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Bi-Bazilevič functions of complex order involving Ruscheweyh type q-difference operator

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Abstract. In this paper, we define a new subclass of bi-univalent functions involving q-difference operator in the open unit disk. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\triangle = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as $(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, where $f(z)$ is given by (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

An analytic function f is subordinate to an analytic function h, written $f(z) \prec h(z)$ $(z \in \triangle)$, provided there is an analytic function w defined on $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = h(w(z))$.

By $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\triangle$. Some of the important and well-investigated subclasses of the univalent function

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class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\triangle$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\triangle$.

Ma and Minda [13] unified various subclasses of starlike functions and convex functions which consist of functions $f \in \mathcal{A}$ satisfying the subordinations

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad \text{and} \qquad 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \tag{2}$$

respectively, here (and throughout this paper) ϕ with positive real part in the unit disk $\triangle$, $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps $\triangle$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \qquad B_1 > 0.$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad z \in \triangle$$

and

$$f(f^{-1}(w)) = w, \qquad |w| < r_0(f),\ r_0(f) \geq \frac{1}{4},$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both f and f^{-1} are univalent in $\triangle$, in the sense that f^{-1} has a univalent analytic continuation to $\triangle$. Let Σ denote the class of bi-univalent functions in $\triangle$ given by (1).

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.

Now we recall here the notion of q-operator i.e. q-difference operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of q-calculus was initiated by Jackson [8], recently Kanas and Răducanu [11] have used the fractional q-calculus operators in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

Let $0 < q < 1$. For any non-negative integer n, the q-integer number n is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-1}, \qquad [0]_q = 0. \tag{4}$$

In general, we will denote

$$[x]_q = \frac{1 - q^x}{1 - q}$$

for a non-integer number x. Also the q-number shifted factorial is defined by

$$[n]_q! = [n]_q [n-1]_q \ldots [2]_q [1]_q, \qquad [0]_q! = 1.$$

Clearly,

$$\lim_{q \to 1^-} [n]_q = n \qquad \text{and} \qquad \lim_{q \to 1^-} [n]_q! = n!.$$

For $0 < q < 1$, the Jackson's q-derivative operator (or q-difference operator) of a function $f \in \mathcal{A}$ given by (1) defined as follows [8]

$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \tag{5}$$

$$\mathcal{D}_q^0 f(z) = f(z),$$

$$\mathcal{D}_q^m f(z) = \mathcal{D}_q(\mathcal{D}_q^{m-1} f(z)) \qquad \text{for } m \in \mathbb{N} = \{1, 2, \ldots\}.$$

From (5), we have

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \qquad z \in \triangle,$$

where $[n]_q$ is given by (4).

For a function $h(z) = z^n$ we obtain

$$D_q h(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$$

and

$$\lim_{q \to 1^-} \mathcal{D}_q h(z) = \lim_{q \to 1^-} ([n]_q z^{n-1}) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative.

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The q-generalized Pochhammer symbol is defined by

$$[t; n]_q = [t]_q [t+1]_q [t+2]_q \ldots [t+n-1]_q$$

and for $t > 0$ the q-gamma function is defined by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1.$$

Using the q-difference operator, Kannas and Raducanu [11] defined the Ruscheweyh q-differential operator as below. For $f \in \mathcal{A}$,

$$\mathcal{R}_q^\delta f(z) = f(z) * F_{q,\delta+1}(z), \qquad \delta > -1, \ z \in \triangle, \tag{6}$$

where

$$F_{q,\delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q! \, \Gamma_q(1+\delta)} z^n = z + \sum_{n=2}^{\infty} \frac{[\delta+1; n]_q}{[n-1]_q!} z^n. \tag{7}$$

We note that

$$\lim_{q \to 1^-} F_{q,\delta+1}(z) = \frac{z}{(1-z)^{\delta+1}}, \qquad \lim_{q \to 1^-} \mathcal{R}_q^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}.$$

Making use of (6) and (7), we have

$$\mathcal{R}_q^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q! \, \Gamma_q(1+\delta)} a_n z^n, \qquad z \in \triangle. \tag{8}$$

From (8), we note that

$$\mathcal{R}_q^0 f(z) = f(z),$$
$$\mathcal{R}_q^1 f(z) = z\mathcal{D}_q f(z),$$
$$\mathcal{R}_q^m f(z) = \frac{z\mathcal{D}_q^m(z^{m-1}f(z))}{[m]_q!} \qquad \text{for } m \in \mathbb{N}.$$

Also we have

$$\mathcal{D}_q(\mathcal{R}_q^\delta f(z)) = 1 + \sum_{n=2}^{\infty} \Theta_n(q,\delta) a_n z^{n-1}, \tag{9}$$

where

$$\Theta_n := \Theta_n(q,\delta) = \frac{[n]_q\,\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)}. \tag{10}$$

For our study, we will use the short presentation

$$\Theta_2 = \Theta_2(q,\delta) = \frac{[2]_q\,\Gamma_q(2+\delta)}{\Gamma_q(1+\delta)},$$
$$\Theta_3 = \Theta_3(q,\delta) = \frac{[3]_q\,\Gamma_q(3+\delta)}{[2]_q!\,\Gamma_q(1+\delta)}.$$

Recently there has been triggering interest to study bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1,2,3\}$$

is still an open problem (see [2, 3, 4, 12, 14, 18]). Many researchers (see [1, 7, 9, 17]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ.

Motivated by the earlier work of Bulut [5], Deniz [6], Inayat Noor [10] and Srivastava et al. [16], in the present paper we introduce new families of Bazilevič functions of complex order of the function class Σ, involving the operator $\mathcal{D}_q(\mathcal{R}_q^\delta f(z))$, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass of function class Σ. Several related classes are also considered, and connection to earlier known results are made.

Definition 1.1
A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_\Sigma^q(\gamma,\lambda,\delta;\phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1\right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1\right) \prec \phi(w),$$

where $z, w \in \triangle$, $\gamma \in \mathbb{C} \setminus \{0\}$, $\delta > -1$, $\lambda \geq 0$ and the function $g = f^{-1}$ is given by (3).

Remark 1.1
The following special cases of Definition 1.1 are worthy of note:

(i) A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, 0, \delta; \phi) \equiv \mathcal{S}_{\Sigma}^{q}(\gamma, \delta; \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\Big(\frac{z\mathcal{D}_q(\mathcal{R}_q^{\delta} f(z))}{\mathcal{R}_q^{\delta} f(z)} - 1\Big) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{w\mathcal{D}_q(\mathcal{R}_q^{\delta} g(w))}{\mathcal{R}_q^{\delta} g(w)} - 1\Big) \prec \phi(w),$$

where $z, w \in \triangle$, $\gamma \in \mathbb{C} \setminus \{0\}$, $\delta > -1$ and the function $g = f^{-1}$ is given by (3).

(ii) A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, 1, \delta; \phi) \equiv \mathcal{H}_{\Sigma}^{q}(\gamma, \delta; \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}(\mathcal{D}_q(\mathcal{R}_q^{\delta} f(z)) - 1) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}(\mathcal{D}_q(\mathcal{R}_q^{\delta} g(w)) - 1) \prec \phi(w),$$

where $z, w \in \triangle$, $\gamma \in \mathbb{C} \setminus \{0\}$, $\delta > -1$ and the function $g = f^{-1}$ is given by (3).

(iii) If we set $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, then the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \phi) \equiv \mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; A, B)$ which is defined as $f \in \Sigma$,

$$1 + \frac{1}{\gamma}\Big(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^{\delta} f(z))}{(\mathcal{R}_q^{\delta} f(z))^{1-\lambda}} - 1\Big) \prec \frac{1+Az}{1+Bz}$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{w^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^{\delta} g(w))}{(\mathcal{R}_q^{\delta} g(w))^{1-\lambda}} - 1\Big) \prec \frac{1+Aw}{1+Bw},$$

where $z, w \in \triangle$, $\gamma \in \mathbb{C} \setminus \{0\}$, $\delta > -1$, $\lambda \geq 0$ and the function $g = f^{-1}$ is given by (3).

(iv) If we set $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, then the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \phi) \equiv \mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \beta)$ which is defined as $f \in \Sigma$,

$$\Re\Big[1 + \frac{1}{\gamma}\Big(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^{\delta} f(z))}{(\mathcal{R}_q^{\delta} f(z))^{1-\lambda}} - 1\Big)\Big] > \beta$$

and

$$\Re\Big[1 + \frac{1}{\gamma}\Big(\frac{w^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^{\delta} g(w))}{(\mathcal{R}_q^{\delta} g(w))^{1-\lambda}} - 1\Big)\Big] > \beta,$$

where $z, w \in \triangle$, $\gamma \in \mathbb{C} \setminus \{0\}$, $\delta > -1$, $\lambda \geq 0$ and the function $g = f^{-1}$ is given by (3).

On specializing the parameters λ and δ, one can state the various new subclasses of Σ.

2. Coefficient Bounds for the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \phi)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \phi)$.

In order to derive our main results, we shall need the following lemma.

Lemma 2.1 (see [15])
If $p \in \mathcal{P}$, then $|p_k| \le 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\triangle$ for which $\Re(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ for $z \in \triangle$.

Theorem 2.1
Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \phi)$. Then

$$|a_2| \le \sqrt{\frac{N}{D}}, \tag{11}$$

where

$$\begin{aligned} N &= 2|\gamma|^2 B_1^3 (1+q)^2 (1+q+q^2), \\ D &= |\gamma B_1^2 [2(1+q)^2(\lambda+q+q^2)\Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_2^2] \\ &\quad - 2(B_2 - B_1)(\lambda+q)^2(1+q+q^2)\Theta_2^2| \end{aligned}$$

and

$$|a_3| \le \left(\frac{|\gamma| B_1 (1+q)}{(\lambda+q)\Theta_2}\right)^2 + \frac{|\gamma| B_1 (1+q+q^2)}{(\lambda+q+q^2)\Theta_3}. \tag{12}$$

Proof. Let $f \in \mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \phi)$ and $g = f^{-1}$ be given by (3). Then there are analytic functions $u, v \colon \triangle \to \triangle$ with $u(0) = 0 = v(0)$, satisfying

$$1 + \frac{1}{\gamma}\left(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^{\delta} f(z))}{(\mathcal{R}_q^{\delta} f(z))^{1-\lambda}} - 1\right) = \phi(u(z)) \tag{13}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^{\delta} g(w))}{(\mathcal{R}_q^{\delta} g(w))^{1-\lambda}} - 1\right) = \phi(v(w)). \tag{14}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \dots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \dots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \dots\right] \tag{15}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\Big[q_1 z + \Big(q_2 - \frac{q_1^2}{2}\Big)z^2 + \dots\Big]. \tag{16}$$

Then $p(z)$ and $q(z)$ are analytic in $\triangle$ with $p(0) = 1 = q(0)$. Since $u, v\colon \triangle \to \triangle$, the functions $p(z)$ and $q(z)$ have a positive real part in $\triangle$, and $|p_k| \le 2$ and $|q_k| \le 2$ for each k. Using (15) and (16) in (13) and (14) respectively, we have

$$1 + \frac{1}{\gamma}\Big(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1\Big) = \phi\Big(\frac{1}{2}\Big[p_1 z + \Big(p_2 - \frac{p_1^2}{2}\Big)z^2 + \dots\Big]\Big) \tag{17}$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{z^{1-\lambda}\mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1\Big) = \phi\Big(\frac{1}{2}\Big[q_1 w + \Big(q_2 - \frac{q_1^2}{2}\Big)w^2 + \dots\Big]\Big). \tag{18}$$

In light of (8)–(10), from (17) and (18), it is evident that

$$\begin{aligned}1 + \frac{1}{\gamma}\frac{\lambda+q}{1+q}\Theta_2 a_2 z + \frac{1}{\gamma}\Big[\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 a_3 + \frac{(\lambda-1)(\lambda+2q)}{2(1+q)^2}\Theta_2^2 a_2^2\Big]z^2 + \dots \\ = 1 + \frac{1}{2}B_1 p_1 z + \Big[\frac{1}{2}B_1\Big(p_2 - \frac{p_1^2}{2}\Big) + \frac{1}{4}B_2 p_1^2\Big]z^2 + \dots\end{aligned}$$

and

$$\begin{aligned}1 - \frac{1}{\gamma}\frac{\lambda+q}{1+q}\Theta_2 a_2 w + \frac{1}{\gamma}\Big[-\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 a_3 \\ + \Big(2\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 + \frac{(\lambda-1)(\lambda+2q)}{2(1+q)^2}\Theta_2^2\Big)a_2^2\Big]w^2 + \dots \\ = 1 + \frac{1}{2}B_1 q_1 w + \Big[\frac{1}{2}B_1\Big(q_2 - \frac{q_1^2}{2}\Big) + \frac{1}{4}B_2 q_1^2\Big]w^2 + \dots.\end{aligned}$$

which yields the following relations:

$$\frac{\lambda+q}{1+q}\Theta_2 a_2 = \frac{\gamma}{2}B_1 p_1, \tag{19}$$

$$\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 a_3 + \frac{(\lambda-1)(\lambda+2q)}{2(1+q)^2}\Theta_2^2 a_2^2 = \frac{\gamma}{2}B_1\Big(p_2 - \frac{p_1^2}{2}\Big) + \frac{\gamma}{4}B_2 p_1^2, \tag{20}$$

$$-\frac{\lambda+q}{1+q}\Theta_2 a_2 = \frac{\gamma}{2}B_1 q_1 \tag{21}$$

and

$$\begin{aligned}-\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 a_3 + \Big(2\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 + \frac{(\lambda-1)(\lambda+2q)}{2(1+q)^2}\Theta_2^2\Big)a_2^2 \\ = \frac{\gamma}{2}B_1\Big(q_2 - \frac{q_1^2}{2}\Big) + \frac{\gamma}{4}B_2 q_1^2.\end{aligned} \tag{22}$$

From (19) and (21), it follows that

$$p_1 = -q_1 \tag{23}$$

and

$$8\frac{(\lambda+q)^2}{(1+q)^2}\Theta_2^2 a_2^2=\gamma^2 B_1^2(p_1^2+q_1^2). \tag{24}$$

Adding (20) and (22), we obtain

$$\begin{aligned}\Big(2\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3+\frac{(\lambda-1)(\lambda+2q)}{(1+q)^2}\Theta_2^2\Big)a_2^2\\ =\frac{\gamma B_1}{2}(p_2+q_2)+\frac{\gamma}{4}(B_2-B_1)(p_1^2+q_1^2).\end{aligned} \tag{25}$$

Using (24) in (25), we get

$$a_2^2=\frac{N_0}{D_0},$$

where

$$\begin{aligned}N_0&=\gamma^2 B_1^3(1+q)^2(1+q+q^2)(p_2+q_2),\\ D_0&=2\gamma B_1^2[2(1+q)^2(\lambda+q+q^2)\Theta_3+(\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_2^2]\\ &\quad-4(B_2-B_1)(\lambda+q)^2(1+q+q^2)\Theta_2^2.\end{aligned}$$

Applying Lemma 2.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2\le\frac{N}{D},$$

where

$$\begin{aligned}N&=2|\gamma|^2 B_1^3(1+q)^2(1+q+q^2),\\ D&=|\gamma B_1^2[2(1+q)^2(\lambda+q+q^2)\Theta_3+(\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_2^2]\\ &\quad-2(B_2-B_1)(\lambda+q)^2(1+q+q^2)\Theta_2^2|.\end{aligned}$$

This gives the bound on $|a_2|$ as asserted in (11).

Next, in order to find the bound on $|a_3|$, by subtracting (22) from (20), we get

$$\begin{aligned}2\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 a_3-2\frac{\lambda+q+q^2}{1+q+q^2}\Theta_3 a_2^2\\ =\frac{\gamma B_1}{2}\Big[(p_2-q_2)-\frac{1}{2}(p_1^2-q_1^2)\Big]+\frac{\gamma B_2}{4}(p_1^2-q_1^2).\end{aligned} \tag{26}$$

Using (23) and (24) in (26), we get

$$a_3=\frac{\gamma^2 B_1^2(1+q)^2(p_1^2+q_1^2)}{8(\lambda+q)^2\Theta_2^2}+\frac{\gamma B_1(1+q+q^2)(p_2-q_2)}{4(\lambda+q+q^2)\Theta_3}. \tag{27}$$

Applying Lemma 2.1 once again for the coefficients p_1, q_1, p_2 and q_2, we readily get (12). This completes the proof of Theorem 2.1.

3. Corollaries and Consequences

By setting $\lambda = 0$ in Theorem 2.1, we have the following Theorem.

Theorem 3.1
Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \delta; \phi)$. Then

$$|a_2| \leq \sqrt{\frac{|\gamma|^2 B_1^3 (1+q)^2 (1+q+q^2)}{q|\gamma B_1^2[(1+q)^3\Theta_3 - (1+q+q^2)\Theta_2^2] - (B_2 - B_1) q (1+q+q^2)\Theta_2^2|}}$$

and

$$|a_3| \leq \Big(\frac{|\gamma| B_1 (1+q)}{q\Theta_2}\Big)^2 + \frac{|\gamma| B_1 (1+q+q^2)}{q(1+q)\Theta_3}.$$

By setting $\lambda = 1$ in Theorem 2.1, we have the following result.

Theorem 3.2
Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_{\Sigma}^{q}(\gamma, \delta; \phi)$. Then

$$|a_2| \leq \sqrt{\frac{2|\gamma|^2 B_1^3 (1+q)^2}{|\gamma B_1^2[2(1+q)^2\Theta_3 + (1+2q)\Theta_2^2] - 2(B_2 - B_1)(1+q)^2\Theta_2^2|}}$$

and

$$|a_3| \leq \Big(\frac{|\gamma| B_1}{\Theta_2}\Big)^2 + \frac{|\gamma| B_1}{\Theta_3}.$$

By setting $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.1, we state the following Theorem.

Theorem 3.3
Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; A, B)$. Then

$$|a_2| \leq \sqrt{\frac{N}{D}},$$

where

$$\begin{aligned} N &= 2|\gamma|^2 (A-B)^2 (1+q)^2 (1+q+q^2), \\ D &= |\gamma(A-B)[2(1+q)^2(\lambda+q+q^2)\Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_2^2] \\ &\quad + 2(B+1)(\lambda+q)^2(1+q+q^2)\Theta_2^2| \end{aligned}$$

and

$$|a_3| \leq \Big(\frac{|\gamma|(A-B)(1+q)}{(\lambda+q)\Theta_2}\Big)^2 + \frac{|\gamma|(A-B)(1+q+q^2)}{(\lambda+q+q^2)\Theta_3}.$$

Further, by setting $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ in Theorem 2.1 we get the following result.

Theorem 3.4
Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \delta; \beta)$. Then

$$|a_2| \leq \sqrt{\frac{4|\gamma|(1-\beta)(1+q)^2(1+q+q^2)}{|2(1+q)^2(\lambda+q+q^2)\Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_2^2|}}$$

and

$$|a_3| \leq \Big(\frac{2|\gamma|(1-\beta)(1+q)}{(\lambda+q)\Theta_2}\Big)^2 + \frac{2|\gamma|(1-\beta)(1+q+q^2)}{(\lambda+q+q^2)\Theta_3}.$$

Concluding Remarks. By taking $\delta = 0$ and specializing the parameters λ and γ, various other interesting corollaries and consequences of our main results (which are asserted by Theorem 2.1 above) can be derived easily hence we omit the details.

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