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*Hassan Haghighi, Mohammad Mosakhani  
and Mohammad Zaman Fashami*

### Resurgence and Waldschmidt constant of the ideal of a fat almost collinear subscheme in $\mathbb{P}^2$

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**Abstract.** Let  $Z_n = p_0 + p_1 + \cdots + p_n$  be a configuration of points in  $\mathbb{P}^2$ , where all points  $p_i$  except  $p_0$  lie on a line, and let  $I(Z_n)$  be its corresponding homogeneous ideal in  $\mathbb{K}[\mathbb{P}^2]$ . The resurgence and the Waldschmidt constant of  $I(Z_n)$  in [5] have been computed. In this note, we compute these two invariants for the defining ideal of a fat point subscheme  $Z_{n,c} = cp_0 + p_1 + \cdots + p_n$ , i.e. the point  $p_0$  is considered with multiplicity  $c$ . Our strategy is similar to [5].

#### 1. Introduction

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero and let  $\mathbb{P}^N$  be the projective  $N$ -space over  $\mathbb{K}$ . Let  $I$  be a non-trivial homogeneous ideal of  $R = \mathbb{K}[\mathbb{P}^N] = \mathbb{K}[x_0, \dots, x_N]$  and let  $m$  be a positive integer. The  $m^{\text{th}}$  symbolic power of  $I$  is defined to be the ideal

$$I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (R \cap I^m R_P),$$

where  $\text{Ass}(I)$  is the set of associated prime ideals of  $I$  and the intersection is taken in the field of fractions of  $R$ .

Recently, comparing two ideals  $I^{(m)}$  and  $I^r$ , for all pairs of positive integers  $(m, r)$ , has raised a great deal of interest among the algebraic geometers and

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commutative algebraists (see e.g. [2, 5, 8] and references therein). It can be easily proved that  $I^r \subseteq I^{(m)}$  if and only if  $r \geq m$  (see [1, Lemma 8.1.4]). In addition, if  $I^{(m)} \subseteq I^r$ , then  $m \geq r$ . However, determining all positive integers  $m$  and  $r$  to insure the containment  $I^{(m)} \subseteq I^r$  is a widely open problem, which is known as the *containment problem*.

Bocci and Harbourne [2] in order to capture more precise information about the containment problem, introduced a number of numerical invariants attached to  $I$ . One of these invariants is the resurgence of  $I$ . It is defined as

$$\rho(I) = \sup \left\{ \frac{m}{r} : I^{(m)} \not\subseteq I^r \right\}.$$

Another invariant is the *Waldschmidt constant* of  $I$ , and it is defined to be

$$\widehat{\alpha}(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m} = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m},$$

where  $\alpha(I)$  is the *initial degree* of  $I$ , i.e. the least degree  $t$  for which  $(I)_t \neq 0$ . Computing  $\rho(I)$  and  $\widehat{\alpha}(I)$  is a hard problem and except some special cases (see for example [2, 3, 6, 7]), they are not known.

Let  $\{p_1, \dots, p_n\}$  be a finite set of points in  $\mathbb{P}^N$  and let  $m_1, \dots, m_n$  be non-negative integers. Let  $I(p_i)$  be the ideal of forms in  $R$  which vanish at the point  $p_i$ . The ideal  $I = I(p_1)^{m_1} \cap \dots \cap I(p_n)^{m_n}$  is called a fat point ideal and defines a subscheme of  $\mathbb{P}^N$ , which is known as a fat point subscheme and formally we denote it by  $Z = m_1 p_1 + \dots + m_n p_n$ . Moreover, the  $m^{\text{th}}$  symbolic power of  $I$  is the ideal defined by  $I^{(m)} = I(p_1)^{mm_1} \cap \dots \cap I(p_n)^{mm_n}$ .

In one part of his PhD thesis [10], Janssen studied the containment problem of a special zero dimensional subscheme  $Z_n = p_0 + p_1 + \dots + p_n$  in  $\mathbb{P}^2$ , which he called an almost collinear subscheme. Along the way, he showed that  $\rho(I(Z_n)) = n^2/(n^2 - n + 1)$  [5, Theorem 2.7] and  $\widehat{\alpha}(I(Z_n)) = 2 - 1/n$  [5, Lemma 3.1]. Now it would be interesting to see the effect of fattening of points of  $Z_n$  in the resurgence and the Waldschmidt constant of  $I(Z_n)$ . Due to this interest, we study these two invariants for the ideal of a fat almost collinear subscheme  $Z_{n,c} = cp_0 + p_1 + \dots + p_n$  (see Definition 2.1).

The main result of this note is the following theorem.

#### THEOREM A

Let  $I$  be the defining ideal of the fat almost collinear subscheme  $Z_{n,c}$ , where  $c \leq n$ , in  $\mathbb{P}^2$ . Then  $\rho(I) = \frac{n^2}{n^2 - nc + c^2}$ .

#### REMARK 1.1

Fat point ideals  $I$  in  $\mathbb{K}[\mathbb{P}^2]$  for which  $\rho(I) = 1$  are of interest. As a consequence of the above theorem we have  $\rho(I(Z_{n,n})) = 1$ .

Also, as another result we prove:

#### THEOREM B

Let  $I$  be the defining ideal of the fat almost collinear subscheme  $Z_{n,c}$ , where  $c \leq n$ , in  $\mathbb{P}^2$ . Then  $\widehat{\alpha}(I) = (1 + c) - \frac{c}{n}$ .

The proof of Theorem A is given in Section 2. Also see Section 3 for the proof of Theorem B.

## 2. Construction and resurgence of $I(Z_{n,c})$

The goal of this section is to prove Theorem A. We start by recalling the definition of an almost collinear subscheme in  $\mathbb{P}^2$ , which was introduced in [5]. We also define a fat almost collinear subscheme.

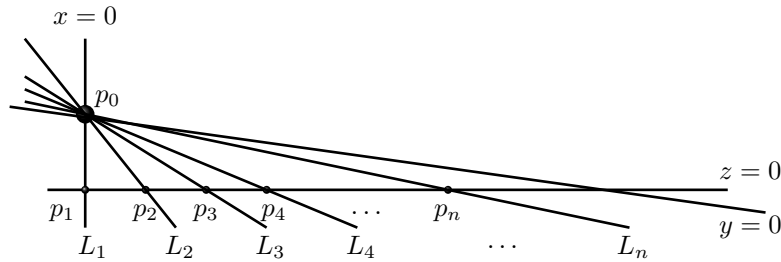
DEFINITION 2.1 ([5, Definition 1.2])

Let  $Z_n = p_0 + p_1 + \cdots + p_n$ , where  $n \geq 2$ , be a zero dimensional subscheme in the projective plane.  $Z_n$  is called an almost collinear subscheme if all these points except  $p_0$  lie on a line  $L$ . Moreover, we call the zero dimensional subscheme  $Z_{n,c} = cp_0 + p_1 + \cdots + p_n$  a fat almost collinear subscheme.

For the remainder of this section, we keep the letter  $R$  to denote the graded ring  $\mathbb{K}[\mathbb{P}^2] = \mathbb{K}[x, y, z]$  and also we assume that  $I = I(Z_{n,c})$  is the defining ideal of the fat almost collinear subscheme  $Z_{n,c}$ .

Let  $Z_{n,c} = cp_0 + p_1 + \cdots + p_n$  be as above. Without loss of generality, we may assume that all collinear points  $p_1, \dots, p_n$  lie on the line  $z = 0$ , and the point  $p_1$  is the intersection point of the lines  $L_1 = x$  and  $z = 0$ . For each  $2 \leq i \leq n$ , let  $p_i$  be the intersection point of the lines  $L_i = x - \ell_i y$  and  $z = 0$ , where  $\ell_i$ s are non-zero distinct elements of  $\mathbb{K}$ . Moreover, we may assume that  $p_0$  is the intersection point of the lines  $x = 0$  and  $y = 0$ . Then  $I = (x, y)^c \cap (z, F)$ , where  $F = L_1 \cdots L_n = x(x - \ell_2 y) \cdots (x - \ell_n y)$  is a homogeneous polynomial in  $x, y$  of degree  $n$ . Since the ideals  $(x, y)$  and  $(z, F)$  are complete intersection ideals, by the unmixedness theorem, we have  $I^{(m)} = (x, y)^{cm} \cap (z, F)^m$ .

The above situation is illustrated in the following figure.



The fat almost collinear subscheme  $Z_{n,c}$

REMARK 2.2

In Theorem A and Theorem B, we assumed that the multiplicity of  $p_0$  is  $c \leq n$ . We need this assumption for computational purposes. In fact, since  $c \leq n$ , we have  $(x, y)^n \subset (x, y)^c$ , and since  $F = x(x - \ell_2 y) \cdots (x - \ell_n y) \in (x, y)^n$ , we have  $F \in (x, y)^c$ . Therefore  $I$  has the following simple description.

$$I = (x, y)^c \cap (z, F) = (zx^c, zx^{c-1}y, \dots, zxy^{c-1}, zy^c, F).$$

Turning to the proof of Theorem A, let  $i$  be a non-negative integer, then by the division algorithm  $i = an + e$ , with  $0 \leq e < n$ . We denote the polynomial  $x^e F^a$  by  $H_i$ . In the sequel, we use [5, Lemma 2.4], stated here as Lemma 2.3, to give a  $\mathbb{K}$ -vector space basis for the ring  $R = \mathbb{K}[x, y, z]$  consisting of elements of the form  $H_i y^j z^l$ .

LEMMA 2.3 ([5, Lemma 2.4])

A  $\mathbb{K}$ -basis of  $R$  is given by  $B_R = \bigcup_{i \geq 0} B_i$ , where

$$B_i = \{H_i y^j z^l : j, l \in \mathbb{N}_0, i = an + e, 0 \leq e < n, H_i = x^e F^a\}.$$

Now, using the same strategy as in [5], we restrict the vector space basis  $B_R$  to obtain  $\mathbb{K}$ -bases for the ideals  $I^{(m)}$  and  $I^r$  of the form  $H_i y^j z^l$  with different conditions on  $i, j$  and  $l$ , which makes it easy to compare  $I^{(m)}$  and  $I^r$  in order to obtain the resurgence of  $I$ .

LEMMA 2.4

Let  $m \geq 1$  be an integer.

- (a) Then  $H_i y^j z^l \in I^{(m)}$  if and only if  $i, j, l \geq 0$ ,  $i + ln \geq mn$  and  $i + j \geq cm$ .
- (b) Moreover,  $I^{(m)}$  is the  $\mathbb{K}$ -vector space span of the elements of the form  $H_i y^j z^l$  contained in  $I^{(m)}$ .

In the following lemma we describe  $I^r$  similarly to  $I^{(m)}$ .

LEMMA 2.5

Let  $r \geq 1$  be an integer.

- (a) The ideal  $I^r$  is the span of the elements of the form  $H_i y^j z^l \in I^r$ . In addition, if  $H_i y^j z^l \in I^r$ , then  $H_i y^j z^l$  is a product of  $r$  elements of  $I$ .
- (b) Moreover,  $H_i y^j z^l \in I^r$  if and only if  $i, j, l \geq 0$ , and either:
  - (1)  $l < j/c$  and  $i + nl \geq rn$ , or
  - (2)  $j/c \leq l < (i + j)/c$  and  $i + j + (n - c)l \geq rn$ , or
  - (3)  $(i + j)/c \leq l$  and  $r \leq (i + j)/c$ .

REMARK 2.6

Since with some changes in the proofs of [5, Lemma 2.5] and [5, Lemma 2.6] one can prove Lemma 2.4 and Lemma 2.5 respectively, we omit the proof of these two lemmas.

Now with the aid of Lemmas 2.4 and 2.5, we are able to prove the main theorem of this note.

*Proof of Theorem A.* Consider  $H_i y^j z^l$ , where  $i = ctn^2$ ,  $j = 0$  and  $l = tn^2 - ctn$ , and let  $m = tn^2$  and  $r = tn^2 - ctn + c^2 t + 1$ . We have  $i + j \geq cm$  and  $i + nl \geq mn$  then  $H_i y^j z^l \in I^{(m)}$  for every  $t \geq 1$  by Lemma 2.4(a), but  $i + j + (n - c)l < rn$ , so  $I^{(m)} \not\subseteq I^r$  by Lemma 2.5(b)(2). Hence  $m/r \leq \rho(I)$  for all  $t$ . Taking the limit as  $t \rightarrow \infty$  gives  $n^2/(n^2 - nc + c^2) \leq \rho(I)$ .

For the upper bound of  $\rho(I)$ , suppose  $m/r \geq n^2/(n^2 - nc + c^2)$  and hence  $m \geq r$ . Consider  $H_i y^j z^l \in I^{(m)}$ . Then  $i + j \geq cm$  and  $i + nl \geq mn$ , by Lemma 2.4(a). Now we consider the following cases.

- (a) If  $l < j/c$ , then  $i + nl \geq mn \geq nr$ , so  $H_i y^j z^l \in I^r$ , by Lemma 2.5(b)(1).
- (b) If  $j/c \leq l < (i+j)/c$ , use  $i+j \geq cm \geq crn^2/(n^2 - nc + c^2)$  and  $i+nl \geq mn \geq rn^3/(n^2 - nc + c^2)$ . Assume to the contrary that  $i+j + (n-c)l < rn$ . Then  $rn^2 > (n-c)i + ci + nj + n(n-c)l = (n-c)(i+nl) + ci + nj \geq rn^3(n-c)/(n^2 - nc + c^2) + ci + nj$ , so  $rn^2(n^2 - nc + c^2) > rn^3(n-c) + (ci + nj)(n^2 - nc + c^2)$  which simplifies to  $rc^2n^2 > (ci + nj)(n^2 - nc + c^2)$ , so  $rc^2n^2/(n^2 - nc + c^2) > ci + nj = c(i+j) + (n-c)j$ . Using  $i+j \geq crn^2/(n^2 - nc + c^2)$ , this gives  $rc^2n^2/(n^2 - nc + c^2) > rc^2n^2/(n^2 - nc + c^2) + (n-c)j$ , which is impossible. Thus  $i+j + (n-c)l \geq rn$  so  $H_i y^j z^l \in I^r$ , by Lemma 2.5(b)(2).
- (c) If  $(i+j)/c \leq l$ , then  $i+j \geq cm \geq cr$ , which gives  $r \leq (i+j)/c$ , so  $H_i y^j z^l \in I^r$ , by Lemma 2.5(b)(3).

Thus  $m/r \geq n^2/(n^2 - nc + c^2)$  implies  $I^{(m)} \subseteq I^r$  by Lemma 2.4(b), and so  $\rho(I) \leq n^2/(n^2 - nc + c^2)$ , that is,  $\rho(I) = n^2/(n^2 - nc + c^2)$ .

### 3. The Waldschmidt constant of $I(Z_{n,c})$

In this section we compute  $\widehat{\alpha}(I(Z_{n,c}))$ . First, let us recall some notation and definitions related to blowing up  $\mathbb{P}^2$  at a finite set of points in  $\mathbb{P}^2$ .

Let  $Z = m_1 p_1 + \cdots + m_t p_t$  be a fat point subscheme of  $\mathbb{P}^2$  and let  $I(Z)$  be its defining ideal. Let  $\pi : X \rightarrow \mathbb{P}^2$  be the morphism obtained by blowing up at the points  $\{p_1, \dots, p_t\}$ . Let  $E_i = \pi^{-1}(p_i)$ , with  $i = 1, \dots, t$ , be the exceptional curve and let  $L$  be the total transform to  $X$  of a general line in  $\mathbb{P}^2$ . Then  $L$  and  $E_i$ s give an orthogonal basis for the divisor class group of  $X$  such that  $-L^2 = E_i^2 = -1$ , and  $E_i \cdot E_j = E_i \cdot L = 0$ , when  $i \neq j$ .

We need the following lemma to prove Theorem B.

LEMMA 3.1

*Keep the above notation and let  $a, b$  be two positive integers. Also, let  $d_i$ , with  $i = 1, \dots, t$ , be non-negative integer. Let  $N = aL - b(d_1 E_1 + \cdots + d_t E_t)$  be an effective divisor and let  $P \neq 0$  be a nef divisor on  $X$  such that  $P \cdot N = 0$ . Then  $\widehat{\alpha}(I(Z)) = a/b$ .*

*Proof.* The proof is similar to the proof of [9, Proposition 1.4.8].

Now we are ready to use Lemma 3.1 to compute integers  $\widehat{\alpha}(I(Z_{n,c}))$ .

*Proof of Theorem B.* Let  $\widetilde{L}_i = L - E_0 - E_i$ , with  $i = 1, \dots, n$ , be the proper transform of the line passing through  $p_0$  and  $p_i$  and let  $\widetilde{L} = L - (E_1 + \cdots + E_n)$  be the proper transform of the line passing through  $p_1, \dots, p_n$ . Consider the divisor  $N$  on  $X$  as

$$(n + cn - c)L - n(cE_0 + E_1 + \cdots + E_n) = c(\widetilde{L}_1 + \cdots + \widetilde{L}_n) + (n - c)\widetilde{L},$$

which is an effective divisor. Also, let  $P = nL - ((n-1)E_0 + E_1 + \cdots + E_n) = \widetilde{L}_1 + \cdots + \widetilde{L}_n + E_0$ . Since  $P$  is a sum of prime divisors  $\widetilde{L}_i$ , with  $1 \leq i \leq n$ , and  $E_0$  each of which  $P$  meets non-negatively, it is nef. It is easy to see that  $P \cdot N = 0$ . Thus, by Lemma 3.1, we have  $\widehat{\alpha}(I(Z_{n,c})) = (n + cn - c)/n = (1 + c) - c/n$ .

#### 4. A question

In this section we pose a question arising from this paper.

Let  $W = cp_0 + p_1 + \cdots + p_n$  be a zero dimensional subscheme in the projective plane, where all points  $p_i$  except  $p_0$  lie on a line. When  $1 \leq c \leq n$ , we computed  $\rho(I(W))$  in Theorem A. Now, if  $c > n$ , what can be said about  $\rho(I(W))$ ? In what follows we discuss this issue in more general setting.

Let  $\mathcal{Z} = m_1p_1 + m_2p_2 + \cdots + m_s p_s$  be a fat point subscheme in  $\mathbb{P}^N$ , where  $m_1 \geq m_2 + \cdots + m_s$ , and let  $I = I(\mathcal{Z})$ . A necessary condition to have  $I^{(m)} = I^m$  is that  $\alpha(I^{(m)}) = \alpha(I^m)$  for all  $m \geq 1$ . The following theorem shows that  $I$  has this necessary condition.

**THEOREM 4.1**

*Let the ideal  $I$  be as above. Then  $\alpha(I^{(m)}) = \alpha(I^m)$  for all  $m \geq 1$ .*

*Proof.* It is obvious that  $\alpha(I) = m_1$ . Since  $I^m \subseteq I^{(m)}$ , we have  $\alpha(I^{(m)}) \leq \alpha(I^m)$  for all  $m \geq 1$ . Let there exist an integer  $k$  such that  $\alpha(I^{(k)}) < \alpha(I^k)$ . Therefore

$$\hat{\alpha}(I) \leq \frac{\alpha(I^{(k)})}{k} < \frac{\alpha(I^k)}{k} = \frac{k\alpha(I)}{k} = \alpha(I) = m_1.$$

Since  $I \subseteq I(p_1)^{m_1}$ , we have  $\hat{\alpha}(I) \geq \hat{\alpha}(I(p_1)^{m_1}) = m_1$ , which contradicts  $\hat{\alpha}(I) < m_1$ .

Computer calculations using *Singular* [4] which we have carried out in  $I$  suggest that  $I^{(m)} = I^m$  for all  $m \geq 1$ .

With Theorem 4.1 and computer calculations, we pose the following question:

**QUESTION 4.2**

Let  $I$  be the defining ideal of a fat point subscheme  $\mathcal{Z} = m_1p_1 + m_2p_2 + \cdots + m_s p_s$  in  $\mathbb{P}^N$ , where  $m_1 \geq m_2 + \cdots + m_s$ . Is then  $I^{(m)} = I^m$  for all  $m \geq 1$ ? In particular, do we have  $\rho(I) = 1$ ?

As a special case of Question 4.2, we can ask:

**QUESTION 4.3**

For the fat point subscheme  $W = cp_0 + p_1 + \cdots + p_n$ , where  $c > n$ , is it true that  $I(W)^{(m)} = I(W)^m$  for all  $m \geq 1$ ? In particular, do we have  $\rho(I(W)) = 1$ ?

We conclude this note with the following remark.

**REMARK 4.4**

Consider the zero dimensional subscheme  $W = cp_0 + p_1 + \cdots + p_n$  in the projective plane. Let  $I = I(W) = (x, y)^c \cap (z, F)$  be the defining ideal of  $W$ . If we assume  $c = 0$ , then  $I = (z, F)$ , a complete intersection ideal, which implies  $\rho(I) = 1$ . Whenever,  $1 \leq c \leq n$ ,  $W$  is the fat almost collinear subscheme  $Z_{n,c}$ . Thus, by Theorem A, we have  $\rho(I) = n^2/(n^2 - nc + c^2)$ . In particular, for  $c = n$  we get  $\rho(I) = 1$ . In the case of  $c > n$ , if the Question 4.3 has a positive answer, we get  $I^{(m)} = I^m$  for all  $m \geq 1$ . In particular, we again obtain  $\rho(I) = 1$ . To sum up, we may expect:

$$\rho(I) = \begin{cases} 1, & c = 0, \\ \frac{n^2}{(n^2 - nc + c^2)}, & 1 \leq c \leq n - 1, \\ 1, & c \geq n. \end{cases}$$

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*Faculty of Mathematics*  
*K. N. Toosi University of Technology*  
*Tehran*  
*Iran*  
*E-mail:* Haghghi@kntu.ac.ir  
 mosakhani@aut.ac.ir  
 zamanfashami65@yahoo.com

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