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Oriented angles in affine space

To Andrzej Zajtz, on the occasion of His 70th birthday

Abstract. The concept of a smooth oriented angle in an arbitrary affine space is introduced. This concept is based on a kinematics concept of a run. Also, a concept of an oriented angle in such a space is considered. Next, it is shown that the adequacy of these concepts holds if and only if the affine space, in question, is of dimension 2 or 1.

0. Preliminaries

Let us consider an arbitrary affine space, i.e. a triple

$$(E, V, \vec{\ }), \tag{0}$$

(see [B–B]), where E is a set, V is an arbitrary vector space over reals and $\vec{\ }$ is a function which to any points $p, q \in E$ assigns a vector $\overrightarrow{pq}$ of V in such a way that

1) $\overrightarrow{pq} + \overrightarrow{qr} = \overrightarrow{pr}$ for $p, q, r \in E$,

2) $\overrightarrow{pq} = 0$ iff $p = q$ for $p, q \in E$,

3) for any $p \in E$ and any vector x of V there exists $q \in E$ with $\overrightarrow{pq} = x$.

The unique point q for which $\overrightarrow{pq} = x$ will be denoted by $p + x$. The set of all vectors of the vector space V will be denoted by $\underline{V}$. The fact that W is a vector subspace of V will be written as $W \leq V$. For any sets M, N, X, Y, P such that $M \cup N \subset \mathbb{R}$, $X \cup Y \subset \underline{V}$, $P \subset E$, any $b \in \mathbb{R}$, $y \in \underline{V}$ and $p \in E$ we set

$$\begin{aligned} M + N &= \{a + b;\ a \in M \ \&\ b \in N\}, & M + b &= M + \{b\}, \\ MN &= \{ab;\ a \in M \ \&\ b \in N\}, & bM &= \{b\}\, M, \\ MX &= \{ax;\ a \in M \ \&\ x \in X\}, & bX &= \{b\}\, X, \\ X + Y &= \{x + y;\ x \in X \ \&\ y \in Y\}, & & \\ P + X &= \{p + x;\ p \in P \ \&\ x \in X\}, & p + X &= \{p\} + X. \end{aligned}$$

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A subset H of E is a hyperplane in an affine space (0) iff there exist $p \in E$ and $W \leq V$ such that

$$H = p + \underline{W}. \tag{1}$$

The subspace W of V for which (1) holds will be denoted by V_H. The affine space

$$(H, V_H, \xrightarrow{}^{H}), \tag{2}$$

where $\xrightarrow{}^{H}$ is the restriction of the function $\xrightarrow{}$ to the set $H \times H$, is called the *subspace of* (0) *determined by the hyperplane* H. The triple (2), where $H = \emptyset$, $V_H \leq V$, $\underline{V_H} = \{0\}$ and $\xrightarrow{}^{H} = \emptyset$ is an affine space and will be treated as a subspace of (0) as well. Also, the set $\emptyset$ will be considered as a *hyperplane* in (0). We will write $W \leq_k V$ instead of to state that a vector subspace W of V is of codimension k in V. In particular, $W \leq_1 V$ means that W is of codimension 1 in V. We say that H is a *hyperplane of codimension* k in the affine space (0) iff $V_H \leq_k V$.

Any set P of points of the affine space (0), i.e. $P \subset E$, such that

$$P = H + \mathbb{R}_+ e, \tag{3}$$

where H is a hyperplane of codimension 1 in (0), $e \in \underline{V} \setminus \underline{V_H}$, $\mathbb{R}_+ = \langle 0; +\infty)$, is said to be a *halfspace* of (0). The hyperplane H in (3) uniquely determined by P is called the *shore* of the halfspace P and denoted by P^o. The set $P \setminus P^o$ will be called the *interior* of the halfspace P and denoted by P_+. It is easy to check that the set P^- of the form $E \setminus P_+$ is also a halfspace and the equalities

$$\left(P^-\right)^o = P^o \qquad \text{and} \qquad \left(P^-\right)_+ = E \setminus P \tag{4}$$

hold. The set $E \setminus P$ will be denoted by P_-. The halfspace P^- is called the *opposite* one to P. It is easy to verify that (3) yields also

$$P_+ = P^o + (0; +\infty)\, e, \quad P^- = P^o + \mathbb{R}_+ (-e)\,, \quad P_- = P^o + (-\infty; 0)\, e \tag{5}$$

where $e \in \underline{V} \setminus \underline{V_H}$ and $H = P^o$.

Let B be a base of a vector space V. For any $\mathrm{v} \in \underline{V}$ there exists a unique real function v_B defined on B such that $\{e;\ e \in B \;\&\; \mathrm{v}_B(e) \neq 0\}$ is finite and

$$\mathrm{v} = \sum_{e \in B} \mathrm{v}_B(e)\, e, \tag{6}$$

where the sign of addition in (6) denotes of course a finite operation. This formula will be very useful.

For any topology $\mathcal{T}$ (see [K]) the set of all points of $\mathcal{T}$ will be denoted by $\underline{\mathcal{T}}$, i.e. by definition we have

$$\underline{\mathcal{T}} = \bigcup \mathcal{T}. \tag{7}$$

For any set $A \subset \underline{\mathcal{T}}$ the induced to A topology from the topology $\mathcal{T}$ will be denoted by $\mathcal{T}|A$, i.e. $\mathcal{T}|A = \{B \cap A;\ B \in \mathcal{T}\}$.

For any affine space (0) the smallest topology containing the set of all sets P_+, where P is a halfspace of (0) will be called the *topology of the affine space* (0) and denoted by $\mathrm{top}(E, V, \rightarrow)$. It is easy to check that for any hyperplane H in (0) we have

$$\mathrm{top}(H, V_H, \rightarrow^H) = \mathrm{top}(E, V, \rightarrow)|H. \tag{8}$$

Let f be any function. The domain of f will be denoted by D_f. For any $A \subset D_f$ the restriction of the function f to the set A and the f-image of A will be denoted by $f|A$ and fA, respectively. Any function may be treated as a set of ordered pairs, and then

$$D_f = \{x;\ \exists y\ \ ((x,y) \in f)\}, \qquad f|A = \{(x,y);\ (x,y) \in f\ \&\ x \in A\}$$

and

$$fA = \{y;\ \exists x \in A\ \ ((x,y) \in f)\}.$$

For any set B the f-preimage $f^{-1}B$ is defined by

$$f^{-1}B = \{x;\ \exists y \in B\ \ ((x,y) \in f)\}$$

or, equivalently, $f^{-1}B = \{x;\ x \in D_f\ \&\ f(x) \in B\}$.

Let f be a function with $D_f \subset \mathbb{R}$, $fD_f \subset E$, $t \in \mathbb{R}$ and $p \in E$. We say that f tends to p at t in the affine space (0) and we write

$$f(x) \xrightarrow[x \longrightarrow t]{} p \qquad (\text{in } (E, V, \rightarrow)) \tag{9}$$

iff for any $U \in \mathrm{top}(E, V, \rightarrow)$ such that $p \in U$ there exists $\delta > 0$ for which $f(x) \in U$ whenever $0 < |x - t| < \delta$. It is easy to prove the following

Proposition 1

For any function f with $D_f \subset \mathbb{R}$, $fD_f \subset E$, any $t \in \mathbb{R}$ and $p \in E$ we have (9) *if and only if for any base B of vector space V and any $e \in B$ we have*

$$\overrightarrow{pf(x)}_B(e) \xrightarrow[x \longrightarrow t]{} 0. \tag{10}$$

For any vector space V we have well defined the affine space $\mathrm{aff}\, V$ as $(\underline{V}, V, \rightarrow)$, where $\overrightarrow{\mathrm{vw}} = \mathrm{w} - \mathrm{v}$ for v,w$\in \underline{V}$. Let $D_f \subset \mathbb{R}$ and $fD_f \subset E$. Setting

$$f' = \left\{(t, \mathrm{v});\ t \in D_f \cap (D_f)'\ \&\ \tfrac{1}{x-t}\,\overrightarrow{f(t)f(x)} \xrightarrow[x \longrightarrow t]{} \mathrm{v}\ (\text{in } \mathrm{aff}\, V)\right\}, \tag{11}$$

where for any set $A \subset \mathbb{R}$, A' denotes the set of all cluster points of A, we have defined the derivative function f' of a function f. A function $f\colon D_f \to E$, $D_f \subset \mathbb{R}$, is differentiable iff

$$D_{f'} = D_f. \tag{12}$$

Denoting the natural topology of $\mathbb{R}$ by $\mathcal{R}$ we have the topology $\mathcal{R}|D_f$. The function f satisfying (12) and having the continuous derivative function f' from $\mathcal{R}|D_f$ to $\operatorname{top}\operatorname{aff} V$ is said to be *smooth* in $(E, V, \rightarrow)$.

1. Runs, o-turns, and smooth oriented angles

Before introducing the concept of smooth oriented angle in an arbitrary affine space we introduce a concept of a run and a turn. Any function f smooth in $(E, V, \rightarrow)$ with $D_f = \langle a; b\rangle$, $a < b$, is said to be a *run* in $(E, V, \rightarrow)$. Let $o \in E$. Any run f satisfying one of the following conditions:

$$f(t) = f(u) \neq o \qquad \text{for } t, u \in D_f, \tag{$o1f$}$$

or

$$f'(t),\ \overrightarrow{of(t)} \ \text{are linearly independent for } t \in D_f, \tag{$o2f$}$$

is said to be an *o-turn* in $(E, V, \rightarrow)$. The set of all o-turns in $(E, V, \rightarrow)$ will be denoted by $T_o(E, V, \rightarrow)$. In this set we introduce an equivalence $\equiv_o$ setting $f \equiv_o g$ iff $f, g \in T_o(E, V, \rightarrow)$ and there exist real smooth functions λ and φ such that

(i) $D_\varphi = D_\lambda = D_f$ and $\varphi D_\varphi = D_g$,

(ii) $\lambda(t) > 0$, $\varphi'(t) > 0$ and $\overrightarrow{og(\varphi(t))} = \lambda(t)\,\overrightarrow{of(t)}$ for $t \in D_f$.

Denoting by $T_o(E, V, \rightarrow)/\equiv_o$ the set of all cosets in $T_o(E, V, \rightarrow)$ given by the equivalence $\equiv_o$ we may define the set $\operatorname{soa}(E, V, \rightarrow)$ by the equality

$$\operatorname{soa}(E, V, \rightarrow) = \bigcup_{o \in E} T_o(E, V, \rightarrow)/\equiv_o .$$

Any element of this set is said to be a *smooth oriented angle* in the affine space $(E, V, \rightarrow)$.

Proposition 2
For any $o \in E$, $\mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$ and $g \in \mathfrak{a}$ we have

$$\underline{\mathfrak{a}} = \bigcup_{p \in gD_g} (o\,p\,\infty),$$

where

$$\underline{\mathfrak{a}} = \bigcup_{f \in \mathfrak{a}} fD_f \qquad \textit{and} \qquad (o\,p\,\infty) = \{o + t\,\overrightarrow{op};\ t > 0\}.$$

Proof. Let $f \in \mathfrak{a}$. We have then $f \equiv_o g$. Taking any $q \in fD_f$ we get $q = f(t)$, $t \in D_f$. Then there exist functions λ, φ such that (i) and (ii) hold. Setting $p = g(\varphi(t))$ we get $\overrightarrow{oq} = \frac{1}{\lambda(t)}\overrightarrow{op}$, which yields $q \in (o\,p\,\infty)$, where $p \in gD_g$. Now, let $q \in (o\,p\,\infty)$, where $p \in gD_g$. We have then $\overrightarrow{oq} = s\,\overrightarrow{op}$, where $p = g\,(u)$, $u \in D_g$ and $s > 0$. Setting $D_f = D_g$ and $f(t) = o + s\,\overrightarrow{og(t)}$ for $t \in D_f$ we get $f \equiv_o g$ and $q = o + s\,\overrightarrow{op} = o + s\,\overrightarrow{og(u)} = f(u) \in fD_f$, so $(o\,p\,\infty) \subset \underline{\mathfrak{a}}$.

Proposition 3
For any $o \in E$ and $\mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$ if $o \in U \in \mathrm{top}(E, V, \rightarrow)$, then there exists $g \in \mathfrak{a}$ such that $gD_g \subset U$.

Proof. Let $f \in \mathfrak{a}$ and $s > 0$. Setting $D_{f_s} = D_f$ and

$$f_s(t) = o + s\,\overrightarrow{of}\,(t) \qquad \text{for } t \in D_f$$

we have, of course, $f_s \equiv_o f$, so $f_s \in \mathfrak{a}$. We will prove that

for any halfspace P with $o \in P_+$ there exists $\varepsilon > 0$ such that for any $s \in (0; \varepsilon)$ the relation $f_s D_f \subset P_+$ holds. $(\star)$

Let P be a halfspace such that $o \in P_+$. Then we have $P = o + \underline{W} + \langle -\beta; +\infty)\,e$, where $W \leq_1 V$, $e \in \underline{V} \backslash \underline{W}$ and $\beta > 0$. Then $P_+ = o + \underline{W} + (-\beta; +\infty)e$. For any $t \in D_f$ we have $\overrightarrow{of(t)} = \mathrm{w}(t) + \mu\,(t)\,e$. From continuity of f by Proposition 1 it follows that μ is continuous. Thus, μ is bounded. So, there exists $m > 0$ such that $|\mu(t)| < m$ for $t \in D_f$. Hence it follows that $\overrightarrow{of_s(t)} = s\,\mathrm{w}(t) + s\,\mu(t)e \in \underline{W} + (-sm; +\infty)e$, so $f_s(t) \in o + \underline{W} + (-sm; +\infty)e \subset P_+$ for $t \in D_f$, as $0 < s < \frac{\beta}{m}$.

Now, assume that $o \in U \in \mathrm{top}(E, V, \rightarrow)$. Then there exist halfspaces $P_1, \ldots, P_n$ such that $o \in P_{1+} \cap \ldots \cap P_{n+} \subset U$. By $(\star)$ for any $j \in \{1, \ldots, n\}$ we get $\varepsilon_j > 0$ such that $f_s D_f \subset P_{j+}$ as $s \in (0; \varepsilon_j)$. Setting $g = f_s$, where $0 < s < \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, we get $gD_g \subset U$.

Proposition 4
If $o, q \in E$ and $\mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o \cap\, T_q(E, V, \rightarrow)/\equiv_q$, then $o = q$.

Proof. Let us suppose that $o \neq q$. Take any $U \in \mathrm{top}(E, V, \rightarrow)$ such that $q \in U$. Since $\mathfrak{a} \in T_q(E, V, \rightarrow)/\equiv_q$, by Proposition 3 there exists $g \in \mathfrak{a}$ such that $gD_g \subset U$. From the condition $\mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$ it follows that $\mathfrak{a} \subset T_o(E, V, \rightarrow)$. Therefore $g \in T_o(E, V, \rightarrow)$, so $gD_g \subset U \setminus \{o\}$, and by Proposition 2 we get

$$\underline{\mathfrak{a}} \subset A \qquad \text{where } A = \bigcap_{q \in U \in \mathrm{top}(E, V, \rightarrow)} \; \bigcup_{p \in U \setminus \{o\}} (o\,p\,\infty).$$

Now, we will prove that $A \subset (o\,q\,\infty)$. Assume that there exists a point $x \in A \setminus (o\,q\,\infty)$. Let us set $C = \{\overrightarrow{oq}, \overrightarrow{ox}\}$, whenever $\overrightarrow{ox}$ and $\overrightarrow{oq}$ are linearly independent and $C = \{\overrightarrow{oq}\}$ in the opposite case. Then there exists a base B of V with $C \subset B$. Let W be the vector subspace of V generated by $B \setminus \{e\}$, where $e = \overrightarrow{oq}$. Let us set

$$P = o + \underline{W} + \mathbb{R}_+ e.$$

So, we have $P^o = o + \underline{W}$ and $P_+ = o + \underline{W} + (0;+\infty)e$. First, we suppose that $\overrightarrow{ox}$ and $\overrightarrow{oq}$ are linearly independent. Then $x = o + \overrightarrow{ox} \in o + \underline{W} = P^o$. If we assume that $x \in \bigcup_{p \in P_+}(o\,p\,\infty)$, then we get $p \in P_+$ with $x \in (o\,p\,\infty)$. Then it should be in turn, $p = o + w + te$, $w \in \underline{W}$, $t > 0$, $x = o + u\,\overrightarrow{op}$, $u > 0$, $x = o + uw + ute \in P_+$, which is impossible. Therefore we have $x \notin \bigcup_{p \in P_+}(o\,p\,\infty) \supset A$. So, $\overrightarrow{ox}$ and $\overrightarrow{oq}$ should be linearly dependent. Thus, $\overrightarrow{ox} = a \cdot \overrightarrow{oq}$, $a \in \mathbb{R}$. Because of $x \notin (o\,q\,\infty)$ we get $a \leq 0$. Thus $x \in P_-$. By definition of P_- we have

$$P_- \cap \bigcup_{p \in P_+} (o\,p\,\infty) = \emptyset,$$

what yields $x \notin A$. So, we have $A \subset (o\,q\,\infty)$. Hence it follows that $\underline{\mathfrak{a}} \subset (o\,q\,\infty)$ and similarly $\underline{\mathfrak{a}} \subset (q\,o\,\infty)$. By Proposition 2 we get $(o\,p\,\infty) \subset \underline{\mathfrak{a}}$ for some $p \in gD_g$. This yields $(o\,p\,\infty) \subset (o\,q\,\infty) \cap (q\,o\,\infty)$, which is impossible.

The point $o \in E$ such that $\mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$ is called the *vertex* of $\mathfrak{a}$.

Notice that if $f, g \in \mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$, $D_f = \langle a; b\rangle$, and $D_g = \langle c; d\rangle$, then $\langle o\,f(a)\,\infty) = \langle o\,g(c)\,\infty)$ and $\langle o\,f(b)\,\infty) = \langle o\,g(d)\,\infty)$, where

$$\langle o\,p\,\infty) = \{o + s\,\overrightarrow{op};\ s \geq 0\} \qquad \text{for } p \in E. \tag{13}$$

The sets $\langle o\,f(a)\,\infty)$ and $\langle o\,f(b)\,\infty)$ we called the *former side* and the *latter one* of $\mathfrak{a}$, respectively.

2. Oriented angles

Consider any affine space (0) and any $o \in E$. The set of all functions L such that D_L is a closed segment in $\mathbb{R}$ and there exists a function f with $D_f = D_L$, continuous from $\mathcal{R}|D_f$ to $\operatorname{top}(E, V, \rightarrow)$ such that for any $t \in D_f$ we have

$$o \neq f(t) \qquad \text{and} \qquad L(t) = \langle o\,f(t)\,\infty)\,, \tag{L}$$

$\langle o\,f\,(t)\,\infty)$ is defined by (13), and one of the following two conditions

(1 L) $L(t) = L(u)$ for $t, u \in D_L$,

(2 L) for any $t \in D_L$ there exists $\delta > 0$ for which

$$L|D_L \cap (t-\delta; t+\delta) \text{ is 1–1},$$

is satisfied will be denoted by $\langle o; E, V, \stackrel{}{\to}\rangle$. We set

$$\langle E, V, \stackrel{}{\to}\rangle = \bigcup_{o \in E} \langle o; E, V, \stackrel{}{\to}\rangle$$

and $L \equiv M$ iff $L, M \in \langle E, V, \stackrel{}{\to}\rangle$ and there exists a real continuous increasing function φ such that $D_\varphi = D_L$, $\varphi D_\varphi = D_M$ and $M \circ \varphi = L$. It is easy to see that $\equiv$ is an eqiuvalence.

Elements of the set $\langle E, V, \stackrel{}{\to}\rangle / \equiv$ of all cosets of $\equiv$ will be called *oriented angles* in the affine space (0). The point o such that the equality in (L) is satisfied depending only on the oriented angle for which L belongs is called the *vertex* of this oriented angle. Any oriented angle for which constant function L belongs is said to be zero angle in the affine space (0).

Proposition 5

For any smooth oriented angle $\mathfrak{a}$ *in the affine space* (0) *we have the oriented angle* $<\mathfrak{a}>$ *well defined by the formula*

$$<\mathfrak{a}> = [f_o] \tag{14}$$

where $f_o(t) = \langle o\, f(t)\, \infty\rangle$ *for* $t \in D_f$, $f \in \mathfrak{a} \in T_o(E, V, \stackrel{}{\to})/ \equiv_o$, $L \in [L] \in \langle E, V, \stackrel{}{\to}\rangle / \equiv$ *for* $L \in \langle E, V, \stackrel{}{\to}\rangle$. *The function*

$$\mathrm{soa}(E, V, \stackrel{}{\to}) \ni \mathfrak{a} \longmapsto <\mathfrak{a}> \tag{15}$$

is 1–1. *If* $\dim V > 2$, *then there exists an oriented angle in* (0) *which is not of the form* $<\mathfrak{a}>$, *where* $\mathfrak{a}$ *is a smooth oriented angle in* (0).

Lemma

If l_1, l_2 *are real functions,* f_1, f_2 *are vector ones with* $D_{l_1} = D_{l_2} = D_{f_1} = D_{f_2} \subset \mathbb{R}$, $f_j(x) \xrightarrow[x \to t]{} e_j$ *(in* $\mathrm{aff}(V)$*),* $j \in \{1,2\}$, e_1, e_2 *are linearly independent in* V *and*

$$l_1(x) f_1(x) + l_2(x) f_2(x) \xrightarrow[x \to t]{} \mathrm{v} \quad (\text{in } \mathrm{aff}\, V),$$

then there exist reals c_1, c_2 *such that* $l_j(x) \xrightarrow[x \to t]{} c_j$, $j \in \{1,2\}$.

Proof. There exists a base B in V containing $\{e_1, e_2\}$. By Proposition 1 we have $g_i(x) \xrightarrow[x \to t]{} \mathrm{v}_B(e_i)$ where

$$g_i(x) = l_1(x) f_1(x)_B(e_i) + l_2(x) f_2(x)_B(e_i) \tag{16}$$

and

$$f_j(x)_B(e_i) \xrightarrow[x \longrightarrow t]{} e_{jB}(e_i) = \delta_{ji} \qquad (\delta_{ji} \text{ — Kronecker's delta}),$$

so $\det[f_j(x)_B(e_i);\ i,j \leq 2] \xrightarrow[x\longrightarrow t]{} 1$. Therefore, by (16),

$$l_1(x) = \begin{vmatrix} g_1(x) & f_2(x)_B(e_1) \\ g_2(x) & f_2(x)_B(e_2) \end{vmatrix} m(x) \xrightarrow[x\longrightarrow t]{} \begin{vmatrix} \mathrm{v}_B(e_1) & \delta_{21} \\ \mathrm{v}_B(e_2) & \delta_{22} \end{vmatrix} = c_1$$

and

$$l_2(x) = \begin{vmatrix} f_1(x)_B(e_1) & g_1(x) \\ f_1(x)_B(e_2) & g_2(x) \end{vmatrix} m(x) \xrightarrow[x\longrightarrow t]{} \begin{vmatrix} \delta_{11} & \mathrm{v}_B(e_1) \\ \delta_{12} & \mathrm{v}_B(e_2) \end{vmatrix} = c_2,$$

where $m(x) = 1/\det[f_j(x)_B(e_i);\ i,j \leq 2]$ and $c_i = \mathrm{v}_B(e_i)$.

Proof of Proposition 5. Correctness of the definition of $<\mathfrak{a}>$ by (14) is evident. To prove that (15) is 1–1 assume that $<\mathfrak{a}> = <\mathfrak{b}>$, where $\mathfrak{a} \in T_o(E,V,\overset{\rightarrow}{})/\equiv_o$ and $\mathfrak{b} \in T_q(E,V,\overset{\rightarrow}{})/\equiv_q$. We have (14) and

$$<\mathfrak{b}> = [g_q], \qquad \text{where } g_q(u) = \langle q\, g(u)\, \infty\rangle \ \text{ for } u \in D_g,\ g \in \mathfrak{b}. \tag{14'}$$

By definition of $\equiv$ we get a continuous increasing function φ such that $D_\varphi = D_f$, $\varphi D_\varphi = D_g$ and $g_q \circ \varphi = f_o$, i.e. by (14) and (14′), $\langle q\, g(\varphi(t))\, \infty\rangle = \langle o\, f(t)\, \infty\rangle$ for $t \in D_f$. Hence $q = o$ and for any $t \in D_f$ there is

$$\lambda(t) > 0 \qquad \text{with } \overrightarrow{og(\varphi(t))} = \lambda(t)\,\overrightarrow{of(t)}\,. \tag{17}$$

This yields, in turn,

$$\lambda(t+s)\,\overrightarrow{of(t+s)} = \overrightarrow{og(\varphi(t+s))} \xrightarrow[s\longrightarrow 0]{} \overrightarrow{og(\varphi(t))} = \lambda(t)\,\overrightarrow{of(t)}$$

and

$$\overrightarrow{of(t+s)} \xrightarrow[s\longrightarrow 0]{} \overrightarrow{of(t)} \neq 0.$$

According to Lemma we get $\lambda(t+s) \xrightarrow[s\longrightarrow 0]{} \lambda(t)$. So, λ is continuous. We have also

$$\tfrac{1}{s}(\varphi(t+s)-\varphi(t)) \cdot \tfrac{1}{\varphi(t+s)-\varphi(t)}\,\overrightarrow{g(\varphi(t))g(\varphi(t+s))} - \tfrac{1}{s}(\lambda(t+s)-\lambda(t))\,\overrightarrow{of(t)}$$
$$= \lambda\,(t+s) \cdot \tfrac{1}{s}\,\overrightarrow{f(t)\ f(t+s)}\,,$$

$$\tfrac{1}{\varphi(t+s)-\varphi(t)}\,\overrightarrow{g(\varphi(t))g(\varphi(t+s))} \xrightarrow[s\longrightarrow 0]{} g'(\varphi(t))$$

and

$$\tfrac{1}{s}\,\overrightarrow{f(t)f(t+s)} \xrightarrow[s\longrightarrow 0]{} f'(t).$$

First, we consider the case when o–turns f and g satisfy conditions $(o2f)$ and $(o2g)$, respectively. Then by Lemma we have

$$\frac{\varphi(t+s)-\varphi(t)}{s} \xrightarrow[s\longrightarrow 0]{} \varphi'(t) \qquad \text{and} \qquad \frac{\lambda(t+s)-\lambda(t)}{s} \xrightarrow[s\longrightarrow 0]{} \lambda'(t).$$

Thus,

$$\varphi'(t)g'(\varphi(t)) - \lambda'(t)\overrightarrow{of(t)} = \lambda(t)f'(t) \qquad \text{for } t \in D_f. \tag{18}$$

From the fact that φ is increasing it follows that $\varphi'(t) \geq 0$. By $(o2f)$ we have $\varphi'(t) > 0$. According to Lemma by (18) and $(o2f)$ we conclude that the functions φ' and λ' are continuous. In other words, φ and λ are smooth. So, $f \equiv_o g$ and we have $\mathfrak{a} = \mathfrak{b}$.

Now, let us assume $(o1f)$. Setting $\overrightarrow{of(t)} = e$, by (17), we get $\overrightarrow{og(u)} = \mu(u)e$, where $\mu(u) = \lambda(\varphi^{-1}(u))$ for $u \in D_g$. Thus

$$\tfrac{1}{s}\left(\mu(u+s) - \mu(u)\right)\cdot e = \tfrac{1}{s}\overrightarrow{g(u)g(u+s)} \xrightarrow[s\longrightarrow 0]{} g'(u).$$

By Lemma we get $g'(u) = \mu'(u)e$. Hence it follows that $g'(u)$, $\overrightarrow{og(u)}$ are not linearly independent. Therefore $(o1g)$ holds. Thus, taking any $u, u_1 \in D_g$ by (17) we get $\mu(u_1)e = \overrightarrow{og(u_1)} = \overrightarrow{og(u)} = \mu(u)e$, and $\mu(u) = \mu(u_1)$, which yields $g \equiv_o f$, i.e. $\mathfrak{a} = \mathfrak{b}$. Therefore (15) is 1–1.

Assuming that $\dim V > 2$ we get three vectors e_1, e_2, e_3 linearly independent in V. Let us set

$$\overrightarrow{og(u)} = \begin{cases} e_1 + u(e_2 - e_1), & \text{when } 0 \leq u \leq 1, \\ e_2 + (u-1)(e_3 - e_2), & \text{when } 1 < u \leq 2, \end{cases}$$

and $L(u) = \langle o\, g(u)\, \infty\rangle$ for $u \in \langle 0; 2\rangle$. Let us suppose that there exists $f \in T_o(E, V, \overrightarrow{\ })$ such that $[L] = [f_o]$, where $f_o(t) = \langle o\, f\,(t)\, \infty\rangle$ for $t \in D_f$. Then there exist a continuous and increasing function φ for which $D_\varphi = D_f$, $L \circ \varphi = f_o$, $\varphi D_\varphi = D_L = \langle 0; 2\rangle$. Thus, for some function λ with $D_\lambda = D_\varphi$ (17) holds. Let us set $t_1 = \varphi^{-1}(1)$. Hence it follows that $\overrightarrow{of(t)} = \alpha_1(t)e_1 + \alpha_2(t)e_2$ as $t \in D_f$, $t \leq t_1$ and $\overrightarrow{of(t)} = \beta_2(t)e_2 + \beta_3(t)e_3$ as $t \in D_f$, $t \geq t_1$, where α_1, α_2, β_2, β_3 are real functions. Thus, by Lemma we get

$$f'(t_1) = \alpha_1'(t_1)e_1 + \alpha_2'(t_1)e_2 = \beta_2'(t_1)e_2 + \beta_3'(t_1)e_3.$$

Then $\alpha_1'(t_1) = 0 = \beta_3'(t_1)$. So, $f'(t_1) = \alpha_2'(t_1)e_2$. On the other hand,

$$\overrightarrow{of(t_1)} = \tfrac{1}{\lambda(t_1)}\overrightarrow{og(\varphi(t_1))} = \tfrac{1}{\lambda(t_1)}\overrightarrow{og(1)} = \tfrac{1}{\lambda(t_1)}e_2.$$

The vectors $f'(t_1)$ and $\overrightarrow{of(t_1)}$ are linearly dependent. So, $(o2f)$ does not hold. Therefore $(o1f)$ is satisfied, which yields $\overrightarrow{og(\varphi(t))} = \lambda(t)\overrightarrow{of(t_1)}$ for $t \in D_\varphi$, i.e. $\overrightarrow{og(u)} = \lambda(\varphi^{-1}(u))\overrightarrow{of(t_1)}$ for $u \in \langle 0; 2\rangle$, which is impossible.

3. Oriented angles in an Euclidean plane

Let us consider an Euclidean plane, i.e. an affine space (0), $\dim V = 2$, together with a positively defined scalar product $\underline{V} \times \underline{V} \ni (\mathrm{v},\mathrm{w}) \mapsto \mathrm{v}{\cdot}\mathrm{w} \in \mathbb{R}$. For any $\mathrm{v} \in \underline{V}$ we set $|\mathrm{v}| = \sqrt{\mathrm{v} \cdot \mathrm{v}}$ and for any function f defined on the segment of $\mathbb{R}$ with values in E we set $D_f = \langle a; b\rangle$ and for $t \in D_f$

$$|f|\,(t) = \sup\left\{\sum_{i=0}^{k}\left|\overrightarrow{f(t_i)f(t_{i+1})}\right|;\ a = t_0 < \ldots < t_k = t \ \&\ k \in \mathbb{N}\right\}. \tag{19}$$

The function $|f|$ defined by (19) has values in $\mathbb{R} \cup \{+\infty\}$, in general.

Proposition 6
In the Euclidean plane for any oriented angle $\mathcal{A} \in \langle E, V, \rightarrow)/\equiv$ there exists a unique continuous function $f\colon D_f \to E$ such that $D_f = \langle 0; c\rangle$, $c > 0$, $\langle o\, f\,(\cdot)\ \infty) \in \mathcal{A}$,

$$\left|\overrightarrow{of(s)}\right| = 1 \qquad \textit{for } s \in D_f, \tag{20}$$

o is a vertex of $\mathcal{A}$, and one of the following conditions

$$|f|\,(s) = 0 \qquad \textit{for } s \in D_f, \tag{0; f}$$

$$|f|\,(s) = s \qquad \textit{for } s \in D_f \tag{1; f}$$

is satisfied. We have $f \in \mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$ and $\mathcal{A} = <\mathfrak{a}>$, where $<\mathfrak{a}>$ is the oriented angle defined by (14).

Proof. Let $L \in \mathcal{A} \in \langle E, V, \rightarrow)/\equiv$. Then there exists a continuous function h such that $D_L = D_h = \langle a; b\rangle$ and $L(t) = \langle o\, h(t)\, \infty)$ for $t \in D_h$. We consider two cases. First, when $(1\,L)$ is satisfied. Then, setting $c = b - a$ and

$$f(s) = o + \frac{1}{\left|\overrightarrow{oh(a+s)}\right|}\overrightarrow{oh(a+s)} \qquad \text{for } s \in \langle 0; c\rangle$$

we see that

$$f(s) = f(t) \qquad \text{for } s, t \in D_f \tag{21}$$

and

$$\langle o\, f\,(\cdot)\ \infty) = (s \mapsto L(a+s)) \in \mathcal{A}.$$

The condition $(0; f)$ holds in this case. From $(0; f)$ it follows (21). In the second case we assume $(2\,L)$. Thus, for any $t \in D_h$ we have $\delta_t > 0$ such that the function $L|D_L \cap (t - \delta_t; t + \delta_t)$ is 1–1. Then there exist $\tau_1, \ldots, \tau_l \in D_L$ such that $\tau_1 < \ldots < \tau_l$ and $D_L \subset \bigcup_{j=1}^{l}(a_j; b_j)$, where $a_j = \tau_j - \frac{\delta_{\tau_j}}{2}$, $b_j = \tau_j + \frac{\delta_{\tau_j}}{2}$. We have then 1–1 functions

$$L|D_L \cap \langle a_j; b_j\rangle\,, \qquad j \in \{1,\ldots,l\}\,.$$

Setting, $g(t) = o + \frac{1}{\left|\overrightarrow{oh(t)}\right|}\overrightarrow{oh(t)}$ we get $\left|\overrightarrow{og(t)}\right| = 1$ and $L(t) = \langle o\, g\,(t)\ \infty)$ for $t \in D_L$ and 1–1 functions $g|D_g \cap \langle a_j; b_j\rangle$, $D_g = D_L$. We may assume that $a_1 = a$ and $b_l = b$, so $D_L \cap \langle a_j; b_j\rangle = \langle a_j; b_j\rangle$ and setting $g_j = g|\,\langle a_j; b_j\rangle$ we get

$$|g_j|\,(t) \le 2\pi \qquad \text{for } t \in \langle a_j; b_j\rangle\,.$$

Hence it follows that for any $t \in D_g$ we have

$$|g|\,(t) \le |g|\,(b) \le \sum_{j=1}^{l} |g_j|\,(b_j) \le 2l\pi < +\infty.$$

Then the function $|g|$ is finite continuous and increasing. Taking the inverse function $|g|^{-1}$ to $|g|$ and setting $f = g \circ |g|^{-1}$ we get the continuous function f with $D_f = \langle 0; c\rangle$, where $c = |g|\,(b)$. It is easy to see that $|f|$ is continuous and increasing and $L\left(|g|^{-1}(s)\right) = \langle o\, f(s)\,\infty)$ for $s \in D_f$. Therefore, we have $(1; f)$ and $\langle o\, f\,(\cdot)\ \infty) = L \circ |g|^{-1} \equiv L$, so $\langle o\, f(\cdot)\,\infty) \in \mathcal{A}$. From (20) and $(1; f)$ it follows that there exist orthonormal vectors $e_1, e_2 \in \underline{V}$ such that

$$\overrightarrow{of(s)} = \cos s \cdot e_1 + \sin s \cdot e_2 \qquad \text{for } s \in D_f.$$

Thus f is smooth. Taking $\mathfrak{a} \in T_o(E, V, \rightarrow)/\equiv_o$ such that $f \in \mathfrak{a}$ we get $\mathcal{A} = <\mathfrak{a}>$.

To prove that f is uniquely determined we take a continuous function $f_1\colon D_{f_1} \to E$ with $D_{f_1} = \langle 0; c_1\rangle$, $c_1 > 0$, $\langle o\, f_1\,(\cdot)\ \infty) \in \mathcal{A}$, $\left|\overrightarrow{of_1(t)}\right| = 1$ for $t \in D_{f_1}$ and satisfying $(0; f_1)$ or $(1; f_1)$. Then there exists a real continuous increasing function φ such that $D_\varphi = D_f$ and $\varphi D_\varphi = D_{f_1}$ and $\langle o\, f_1(\varphi(s))\,\infty) = \langle o\, f(s)\,\infty)$ for $s \in D_f$. Thus, $\overrightarrow{of_1(\varphi(s))} = \lambda(s)\,\overrightarrow{of(s)}$, where $\lambda\,(s) > 0$ for $s \in D_f$. Hence it follows that $1 = \left|\overrightarrow{of_1(\varphi(s))}\right| = \lambda(s)\left|\overrightarrow{of(s)}\right| = \lambda(s)$, so $f_1 \circ \varphi = f$. This yields $|f_1| \circ |\varphi| = |f|$. If $(0; f_1)$ holds, then $|f_1| = 0$, so $|f| = 0$. If $(1; f_1)$ is satisfied, then $\varphi = |f| = \mathrm{id}_{\langle 0;c\rangle}$. Therefore $f_1 = f$.

Corollary

If (0) *is an affine plane, i.e.* $\dim V = 2$*, then the function in* (15) *is* 1–1 *and maps* $\mathrm{soa}(E, V, \rightarrow)$ *onto* $\langle E, V, \rightarrow)/\equiv$.

Indeed, taking any positively defined scalar product in V we get an Euclidean space and we may apply Proposition 6.

4. Conclusion

The case when the affine space is 1-dimensional is not of importance however from purely logical point of view the definition of the set $\langle E, V, \overrightarrow{\ }\rangle/\equiv$ is correct.

Remark

If the affine space (0) is 1-dimensional, then all elements of $\langle E, V, \overrightarrow{\ }\rangle/\equiv$ are zero angles and (15) is 1–1 and maps $\mathrm{soa}(E, V, \overrightarrow{\ })$ onto $\langle E, V, \overrightarrow{\ }\rangle/\equiv$.

Indeed, for any $\mathcal{A} \in \langle E, V, \overrightarrow{\ }\rangle/\equiv$ there is $L \in \mathcal{A}$, so $L(t) = \langle o\, f(t)\, \infty\rangle$ and $o \neq f(t)$ for $t \in D_L$, where $f\colon D_L \to E$ is continuous and $(1\, L)$ or $(2\, L)$ holds. Let $0 \neq e \in \underline{V}$. Then $\overrightarrow{of(t)} = \lambda(t)e$, $0 \neq \lambda(t) \in \mathbb{R}$. According to Lemma λ is continuous. Thus $\lambda(t) > 0$ for $t \in D_L$ or $\lambda(t) < 0$ for $t \in D_L$. We may assume that $\lambda(t) > 0$. Therefore $L(t) = \langle o\, p\, \infty\rangle$, where $p = o + e$. Setting $f_1(t) = p$ for $p \in D_L$ we get a smooth function f_1 for which $L(t) = \langle o\, f_1(t)\, \infty\rangle$ as $t \in D_L$. Then we have $(1\, L)$. For $\mathfrak{a} \in T_o(E, V, \overrightarrow{\ })/\equiv_o$ such that $f_1 \in \mathfrak{a}$ we get $<\mathfrak{a}> = \mathcal{A}$.

Proposition 5, Corollary to Proposition 6 and the above Remark allows us to conclude our consideration by

Theorem

For any affine space (0) *the function* (15) *is* 1–1. *This function maps the set* $\mathrm{soa}(E, V, \overrightarrow{\ })$ *of all smooth oriented angles in the affine space* (0) *onto the set* $\langle E, V, \overrightarrow{\ }\rangle/\equiv$ *of all oriented angles in* (0) *if and only if* $\dim V = 2$ *or* $\dim V = 1$.

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