $n^* = n$ for $m, n \in \mathbb{N}$ with $E(S_r) = S_r$ not finite. Hence $l^1(S_r)$ is not pseudo-amenable.

Proposition 4.10
Let S be an inverse semigroup. Then $l^1(S_r)$ is pseudo-amenable if and only if it is character amenable.

Proof. Suppose $l^1(S_r)$ is pseudo-amenable and we have that $l^1(S_r) = l^1(S \cup \{0\})$, then $l^1(S)$ is pseudo-amenable. By Proposition 3.5, S is finite and thus has a finite set of idempotent elements. Using the converse of [13, Proposition 4.2(ii)], we obtain that $l^1(S_r)$ is character amenable.

Conversely, if $l^1(S_r)$ is character amenable, then by [13, Theorem 3.3], it has a bounded approximate identity. Now suppose S is left cancellative, then by Proposition 4.4, $l^1(S_r)$ is pseudo-amenable.

Theorem 4.11
Let S be an inverse semigroup. Then $l^1(S_r)$ is pseudo-amenable if and only if S_r has principal series.

Proof. Let

$$S_r = (S_1 \cup \ldots \cup S_n) \supsetneqq (S_1 \cup \ldots \cup S_{n-1}) \supsetneqq \ldots \supsetneqq (S_1 \cup S_2) \supsetneqq (S_1) \supsetneqq \{0\} \supsetneqq \emptyset$$

be the chain of S. Clearly S_r is finite and so is S, since $(S_r) = S \cup \{0\}$. Thus by Proposition 3.5, $l^1(S)$ is pseudo-amenable. It then suffice to say that if $l^1(S_r) = l^1(S \cup \{0\})$ and $l^1(S)$ is pseudo-amenable, then $l^1(S_r)$ is pseudo-amenable.

Conversely, if S is a Brandt semigroup, and $l^1(S_r)$ is pseudo-amenable, then by Theorem 4.2, S_r has finitely many idempotents. By [14, Lemma 2.3], it follows that S_r has principal series.

4.2. Pseudo-contractibility of restricted semigroup algebras

In this section, we prove some results about pseudo-contractible restricted semigroup algebras and pseudo-contractible semigroup algebras on restricted semigroup.

Proposition 4.12
Let S be an inverse semigroup with finitely non-zero idempotent set. If $l^1_r(S)$ is pseudo-contractible such that there is an epimorphism $\theta\colon l^1_r(S) \to l^1_r(E(S))$, then

- *(i) $l^1_r(S)$ has left identity,*
- *(ii) S is a semilattice.*

Proof. Let M_α a central approximate diagonal for $l^1_r(S)$. Suppose $e \in E(S)$ and $\pi\colon l^1_r(S)\widehat{\otimes}l^1_r(S) \to l^1_r(S)$ is an induced product map. If $\pi M_\alpha \in l^1_r(S)$, then $\theta\pi M_\alpha = \delta_e$ for $\delta_e \in l^1_r(E(S)$. Now for each $\delta_s \in l^1_r(S)$, $\theta\pi M_\alpha\delta_s = \delta_e\delta_s = \delta_s$. Hence we can conclude that δ_e is a left identity in $l^1_r(S)$.

(ii) Clearly $E(S)$ is is a commutative subsemigroup of S. Then it suffices to show that $l^1_r(E(S)) \subset l^1_r(S)$. Let δ_e be a left identity in $l^1_r(S)$. For each $\delta_s \in l^1_r(S)$ we have $\delta_e\delta_s = \theta\pi M_\alpha\delta_s = \delta_s$. If $\delta_e \in l^1_r(E(S))$ and $l^1_r(E(S))$ is closed in $l^1_r(S)$ then $\delta_e\delta_e = \delta_e\delta_s$ We then conclude that S is a semilattice.

Proposition 4.13
Let S be a semilattice. Then $l^1_r(S)$ is pseudo-contractible if and only if $l^1_r(E(S))$ is pseudo-contractible.

Proof. Suppose $l^1_r(S)$ is pseudo-contractible, then $m \in l^1_r(S \times S)$ is a central diagonal for $l^1_r(S)$ such that $m\delta_s = \delta_s m$ and $\pi m\delta_s = \delta_s$. Let $T\colon l^1_r(S) \to l^1_r(E(S))$ be a norm decreasing linear map defined by $T\delta_s = \delta_e$, $\delta_s \in l^1_r(S)$ and $\delta_e \in l^1_r(E(S))$. Let $\pi\colon l^1_r(S \times S) \to l^1_r(S)$ be an induced product map. Clearly, $\pi m \in l^1_r(S)$ and since $\|T\| = 1$, we have

$$\|T(\pi m)\| = \|\delta_e\| = \|T\pi m - T\delta_s\| \leq \|T\|\|\pi m - \delta_s\| \leq \|\pi m - \delta_s\|,$$

thus we get $\pi m - \delta_s \to 0$ and $\pi m = \delta_s$.

Suppose M is a diagonal element for $l^1_r(E(S))$ and $\pi M \in l^1_r(E(S))$, then $T\pi m = \pi M$, so $T\delta_s = \pi M = \delta_e$. Hence for each δ_e in $l^1_r(E(S))$, $\pi M\delta_e = \delta_e$. Then $\pi m = \pi M$. This implies that $m = M$ and thus M is a central diagonal for $l^1_r(E(S))$. Therefore $m\delta_s = M\delta_s = M\delta_e$, since S is a semilattice. Then $M\delta_e = \delta_e M$ and $\pi M\delta_e = \delta_e$. Hence the proof is completed.

Arens in [1] defined two products $\square$ and $\diamond$ on the bidual A'' of Banach algebra A; A'' is a Banach algebra with respect to each of these products and each algebra contains A as a closed subalgebra. These products are called the first and second Arens products on A'', respectively. For the general theory of Arens products see [3, 4, 6].

Now let the restricted semigroup algebra be denoted by $B_r(S)$. In the particular case:

$$\delta_s \bullet \delta_t = \begin{cases} \delta_{st}, & s^*s = tt^*, \\ 0, & \text{otherwise}, \end{cases} \qquad s,t \in S,$$

see [17]. We identify the characteristic function of $\{(s,t)\}$ for $B_r(S\times S)$ by setting $\delta_{(s,t)} = \delta_s \otimes \delta_t$, so this induceds a Banach algebra isometric isomorphism from $B_r(S)\widehat{\otimes}B_r(S)$ onto $B_r(S\times S)$. With this identification $B_r(S\times S)$ is a Banach $B_r(S)$-bimodule. We also identify $(B_r(S)\widehat{\otimes}B_r(S))''$ with $B_r(S\times S)''$.

Now we show the module action of Arens regular restricted semigroup algebra. For $\delta^r_\lambda \in B_r(S)'$ we have

$$\langle \delta_s, \delta_t.\delta^r_\lambda\rangle = \langle \delta_{st}, \delta^r_\lambda\rangle, \quad \langle \delta_s, \delta^r_\lambda \delta_t\rangle = \langle \delta_{ts}, \delta^r_\lambda\rangle, \qquad \delta_s, \delta_t \in B_r(S).$$

Now for $\delta^r_\lambda \in B_r(S)'$ and $\delta^r_\Phi \in B_r(S)''$ we define $\delta^r_\lambda.\delta^r_\Phi$ and $\delta^r_\Phi.\delta^r_\lambda$ by

$$\langle \delta_s, \delta^r_\lambda.\delta^r_\Phi\rangle = \langle \delta^r_\Phi, \delta_s.\delta^r_\lambda\rangle, \quad \langle \delta_s, \delta^r_\Phi.\delta^r_\lambda\rangle = \langle \delta_\Phi, \delta^r_\lambda.\delta_s\rangle, \qquad \delta_s \in B_r(S).$$

Finally, for $\delta^r_\Phi, \delta^r_\Psi \in B_r(S)''$ we define

$$\langle \delta^r_\Phi \Box \delta^r_\Psi, \delta^r_\lambda\rangle = \langle \delta^r_\Phi, \delta^r_\Psi.\delta^r_\lambda\rangle, \quad \langle \delta^r_\Phi \diamond \delta^r_\Psi, \delta^r_\lambda\rangle = \langle \delta^r_\Psi, \delta^r_\lambda.\delta^r_\Phi\rangle \qquad \delta^r_\lambda \in B_r(S)'.$$

Theorem 4.14
Let $B_r(S)$ be an Arens regular restricted semigroup algebra. If $B_r(S)''$ is amenable then $B_r(S)$ is pseudo-contractible.

Proof. Let m^r_α and M^r be an approximate diagonal and virtual diagonal for $B_r(S)$, respectively. Let $\pi\colon B_r(S)\widehat{\otimes}B_r(S) \to B_r(S)$ be an induced product map and let $k_{B_r(S)}\colon B_r(S) \to B_r(S)''$ be a canonical embedding map. We have the composition map $k_{B_r(S)} \circ \pi\colon B_r(S)\widehat{\otimes}B_r(S) \to B_r(S)''$ such that

$$\|k_{B_r(S)} \circ \pi m_\alpha \delta_s\| \le \|\pi'' M^r \delta_s\|, \qquad \delta_s \in B_r(S),\ M^r \in B_r(S\times S)''.$$

Suppose $\|k_{B_r(S)}\| \le 1$, then $\|\pi m^r_\alpha\| = \|\pi'' M^r\|$. Since $\pi''|_{B_r(S\times S)} = \pi$ we have $\pi \subseteq \pi''$. Since m^r_α is weak* convergent to M^r in $(B_r(S)\widehat{\otimes}B_r(S))''$, then $m^r_\alpha \subseteq M^r$. Since $B_r(S)$ is a closed subalgebra of $B_r(S)''$, then it is amenable, this confirms the existence of virtual diagonal M^r in $B_r(S)$. By Goldstein's Theorem, $M^r = w^* \lim_\alpha(\delta_s m^r_\alpha - m^r_\alpha \delta_s) = 0$ for each $\delta_s \in B_r(S)$. Then

$$\pi'' w^* \lim_\alpha(\delta_s m^r_\alpha - m^r_\alpha \delta_s) = w^* \lim_\alpha \pi(\delta_s.m^r_\alpha - m^r_\alpha \delta_s) = 0.$$

Clearly, $\ker\pi \subset B_r(S)''$. Since $\ker\pi$ is closed in $B_r(S\times S)$, then for each $m^r_\alpha \in B_r(S\times S)$ and $M^r \in B_r(S)''$, m^r_α is closed in M^r. This shows that m_α is a central approximate diagonal. Hence if $M^r\delta_s = \delta_s M^r$, then $m^r_\alpha.\delta_s = \delta_s.m^r_\alpha$ and $\pi m^r_\alpha \delta_s = \delta_s$. Therefore, this shows that $B_r(S)$ is pseudo-contractible.

PROPOSITION 4.15
Let $S = M^0(G, I)$ be a Brandt semigroup. Then the following are equivalent

(i) $l^1(S_r)$ *is pseudo-contractible,*

(ii) $l^1(S_r)$ *is pseudo-amenable.*

Proof. $(i) \Rightarrow (ii)$. By Example 1.2 in [14], $S = S_r$ and so $l^1(S) = l^1(S_r)$. Now suppose $l^1(S_r)$ is pseudo-contractible, then G and I are finite [7, Corollary 2.5]. The finiteness of I implies that G is amenable [5, Theorem 7]. Using Theorem 3 [19], yields that $l^1(S_r)$ is pseudo-amenable.

$(ii) \Rightarrow (i)$. Suppose $l^1(S_r)$ is pseudo-amenable and $l^1(S) = l^1(S_r)$ as in the above argument, G is amenable [8, Corollary 3.8]. Hence G and I are finite and so by Corollary 2.5 [7], $l^1(S_r)$ is pseudo-contractible.

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